PERIOD MAPPINGS AND PROPERTIES OF THE HODGE LINE BUNDLE

MARK GREEN, PHILLIP GRIFFITHS, RADU LAZA, AND COLLEEN ROBLES

Abstract. Let $\wp \subseteq \Gamma\backslash D$ be the image of a period map. We discuss progress towards a conjectural Hodge theoretic completion $\overline{\wp}$, an analogue of the Satake-Baily-Borel compactification in the classical case. The set $\overline{\wp}$ is defined, and we conjecture that it admits the structure of a compact complex analytic variety. The conjecture is proved when $\dim \wp = 1, 2$. In general, assuming the conjecture holds, we prove that the augmented Hodge line bundle $\hat{\Lambda}$ extends to an ample line bundle on $\overline{\wp}$, thus giving $\overline{\wp}$ the structure of a projective algebraic variety that compactifies $\wp$.

Contents

1. Introduction

1.1. **Overview.** Let $\mathcal{M}$ be the moduli space for smooth varieties X of general type and with given numerical characters. In a sweeping generalization of the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of the moduli space of curves, Kollár, Shepherd-Barron and Alexeev (KSBA), with contributions of many others, have constructed a canonical projective completion $\overline{\mathcal{M}}$ with geometric meaning (see [Kol13] and the references therein). However, even in the case of surfaces of general type with small invariants, little is known towards a classification of the boundary varieties, and about the global structure of the moduli space and its boundary $\partial\mathcal{M}$. An idea that goes back to the origin of the moduli subject is to study $\mathcal{M}$ and its compactifications by using a natural invariant, namely a period mapping $\Phi : \mathcal{M} \to \Gamma\backslash D$,

Date: July 7, 2020.
Laza was partially supported by NSF grants DMS 1802128 and 1361143.
Robles was partially supported from the NSF via grants DMS 1611939 and 1361120.

which associates to a (smooth) variety its relevant cohomology group endowed with a Hodge structure. Since $\Gamma\backslash D$ has a rich structure given by representation theory and arithmetic, one expects that the period map would be a powerful tool for understanding the structure of $\mathcal{M}$ and $\overline{\mathcal{M}}$.

This is indeed the case for the study of moduli spaces of curves, abelian varieties, $K3$ surfaces, and a few other related cases. Beyond these classical cases, to our knowledge, very little is known in terms of the behavior of period maps for compactified moduli spaces. In fact, arguably, the simplest non-classical case is that of surfaces of general type with small invariants, for instance the case of H-surfaces and I-surfaces (surfaces of general type with $p_g = 2$, $q = 0$, $K^2 = 2$ or 1 respectively). For such surfaces, one has a reasonable hold on the geometry of the KSBA degenerations (see [FPR15a, FPR15b, FPR17]). On the other hand, in current work in progress (parts of it also jointly with M. Franciosi, R. Pardini, and S. Rollenske, the authors of loc. cit.) we have obtained a number of partial results about an extended period map Φ_e for H and I-surfaces. The emerging picture from this investigation is that the period map is a very effective way to organize and give structure to the boundary $\partial\mathcal{M}$ of the compactified moduli spaces for H and I-surfaces. We refer to the announcements [Gri18, Gri19] for a discussion of our program and some specific results.

In this paper, we focus on a piece of this program. Namely, given the image of a period map $\wp := \Phi(\mathcal{M}) \subset \Gamma\backslash D$, we are interested in a Hodge theoretic completion $\overline{\wp}$ of it. Before going into details, we recall that the subject naturally splits into a classical case (essentially abelian varieties, and $K3$ type), when D is a Hermitian symmetric domain, and a non-classical case (encompassing almost everything else). The classical case is well understood: there is a canonical (but quite singular) Satake-Baily-Borel projective compactification of $\Gamma\backslash D$ ([Sat60], [BB66]), which admits various (partial) toroidal resolutions ([AMRT10]). In contrast, the non-classical case is much harder and few results are known. For instance, we note that except the classical cases, $\Gamma\backslash D$ is never an algebraic variety ([GRT14]). Moreover, the global monodromy group Γ may be a thin matrix group of infinite index in an arithmetic group (so that $vol(\Gamma\backslash D) = \infty$). These observations reflect the very different character the subject has from the classical case. One consequence of this is that both analytic and algebraic methods will be required for the study of the period map Φ and its extensions Φ_e.

Remark 1.1.1. We will repeatedly refer to *the classical case*. This is the case when the period domain D is Hermitian symmetric (or more generally, D is an unconstrained Mumford-Tate domain), and Γ is an arithmetic group. Geometrically, this corresponds to period maps for abelian varieties or K3-type objects (e.g. K3's, hyper-Kähler manifolds, cubic fourfolds). Occasionally, when we say classical case, we implicitly assume also $\wp = \Gamma\backslash D$.

1.1.1. *Set-up and Problem.* Concretely, in this paper, we consider a period mapping

$$\Phi : B \to \Gamma\backslash D \tag{1.1.2}$$

with B a smooth, quasi-projective variety. We fix a smooth projective compactification $\overline{B}$ such that $Z = \overline{B}\backslash B$ is a reduced simple normal crossing divisor. We further assume that the local monodromies $T_i = \exp(N_i)$ around the irreducible branches Z_i of Z are unipotent. (For example, this situation might arise starting with a period map $\Phi : \mathcal{M} \to \Gamma\backslash D$ of geometric origin as above and then passing to a finite covering B and suitable completions and desingularizations.) We denote by

$$\wp \; := \; \Phi(B) \; \subset \; \Gamma\backslash D \tag{1.1.3}$$

the image of such a period map. We sometimes assume additionally that Φ is generically injective (i.e. generic Torelli holds); this is essentially the primitive case.

Our goal here is to construct a completion $\overline{\wp}$ of the image of the period map $\wp$, analogous to the Satake-Baily-Borel (SBB) compactification in the classical case. We point out that while the period map is a priori transcendental in nature and $\Gamma\backslash D$ might not be algebraic, various algebraicity properties for $\wp$ and Φ are known (e.g. [Som78]) or conjectured. Most recently, a significant breakthrough has been obtained by Bakker–Brunebarbe–Tsimerman (BBT) [BBT18] who proved using o-minimality techniques that, under the additional assumption that Γ is arithmetic, the image $\wp$ of the period map is a quasi-projective variety. The purpose of this paper is to report on progress towards strengthening the BBT result by constructing a (canonical) SBB type projective compactification $\overline{\wp}$ $(= \Phi(\overline{B})_e)$ of $\wp$. Our techniques are constructive, in line with the standard Hodge theoretic degeneration arguments, and, unlike SSB and BBT, we do not assume that Γ is arithmetic.

1.2. **The conjectural SBB compactification of $\wp$.** In the classical case, the SBB compactification of $\Gamma\backslash D$ is naturally stratified (e.g. $\mathfrak{A}_g^* = \mathfrak{A}_g \sqcup \mathfrak{A}_{g-1} \sqcup \cdots \sqcup \mathfrak{A}_0$) with the various strata encoding the graded pieces of the possible limit mixed Hodge structure (LMHS). While in contrast, the toroidal compactifications keep track (at least partially) of the full LMHS (e.g. see [Cat84]). Inspired by this, partial results towards a toroidal type compactification for general images of period maps were obtained by Kato–Nakayama–Usui [KU09]. Here, we aim for a SBB type compactification for images $\wp$ of period maps, and thus it is natural to construct a completion (at least set-theoretically) by gluing strata corresponding to the possible graded pieces of LMHS as one extends the period map Φ from B to $\overline{B}$.

More precisely, given a period mapping (1.1.2) and a completion $B \subset \overline{B}$ as specified previously, we define

$$Z_I \; := \; \bigcap_{i \in I} Z_i$$

where Z_i are the irreducible components of the divisor at infinity $Z := \overline{B}\backslash B$. This in turn defines a finite stratification of $\overline{B}$ by setting

$$Z_I^* \; := \; Z_I \backslash \left(\cup_{|J|>|I|} Z_J \right),$$

the open strata obtained by removing from Z_I the lower dimensional sub-strata. Naturally, we set $Z_\emptyset^* = B$, the big open stratum on which the period map Φ is defined. It is a consequence of [CKS86] that each open stratum Z_I^* carries a polarizable variation of mixed

Hodge structure (VMHS) with weight filtration $W(N_I)$,

$$N_I \ := \ \sum_{i\in I} N_i \,,$$

where N_i are the log monodromy transformations around each of the boundary divisors Z_i. (In the literature this VMHS is frequently referred to as the *limiting mixed Hodge structures* (LMHS) associated to the VHS over B. For the general theory of variations of mixed Hodge structures we refer to [SZ85].) Passing to the primitive part of the associated graded (which are pure polarized Hodge structures) of this variation of mixed Hodge structure gives period mappings

$$\Phi_I : Z_I^* \to \Gamma_I \backslash D_I \,; \tag{1.2.1}$$

see §2.4 for a brief review.

In general the maps Φ_I may not be proper. However, following [Gri70b], they can be grouped together into proper maps. Namely, for the main stratum B, we extend $\Phi (= \Phi_\emptyset)$ along the divisors with trivial monodromy ($N_i = 0$), resulting into a proper extension of Φ into $\Gamma\backslash D$. We then proceed inductively by grouping together the strata with the same monodromy type. More precisely, suppose that $I \subset J$ and $W(N_I) = W(N_J)$. Then, there are natural maps $\Gamma_J\backslash D_J \to \Gamma_I\backslash D_I$ (with finite fibers), and the period map Φ_I extends to Z_J^* where it coincides with the composition of Φ_J with the finite map $\Gamma_J\backslash D_J \to \Gamma_I\backslash D_I$. Moreover, for all $I \subset I' \subset J$ we have $W(N_I) = W(N_{I'}) = W(N_J)$. And given W, there exists a unique maximal I_W with the property $W = W(N_{I_W})$, a flag domain D_W, and finite maps $\Gamma_I\backslash D_I \to \Gamma_W\backslash D_W$ (for further discussion see [GGR20]). The resulting map

$$\Phi_W^\star \ : \ \bigcup_{W(N_I)=W} Z_I^* \ \to \ \Gamma_W\backslash D_W$$

is proper. Set

$$Z_W^\star \ := \ \bigcup_{W(N_I)=W} Z_I^* \tag{1.2.2}$$

and

$$\wp_W \ := \ \Phi_W^\star(Z_W^\star) \ \subset \ \Gamma_W\backslash D_W \,. \tag{1.2.3}$$

The properness of $\Phi_W^\star$ implies that each $\wp_W$ is a complex analytic variety. (Furthermore, according to [BBT18], at least assuming arithmeticity for each Γ_I, the strata $\wp_W$ are quasi-projective.) By construction, each $\wp_W$ parameterizes the graded pieces of the corresponding LMHS over the strata of $\overline{B}$.

We consider the Stein factorization

$$Z_W^\star \ \xrightarrow{\Phi_W^{\mathrm{c}}} \ \mathcal{S}_W \ \xrightarrow{\Phi_W^{\mathrm{f}}} \ \wp_W$$

of $\Phi^\star_W$. The $\mathcal{S}_W$ are normal complex analytic varieties, the fibres of $\Phi^{\mathrm{c}}_{\mathrm{e}} : Z^\star_W \to \mathcal{S}_W$ are connected, and the fibres of $\Phi^{\mathrm{f}}_{\mathrm{e}} : \mathcal{S}_W \to \wp_W$ are finite ([GR84]). Taking unions

$$\overline{\mathcal{S}} \ := \ \bigcup_W \mathcal{S}_W \quad \text{and} \quad \overline{\wp} \ := \ \bigcup_W \wp_W$$

we obtain set-theoretically the extension

$$\Phi_{\mathrm{e}} : \overline{B} \ \to \ \overline{\wp}\,,$$

of $\Phi : B \to \wp$ by defining $\Phi_{\mathrm{e}}|_{Z^\star_W} := \Phi^\star_W$, and its "Stein factorization"

$$\overline{B} \ \xrightarrow{\Phi^{\mathrm{c}}_{\mathrm{e}}} \ \overline{\mathcal{S}} \ \xrightarrow{\Phi^{\mathrm{f}}_{\mathrm{e}}} \ \overline{\wp}$$

by defining $\Phi^{\mathrm{c}}_{\mathrm{e}}|_{Z^\star_W} := \Phi^{\mathrm{c}}_W$ and $\Phi^{\mathrm{f}}_{\mathrm{e}}\big|_{\mathcal{S}_W} := \Phi^{\mathrm{f}}_W$. The resulting space $\overline{\wp}$ is our proposed SBB type compactification for general images of period maps. As it stands, our construction is set-theoretic with analytic pieces. In order for $\overline{\wp}$ to be a genuine generalization of the SBB construction (and Φ_e to be a Borel type [Bor72] extension of the period map), we expect the following to be true.

Conjecture 1.2.4. *The set $\overline{\mathcal{S}}$ admits the structure of a normal complex analytic variety with the properties that:*

(i) *The extension $\Phi^{\mathrm{c}}_{\mathrm{e}} : \overline{B} \to \overline{\mathcal{S}}$ is an analytic.*

(ii) *The restriction of the analytic structure on $\overline{\mathcal{S}}$ to the strata $\mathcal{S}_W$ coincides with the natural analytic structure on $\mathcal{S}_W$.*

Remark 1.2.5. Both $\overline{\mathcal{S}}$ and $\overline{\wp}$ inherit a topology from $\Phi^{\mathrm{c}}_{\mathrm{e}} : \overline{B} \to \overline{\mathcal{S}}$ and $\Phi_{\mathrm{e}} : \overline{B} \to \overline{\wp}$, respectively. Thus, they are topological spaces that are stratified by complex analytic spaces (or even quasi-projective varieties). Following [BB66], one can define $\mathcal{A}$ to be the sheaf of continuous functions with analytic restrictions on strata. Using Theorem 9.2 of loc. cit., one sees that the key missing ingredient to establish Conjecture 1.2.4 is to show that $\mathcal{A}$ locally separates points in $\overline{\mathcal{S}}$.

Remark 1.2.6 (*The conjecture holds when D is Hermitian).* It is well known that the conjecture holds in the classical case that D is Hermitian symmetric and Γ is arithmetic. In that situation $\Gamma\backslash D$ is a quasi-projective variety, with a projective compactification $(\Gamma\backslash D)^*$ [Sat60, BB66], and the Borel Extension Theorem [Bor72] yields an extension $\Phi_{\mathrm{e}} : \overline{B} \to (\Gamma\backslash D)^*$ of the period map (1.1.2) to an algebraic map. In this situation, we can take $\overline{\wp}$ to be the closure of $\wp$ in $(\Gamma\backslash D)^*$. In general, such an argument would not work: the quotient $\Gamma\backslash D$ is not algebraic, and meaningful compactifications are expected only in the horizontal directions.

Remark 1.2.7. Under the assumption that Γ is an arithmetic group, Kato-Nakayama-Usui [KU09] have constructed an analogue of the toroidal compactification in the horizontal directions (though the existence of a compatible fan is still open in general). In this case it is possible that our completion $\overline{\wp}$ could be obtained by taking closure in this toroidal type compactification, and "forgetting", or quotienting out, the extension data. (The arithmecity assumption of Γ does not seem essential for our construction. Moreover, in some concrete geometric examples Γ is known to be thin [BT14].)

Our first main result is to establish Conjecture 1.2.4 for the case when the base B is at most 2 dimensional. The one-dimensional case (or more precisely, the case $\dim \wp = 1$) is a direct consequence of [Som73] and [CDK95]. Using specific low-dimensional arguments, we establish the two dimensional case in §3.1.

Theorem 1.2.8. *Conjecture 1.2.4 holds when* $\dim B \leq 2$.

Remark 1.2.9. An ongoing study [GGR20] of the global structure of the period mappings at infinity yields further results on $\overline{\wp}$ and in particular a complete analysis of the 2-dimensional case.

Remark 1.2.10. In the Appendix, we comment on the so-called *Siegel property*, which plays a key role in Sommese's 1-dimensional case. The property fails in general, illustrating one of the challenges of passing from the 1-variable case to multiple variables.

1.2.1. *Establishing the conjecture in general.* Conjecture 1.2.4 amounts to an existence theorem, one that may be viewed analytically or algebraically. In both cases the key object is the extended augmented Hodge line bundle $\hat{\Lambda}_{\mathrm{e}} \to \overline{B}$ of §1.3 and its positivity properties.

From the algebraic perspective the conjecture is equivalent to the statement: *The line bundle* $\hat{\Lambda}_{\mathrm{e}} \to \overline{B}$ *is free.* What is known is that $\hat{\Lambda}_{\mathrm{e}}$ is nef and the curves $C \subset \overline{B}$ such that $\deg(\hat{\Lambda}_{\mathrm{e}} \otimes \mathcal{O}_C) = 0$ may be identified. Assuming that the differential Φ_* is injective at one point (e.g. generic Torelli holds), it follows that $\hat{\Lambda}_{\mathrm{e}}$ is also big. Both of these results follow from the discussions of §6. One may ask: *Why don't standard methods from birational geometry, specifically the base-point-free theorem* [KM98] *(or a variant of it), apply?* The answer seems to be that verification of the the sign assumptions required to apply the base-point-free theorem is a subtle matter, requiring good understanding of the global geometry of the fibers of Φ_{e} at infinity (this will be discussed in [GGR20]).

From an analytic perspective, and assuming for simplicity that the differential Φ_* is everywhere injective, the Chern form $\hat{\Omega}_{\mathrm{e}}$ of $\hat{\Lambda}_{\mathrm{e}}$ gives a complete Kähler metric on B whose holomorphic sectional curvatures are negative and bounded from above ($R(\xi) \leq -c$, with $c > 0$). Moreover, the holomorphic bi-sectional curvature is nonpositive $R(\xi, \eta) \leq 0$) and one may describe precisely the zero locus. Given $b_o \in B$ and $r \geq 0$, define an exhaustion of B by $B(b_0, r) = \{b \in B \mid \mathrm{dist}(b_o, b) \leq r\}$. In order to apply $L^2 - \bar{\partial}$ methods to the conjecture one needs to know the geometry of the Levi form of the boundary $\partial B(b_0, r)$. Near the fibres of $\Phi_{\mathrm{e}}|_Z$ this involves second-order curvature properties of the Hodge bundles, a seemingly quite interesting topic in Hodge theory that has yet to be explored.

1.3. **The extended augmented Hodge bundle and the projectivity of $\overline{\wp}$.** The set $\overline{\mathbb{S}}$ carries a natural (augmented Hodge) line bundle. Assuming that Conjecture 1.2.4 holds, we will prove that this line bundle is ample (Theorem 1.3.10), so that $\overline{\mathcal{S}}$ is in fact a projective variety (as it is also the case for SBB compactification in the classical case).

To define the natural line bundle on $\overline{\mathcal{S}}$, we recall that the period domain D is a homogeneous domain $D = G_{\mathbb{R}}/H$ that parametrizes effective, weight n, Q–polarized Hodge structures (PHS) $F^\bullet = \{F^n \subset \cdots \subset F^0\}$ (viewed as Hodge filtrations) with fixed Hodge numbers.

Definition 1.3.1. The *Hodge vector bundle* $\mathcal{F}^n \to D$ is the homogeneous vector bundle whose fiber at $F^\bullet \in D$ is F^n. The *Hodge line bundle* is

$$\Lambda \;:=\; \det \mathcal{F}^n \,.$$

We shall frequently refer to Λ as simply the *Hodge bundle*, but we will always use "vector" when discussing the Hodge vector bundle $\mathcal{F}^n$.

Remark 1.3.2. For applications to moduli of varieties of general type the Hodge vector and line bundles are especially important due to the identification $F^n = H^{n,0}(X) = H^0(K_X)$, where X is a smooth projective variety of dimension n and $H^n(X) = \oplus_{p+q=n} H^{p,q}(X)$ is the Hodge decomposition of its n^{th} cohomology group.

The Hodge vector bundle is one of a family of "quotient Hodge bundles".

Definition 1.3.3. The *p–th graded quotient Hodge bundle* is the holomorphic vector bundle $\mathrm{Gr}^p\mathcal{F} \to D$ whose fiber over $F^\bullet \in D$ is F^p/F^{p+1}. (Note that $F^{n+1} = 0$ implies $\mathrm{Gr}^n\mathcal{F}$ is the Hodge vector bundle $\mathcal{F}^n$.)

Out of these natural vector bundles, we produce a single line bundle as follows.

Definition 1.3.4. The *augmented Hodge (line) bundle*[1] is

$$\begin{aligned}\hat\Lambda \;&:=\; \det(\mathcal{F}^n) \otimes \det(\mathcal{F}^{n-1}) \otimes \cdots \otimes \det(\mathcal{F}^{\lceil (n+1)/2 \rceil})\\ &=\; \det(\mathcal{F}^n)^{f_n} \otimes \det(\mathrm{Gr}^{n-1}\mathcal{F})^{f_{n-1}} \otimes \cdots \otimes \det(\mathrm{Gr}^{\lceil (n+1)/2 \rceil}\mathcal{F})\,,\end{aligned}$$

with $f_p = p + 1 - \lceil (n+1)/2 \rceil$.

Remark 1.3.5. The Hodge bundle and augmented Hodge bundle agree when $n = 1, 2$. The use of the Hodge bundle in the weight 1 case (e.g. for studying moduli of curves) is a classical rich subject. For the geometric applications that we have in mind, we are concerned mostly with the $n = 2$ case, and thus there is no difference in working with the Hodge bundle. However, the difference between $\hat\Lambda$ and Λ becomes relevant starting with dimension 3.

Remark 1.3.6. Very roughly, the difference between the between Λ and $\hat\Lambda$ is that positivity of the augmented Hodge bundle corresponds to injectivity of the differential of the period

[1]For consistency with the existing literature, we point out that in [BBT18] the augmented Hodge bundle is called *Griffiths bundle*.

map, while positivity of the Hodge bundle corresponds to injectivity of only a component of the period map; this is made precise in (1.3.7). Let

$$\Phi_* : TB \;\to\; \bigoplus_{p=n}^{\lceil (n+1)/2 \rceil} \mathrm{Hom}\left(\mathrm{Gr}^p \mathcal{F}, \mathrm{Gr}^{p-1}\mathcal{F}\right)$$

denote the differential of the period map, and let

$$\Phi_{*,n} : TB \;\to\; \mathrm{Hom}(\mathcal{F}^n, \mathcal{F}^{n-1}/\mathcal{F}^n)$$

denote the component taking value in the first summand. (Notice that $\Phi_* = \Phi_{*,n}$ when $n = 1, 2$.) The Hodge metrics on the $\mathcal{F}^p$ induce metrics on Λ and $\hat{\Lambda}$; let Ω and $\hat{\Omega}$ denote the corresponding curvature forms. For $\xi \in TB$ we have

$$\begin{aligned} \Omega(\xi) &= \|\Phi_{*,n}(\xi)\|^2 \\ \hat{\Omega}(\xi) &= \|\Phi_*(\xi)\|^2 . \end{aligned} \tag{1.3.7}$$

The set $\overline{B}$ is stratified by the $Z_W^\star$. Let $B^\star$ be the open strata containing B. Equivalently, $B^\star \supset B$ is the largest subset of $\overline{B}$ to which Φ extends. Likewise, the set $\overline{\mathcal{S}}$ is stratified by the $\mathcal{S}_W$. Let $\mathcal{S} = \Phi_{\mathrm{e}}^{\mathrm{c}}(B^\star) \subset \overline{\mathcal{S}}$ be the strata containing $\Phi_{\mathrm{e}}^{\mathrm{c}}(B)$. The form $\hat{\Omega}$ descends to $\mathcal{S}$, and the discussion above implies $\hat{\Omega}$ is positive on the smooth points of $\mathcal{S}$, while Ω is only non-negative there.

It is standard that the data of a period mapping (1.1.2) is equivalent to that of a (polarized) *variation of Hodge structures* (VHS) $(\mathbb{V}, \mathcal{F}^\bullet, Q, \nabla)$ over B. Here $\mathbb{V}$ is a local system with Gauss-Manin connection $\nabla : \mathcal{O}_B(\mathbb{V}) \to \Omega^1_B(\mathbb{V})$ where $\mathcal{O}_B(\mathbb{V}) = \mathbb{V} \otimes \mathcal{O}_B$, the $\mathcal{F}^p \subset \mathcal{O}_B(\mathbb{V})$ are holomorphic sub-bundles, $Q : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$ is a horizontal bilinear form and where this data induces at each point of B a polarized Hodge structure. The infinitesimal period relation (IPR) is $\nabla \mathcal{F}^p \subseteq \Omega^1_B \otimes \mathcal{F}^{p-1}$. In this context the Hodge vector bundle $\mathcal{F}^n$ is the pull-back of $\mathcal{F}^n$ under the period map. We shall use interchangeably the data of period mappings and of variations of Hodge structure.

Under the assumption that the local monodromies around the branches Z_i of Z are unipotent with logarithms N_i, it is well known ([CKS86] and [PS08]) that there are canonical extensions of the Hodge filtration bundles $\mathcal{F}^p$ to vector bundles $\mathcal{F}^p_{\mathrm{e}} \to \overline{B}$ where the infinitesimal period relation becomes $\nabla \mathcal{F}^p_{\mathrm{e}} \subseteq \Omega^1_{\overline{B}}(\log Z) \otimes \mathcal{F}^{p-1}_{\mathrm{e}}$, and where $\mathrm{Res}_{Z_i} \nabla = N_i$ (up to a factor of $2\pi\sqrt{-1}$).

Definition 1.3.8. We denote by $\mathcal{F}^n_{\mathrm{e}}$ the canonical extension of the Hodge vector bundle, by $\Lambda_{\mathrm{e}} = \det \mathcal{F}^n_{\mathrm{e}}$ the canonically *extended Hodge (line) bundle* and by $\hat{\Lambda}_{\mathrm{e}}$ the canonically *extended augmented Hodge (line) bundle.*

The precise relationship between the canonical extensions of the line bundles over B, and the analogous line bundles over the open strata of $Z = \overline{B} \backslash B$ is given by

$$\Lambda_{\mathrm{e}}\big|_{Z_I^*} \;=\; \Lambda_I \quad \text{and} \quad \hat{\Lambda}_{\mathrm{e}}\big|_{Z_I^*} \;=\; \hat{\Lambda}_I. \tag{1.3.9}$$

Here the left-hand sides of these two expressions are the restrictions to Z_I^*; and the right-hand sides are the Hodge line bundle and augmented Hodge line bundle, respectively, associated to the period mapping Φ_I. The line bundle $\hat{\Lambda}_{\mathrm{e}} \to \overline{B}$ is trivial on the connected fibres of $\Phi_{\mathrm{e}}^{\mathrm{c}}$, and so descends to a line bundle $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$.

As announced, we prove (in Section 6) that the extended augmented line bundle is ample, giving the projectivity of the completed images $\overline{\wp}$ of period maps. More precisely, the following hold:

Theorem 1.3.10. *Assume that Conjecture 1.2.4 holds. Then $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$ is holomorphic and ample.*

Corollary 1.3.11. *The completion* $\overline{\mathcal{S}} = \operatorname{Proj} R(\overline{\mathcal{S}}, \hat{\Lambda}_{\mathrm{e}})$. (As usual, $R(\overline{\mathcal{S}}, \hat{\Lambda}_{\mathrm{e}})$ denotes the ring of sections.)

Remark 1.3.12. In the existing literature, interest has been focused primarily on the positivity properties of the Hodge vector and line bundles (cf. [Kol87, Vie83b] and the references therein). However for the general study of images of period maps it is the augmented Hodge bundle that is particularly relevant. In this regard an interesting question is: *Does either the Hodge bundle or the augmented Hodge bundle live on the KSBA completion $\overline{\mathcal{M}}$?* We conjecture that *the Hodge bundle lives over $\overline{\mathcal{M}}$ when $n = 2$.*

In the classical case $\Lambda_{\mathrm{e}} = \hat{\Lambda}_{\mathrm{e}}$ and this result is a consequence of the properties of the Satake-Baily-Borel construction. That construction is a global one in that sections of $\Lambda_e^{\otimes m}$ that give a projective embedding of $\Gamma\backslash D$ are constructed using modular forms. As explained above, such an approach is not possible in the non-classical case. Our proof of Theorem 1.3.10 is in spirit analogous to the one used by Kodaira to show that over a compact, complex manifold a line bundle with positive Chern class in the differential-geometric sense is ample. The proof of the result here depends on some rather subtle properties of the Chern form $\hat{\Omega}$ of the augmented Hodge bundle. From (1.3.7) we see that the augmented Hodge bundle has positivity properties. It is due to [CKS86] with an important amplification in [Kol87] that $\hat{\Omega}$ defines a closed $(1,1)$ current $\hat{\Omega}_{\mathrm{e}}$ on $\overline{B}$ that represents $c_1(\hat{\Lambda}_{\mathrm{e}})$ in cohomology. For the proof of Theorem 1.3.10 we need to significantly refine this in several ways. That analysis of the Chern form and the base point free theorem yield (in §3.2)

Theorem 1.3.13. *Suppose that $\hat{\Omega}$ is positive on on B (equivalently, Φ_* is everywhere injective), and that $0 \le -K_{\overline{B}} \cdot C$ whenever Φ_{e} collapses the curve $C \subset \overline{B}$ to a point. Then $\hat{\Lambda}_{\mathrm{e}} \to \overline{B}$ is free. In particular,* $\overline{\wp} = \operatorname{Proj} R(\overline{B}, \hat{\Lambda}_{\mathrm{e}})$.

1.4. **Properties of the Chern form.** As discussed above, one of our main techniques is the study of the Chern form $\hat{\Omega}_{\mathrm{e}}$ associated to the extended augmented Hodge line bundle $\hat{\Lambda}_{\mathrm{e}}$. Here, we briefly review the main aspects of this study. First, since currents are differential forms with distribution coefficients, the singular support $\operatorname{sing}\hat{\Omega}_{\mathrm{e}}$ of $\hat{\Omega}_{\mathrm{e}}$ is defined, and assuming (as we may, by passing to the proper extension of the period map) that all monodromy logarithms $N_i \neq 0$ we have

(a) *The singular support is* $\operatorname{sing}\hat{\Omega}_{\rm e} = Z$. (In general, $\operatorname{sing}\hat{\Omega}_{\rm e} \subseteq Z$.)

Next, it is well known that distributions and currents cannot in general be multiplied or restricted to submanifolds. To get around this one needs a more subtle notion than just the singular support. Associated to a current Ψ on a manifold Y is its *wave front set* $\mathrm{WF}(\Psi) \subset T^*Y$.[2] If $W \subset Y$ is a submanifold whose tangent spaces are transverse to the wave front set in the sense that $TW \subset \mathrm{WF}(\Psi)^\perp$, then the restriction $\Psi\big|_W$ is a well-defined current on W. This suggests that one should think of a refined notion of the singularities of $\hat{\Omega}_{\rm e}$ as being in $T^*\overline{B}$; in particular, one would like to assert that

(b) *There exists a well-defined way of defining a restriction* $\hat{\Omega}_{\rm e}\big|_{Z_I^*}$ *as a smooth* $(1,1)$ *form on the open strata* Z_I^*.

That this is possible will be part of the content of Theorem 1.4.1. Intuitively, we may think of this result as having $\mathrm{WF}(\hat{\Omega}_{\rm e}) \subseteq \cup_I N^*_{Z_I^*/\overline{B}}$, at least so far as the restriction property (b) is concerned. Finally, given (b), the last property that we would like is

(c) $\hat{\Omega}_{\rm e}\big|_{Z_I^*} = \hat{\Omega}_I$ *is the Chern form of* the Hodge bundle $\hat{\Lambda}_I \to Z_I^*$.

These desired properties do indeed hold, and we have

Theorem 1.4.1. *The Chern form* $\hat{\Omega}$ *of the augmented Hodge line bundle* $\hat{\Lambda} \to B$ *extends to a current* $\hat{\Omega}_{\rm e}$ *on the completion* $\overline{B}$ *of* B. *There it has singularities as described above. In particular,* (b) *and* (c) *hold, so that in a precise sense* $\hat{\Omega}_{\rm e}$ *represents the Chern class of the augmented Hodge bundle* $\hat{\Lambda}_{\rm e} \to \overline{B}$.

From (1.3.7) and Theorem 1.4.1 we deduce

Corollary 1.4.2. *Assume that Conjecture 1.2.4 holds. Then the Chern form* $\hat{\Omega}_{\rm e}$ *of* $\hat{\Lambda}_{\rm e} \to \overline{B}$ *descends to* $\overline{\mathcal{S}}$ *where it represents the Chern class of* $\hat{\Lambda}_{\rm e} \to \overline{\mathcal{S}}$, *and is positive on the Zariski tangent spaces of* $\overline{\mathcal{S}}$.

Corollary 1.4.2 is proved in §5.5. We will give two arguments for Theorem 1.4.1. The first, in Section 4, will be geometric, applies to the case of algebraic surfaces ($n = 2$) and essentially treats the case of 1-parameter degenerations. One product of the argument is a display of the estimates on the Hodge norms and the resulting connection and curvature forms giving descriptions that are more precise than the ones in the literature. We will show that, in the geometric case, the Hodge theoretically defined polarizations on the limiting mixed Hodge structure coincide, up to constants, with standard ones derived from geometry (Proposition 4.3.5).

The second proof given in Section 5 establishes a more general result for arbitrary variations of Hodge structure: Theorem 1.4.1 holds not only for the Chern form of $\hat{\Lambda}$, but for any Chern form of the quotient Hodge bundles $\mathrm{Gr}^p\mathcal{F}$ (see Theorem 5.1.2 for a precise statement). The proof exhibits in detail how the very special and subtle properties of several

[2] Any regular holonomic $\mathcal{D}$–module has a wave front set. Our notion is somewhat different in that the wave front set is associated to the Chern form, *not* to a $\mathcal{D}$–module. We are using the linear PDE notion of a wave front set as in the work of Hormander [Hor76].

variable degenerations of polarized Hodge structures come into play: the argument is based on an extension of the analysis underlying the Cattani–Kaplan–Schmid estimates of the Hodge metric [CKS86, §5].

Remark 1.4.3. It is well known that distributions cannot in general be multiplied, and similarly for the wedge product of currents. Although not required for our purposes, the methods used to prove Theorem 1.4.1 may be used to show: *The currents defined by the Chern forms of the Hodge bundle may be multiplied.* More precisely, the Chern forms are given by differential forms whose coefficients are locally L^1 functions on B. Then the usual formal expressions for the wedge product of forms are used with the result being again a differential form with locally L^1 functions as coefficients. The resulting current is then closed and its cohomology class is given by cup product of the corresponding Chern classes.

Remark 1.4.4. Regarding the positivity of the extended Chern form $\hat{\Omega}_{\mathrm{e}}$ (Remark 1.3.6), an interpretation of the analysis behind the property (a) and the proof of (b) above may be informally expressed as saying that *the more singular the extended period mapping is, the more positive* $\hat{\Omega}_{\mathrm{e}}$ *is.*

1.5. **Ampleness with zeros at infinity.** We finish with the following variation on Theorem 1.3.10:

Conjecture 1.5.1. *Assume that Conjecture 1.2.4 holds, and that* $\Phi : B \to \wp$ *is locally one-to-one. Then there exists* m_o *so that* $L_m := \hat{\Lambda}_{\mathrm{e}}^m - Z \to \overline{B}$ *is ample for all* $m \geq m_o$*.*

Remark 1.5.2. Conjecture 1.5.1 is related to two results:

(a) The Cornalba–Harris result [CH88] on the ampleness of L_m over $\overline{\mathcal{M}}_g$.
(b) Bakker–Brunebarbe–Tsimerman show that a finite quotient Y of $\mathcal{S}$ $(= Y')$ is realized as a quasi-projective scheme by sections of a power Λ^m that "vanish at the boundary" [BBT18, Theorem 6.2].

Let $\mathrm{Eff}_1(\overline{B})$ be the effective cone of all 1-cycles; these are the finite sums $\sum n_i C_i$, with $0 < n_i \in \mathbb{Z}$ and $C_i \subset \overline{B}$ an irreducible and reduced curve. In §3.3 we discuss how Conjecture 1.5.1 might be established when Conjecture 1.2.4 holds.

Acknowledgements. This is a significant revision of our 2017 draft (arXiv:1708.09523v1). While our claims have been scaled back, we remain optimistic on the validity of the main conjecture. We thank several people for feedback on the original version, particularly Wushi Goldring and Patrick Brosnan for some very relevant comments. Beyond this revision, several significant developments have happened, and they will appear elsewhere (e.g. [GG20, GGR20]). We also point out to the Bakker–Brunebarbe–Tsimerman recent work ([BBT18] and subsequent) which cover some of the same material by different methods.

While the subject of periods and moduli is central in algebraic geometry with a venerable tradition, our interest in the questions addressed here was rekindled by an NSF FRG grant for which two of the authors were PIs (RL/DMS-1361143 and CR/DMS-1361120), and the other two (MG and PG) were frequent collaborators. We acknowledge the NSF support and thank the other PIs (P. Brosnan, M. Kerr, and G. Pearlstein) for many stimulating discussions and sharing with us some related ideas.

2. Background material

We briefly review the behavior of the period map Φ in a (punctured) neighborhood of a point $b_0 \in Z$ "at infinity." References for the definitions and properties that follow include [CKS86, Sch73].

2.1. **Local VHS.** Recollect that D parameterizes weight n, Q–polarized Hodge structures on a rational vector space V. We assume without loss of generality that the monodromy operator $T_i \in \mathrm{Aut}(V,Q)$ about Z_i^* is unipotent, and let $N_i := \log(T_i) \in \mathrm{End}(V,Q)$ denote the nilpotent logarithm. Then $N_I = \sum_{i\in I} N_i$ is the nilpotent monodromy operator about Z_I^*.

Let

$$\Delta \;:=\; \{\zeta \in \mathbb{C} \;:\; |\zeta| < 1\}$$

denote the unit disc, and

$$\Delta^* \;:=\; \{\zeta \in \mathbb{C} \;:\; 0 < |\zeta| < 1\}$$

the punctured unit disc. Fix a point $b_0 \in Z_I^*$. Let $\overline{\mathcal{U}} \simeq \Delta^r = \Delta^k \times \Delta^\ell \ni (t,w)$ be a neighborhood of b_0 in $\overline{B}$ so that $Z \cap \overline{\mathcal{U}} = \{t_1 \cdots t_k = 0\}$; in particular,

$$\mathcal{U} \;:=\; \overline{\mathcal{U}} \cap B \simeq (\Delta^*)^k \times \Delta^\ell\,,$$

with $r = k + \ell$ and $k = |I|$.

Let $\mathcal{H} \subset \mathbb{C}$ denote the upper-half plane, and let

$$\widetilde{\Phi}_{\mathcal{U}} : \mathcal{H}^k \times \Delta^\ell \;\to\; D$$

be a lift of $\Phi|_{\mathcal{U}}$. Fix coordinates $(z,w) \in \mathcal{H}^k \times \Delta^\ell$. Then $(z,w) \mapsto (\exp(2\pi\sqrt{-1}z), w)$ defines the covering map $\mathcal{H}^k \times \Delta^\ell \to (\Delta^*)^k \times \Delta^\ell$. Here we are writing $\exp(2\pi\sqrt{-1}\,z)$ as short-hand for the $(\Delta^*)^k$–valued $(\exp(2\pi\sqrt{-1}\,z_1), \ldots, \exp(2\pi\sqrt{-1}\,z_k))$. Let $\check{D} \supset D$ denote the compact dual of the period domain. Schmid [Sch73] showed that there exists a holomorphic map

$$F : \Delta^r = \Delta^k \times \Delta^\ell \;\to\; \check{D}$$

so that the lifted period map factors as

$$\widetilde{\Phi}_{\mathcal{U}}(z,w) \;=\; \exp\left(\textstyle\sum_{i\in I} z_i N_i\right) \cdot F(\exp(2\pi\sqrt{-1}z), w)\,. \tag{2.1.1}$$

Let

$$\ell(t_j) \;:=\; \frac{\log t_j}{2\pi\sqrt{-1}}\,.$$

Observe that

$$\Phi_{\mathcal{U}}(t,w) \;:= \exp\left(\textstyle\sum_{i\in I} \ell(t_i) N_i\right) \cdot F(t,w) \tag{2.1.2a}$$

defines a *local variation of Hodge structure*

$$\Phi_{\mathcal{U}} : \mathcal{U} = (\Delta^*)^k \times \Delta^\ell \;\to\; \Gamma_{\mathcal{U}}\backslash D\,, \tag{2.1.2b}$$

where $\Gamma_{\mathcal{U}} \subset \Gamma$ is the *local monodromy group* generated by the unipotent monodromy operators $\{T_i\}_{i\in I}$. Notice that (2.1.2) recovers $\Phi|_{\mathcal{U}}$ after quotienting $\Gamma_{\mathcal{U}}\backslash D \to \Gamma\backslash D$ by the full monodromy group Γ.

2.2. **Nilpotent Orbits.** The (lifted) period map $\widetilde{\Phi}_{\mathcal{U}}$ is approximated by the *nilpotent orbit*

$$\widetilde{\vartheta}_{\mathcal{U}}(z,w) \;:=\; \exp\left(\textstyle\sum_{i\in I} z_i N_i\right)\cdot F(0,w) \tag{2.2.1}$$

as $\mathrm{Im}\, z_j \to 0$, with $\mathrm{Re}\, z_j$ bounded. The nilpotent orbit is horizontal, and $N_j\, F^p(0,w) \subset F^{p-1}(0,w)$. Setting

$$\vartheta_{\mathcal{U}}(t,w) \;:=\; \exp\left(\textstyle\sum_{i\in I} \ell(t_i) N_i\right)\cdot F(0,w)$$

yields a well-defined map

$$\vartheta_{\mathcal{U}} : \mathcal{U} \;\to\; \Gamma_{\mathcal{U}}\backslash D\,.$$

Note that the nilpotent orbit (2.2.1) is the lift of $\vartheta_{\mathcal{U}}$.

2.3. **Horizontality.** Shrinking the neighborhood $\overline{\mathcal{U}} \simeq \Delta^r$ if necessary, there exists a canonical choice of holomorphic map $X : \Delta^r \to \mathfrak{g}_{\mathbb{C}}$ so that

$$F(t,w) \;=\; \exp(X(t,w))\cdot F_0\,, \tag{2.3.1}$$

with $F_0 = F(0)$. The map X is determined as follows. (See [Cat14] for further discussion.) Let $N := N_1 + \cdots + N_k$ be the sum of the logarithms of the local unipotent monodromies about b_0. The pair $(W(N), F_0)$ is a MHS. Let $\mathfrak{g}_{\mathbb{C}} = \oplus \mathfrak{g}^{p,q}$ be the Deligne splitting, and define

$$\mathfrak{g}^{p,\bullet} \;:=\; \bigoplus_q \mathfrak{g}^{p,q}\,, \quad \mathfrak{n} \;:=\; \mathfrak{g}^{<0,\bullet} \;=\; \bigoplus_{p<0} \mathfrak{g}^{p,\bullet} \quad\text{and}\quad \mathfrak{p} \;:=\; \mathfrak{g}^{\geq 0,\bullet} \;=\; \bigoplus_{p\geq 0} \mathfrak{g}^{p,\bullet}\,.$$

Then

$$\mathfrak{g}_{\mathbb{C}} \;=\; \mathfrak{p} \oplus \mathfrak{n}\,, \tag{2.3.2}$$

$\mathfrak{p}$ is the Lie algebra of the stabilizer of F_0 in $G_{\mathbb{C}}$, and $\mathfrak{n}$ is a nilpotent Lie algebra. Consequently, there exists a unique holomorphic map

$$X : \Delta^r \;\to\; \mathfrak{n}$$

such that $X(0) = 0$ and (2.3.1) holds. Define holomorphic

$$X^{-p} : \Delta^r \;\to\; \mathfrak{g}^{-p,\bullet}$$

by $X(t,w) = \sum_{p>0} X^{-p}(t,w)$. Horizontality of (2.3.1) implies that:

(i) The subspace of $\mathfrak{g}_{-1}$ spanned by $\{N_i + 2\pi\sqrt{-1}\, t_i \partial_{t_i} X^{-1}(t,w) \mid 1 \leq i \leq k\} \cup \{\partial_{w_j} X^{-1}(t,w) \mid 1 \leq j \leq \ell\}$ is abelian. In particular, $X(t_I, w)$, with $t_I \in \Delta_I^*$, commutes with $\{N_i \mid i \in I\}$, and therefore takes value in $\mathfrak{z}_I := \cap_{i\in I} \ker(\mathrm{ad}\, N_i) \subset \mathfrak{g}_{\mathbb{C}}$.

(ii) The functions $X^{-p}(t,w)$, with $p \geq 2$, are functions of $\sum_{i=1}^k \ell(t_i) N_i + X^{-1}(t,w)$.

2.4. **Limiting mixed Hodge structures.** Let

$$N_I \ := \ \sum_{i \in I} N_i$$

denote the monodromy operator about Z_I^*. Let

$$W_0(N_I) \ \subset \ W_1(N_I) \ \subset \cdots \subset \ W_{2n}(N_I)$$

denote the monodromy (shifted) weight filtration.[3] The triple $(V, W(N_I), F(0,w))$ is a limiting mixed Hodge structure. Let

$$\mathrm{Gr}_a^{W(N_I)} \ := \ W_a(N_I)/W_{a-1}(N_I)\,.$$

Recall that $N_I \in \mathrm{End}(V,Q)$ maps $W_a(N_I) \subset V$ into $W_{a-2}(N_I)$. Consequently there is a well-defined map $N_I : \mathrm{Gr}_a^{W(N_I)} \to \mathrm{Gr}_{a-2}^{W(N_I)}$. The flag $F(0,w) \in \check{D}$ induces a weight $n+a$ Hodge structure on the the *primitive spaces*

$$H_I^{n-a}(-a) \ := \ \ker\{N_I^{a+1} : \mathrm{Gr}_{a+n}^{W(N_I)} \to \mathrm{Gr}_{n-a-2}^{W(N_I)}\}\,, \tag{2.4.1}$$

$0 \le a \le n$, that is polarized by

$$Q_a^I(u,v) \ := \ Q(u, N_I^a v)\,. \tag{2.4.2}$$

In this way we obtain a (local) variation of polarized Hodge structures over Δ^ℓ. Note that the latter is an open neighborhood of b_0 in Z_I^*. And this leads to a (global) variation of polarized Hodge structures

$$\Phi_I : Z_I^* \ \to \ \Gamma_I \backslash D_I\,. \tag{2.4.3}$$

In general, D_I will be a product of period domains.

3. Two and a half proofs

3.1. **The case that B is a surface.** Here we prove Theorem 1.2.8: assume that $\dim B = 2$. Since the theorem holds in the case $\dim \mathcal{S} = 1$ (cf. [Som73], [CDK95]), it suffices to consider the case that $\dim \mathcal{S} = 2$; equivalently, Φ_* is one-to-one on an open subset of B.

Since $\dim B = 2$, the Z_i are smooth, irreducible curves meeting transversally. Each $\Phi_\mathrm{e}(Z_i)$ is either a point or a curve. Let

$$Z' \ := \sum_{\Phi_\mathrm{e}(Z_i) = \mathrm{pt}} Z_i \ = \ \sum_{i=1}^m Z_i$$

be the union of those Z_i that are mapped to a point.

Lemma 3.1.1. *The intersection matrix $\|Z_i \cdot Z_j\|_{i,j=1}^m$ is negative definite.*

[3]Typically, "$W(N)$" denotes a representation-theoretic filtration with indexing that is centered at 0. In this paper, we are letting "$W(N)$" denote the shifted geometric filtration $W(N)[-n]$, with indexing that centered at n. The reasons for this mild abuse of notation is that (i) this is the only filtration we will work with and (ii) the abuse significantly reduces notational clutter.

Proof. By Theorem 1.4.1, the current $\hat{\Omega}_{\rm e}$ represents the Chern class $c_1(\Lambda_{\rm e}) \in H^2(\overline{B})$ of the augmented Hodge line bundle $\hat{\Lambda}_{\rm e} \to \overline{B}$. It follows from (1.3.7) that

- $\hat{\Omega}_{\rm e} \geq 0$, and
- $\hat{\Omega}_{\rm e}\Big|_{Z_i} = 0$, for $i = 1, \ldots, m$, so that $\Lambda_{\rm e} \cdot Z_i = 0$.

Additionally, the assumption that Φ_* is one-to-one on an open subset of B implies that $\hat{\Omega}_{\rm e}^2 > 0$. It follows that $\hat{\Lambda}_{\rm e}$ lies in the positive cone in $\mathrm{Pic}(\overline{B})$. We now infer the lemma from the Hodge index theorem. □

Given the lemma, a result of Grauert [Gra62] asserts that Z' may be contracted to normal singular points on a complex analytic space Y. This completes the proof of Theorem 1.2.8.

Remark 3.1.2. In the classical case that the period domain is Hermitian symmetric, the SBB compactification $\Gamma\backslash D^*$ of $\Gamma\backslash D$ is a normal projective variety. Borel's extension theorem yields the morphism $\Phi_{\rm e} : \overline{B} \to \Gamma\backslash D^*$. By hypothesis this morphism contracts Z' to a set of points. Lemma 3.1.1 then follows from a result of Mumford [Mum61].

3.2. **Proof of Theorem 1.3.13.** Suppose that $\hat{\Omega}$ is positive on on B (equivalently, Φ_* is everywhere injective), and that $0 \leq -K_{\overline{B}} \cdot C$ whenever $\Phi_{\rm e}(C)$ is a point. In order to apply the base point free theorem [KM98] to show that $\hat{\Lambda}_{\rm e} \to \overline{B}$ is free we need to show that

(i) $m\hat{\Lambda}_{\rm e} - K_{\overline{B}}$ is nef for $m \gg 0$, and
(ii) $m\hat{\Lambda}_{\rm e} - K_{\overline{B}}$ is big for $m \gg 0$.

We begin with nef. If $\Phi_{\rm e}(C)$ is a point, then $(m\hat{\Lambda}_{\rm e} - K_{\overline{B}}) \cdot C = -K_{\overline{B}} \cdot C \geq 0$ by assumption. Suppose that $\Phi_{\rm e}(C)$ is a curve. Then (1.3.7) and Theorem 1.4.1 imply $(m\hat{\Lambda}_{\rm e} - K_{\overline{B}}) \cdot C > 0$ for $m \geq m_o(C)$. Lemma 5.4.20 implies that we may choose $m_o \geq m_o(C)$ for all curves C. This establishes (i).

To prove bigness, Theorem 6.1.2 asserts that it suffices to show that at some point $b \in B$ we have $(m\hat{\Lambda}_{\rm e} - K_{\overline{B}})^d > 0$ for $m \gg 0$ and with $d = \dim \overline{B}$. This follows from our assumption on the positivity of $\hat{\Omega}$.

3.3. **Discussion of ampleness with zeros at infinity.** In this section discuss an incomplete proof of Conjecture 1.5.1.

3.3.1. *Sketch of proof.* The conjecture is equivalent to

Lemma 3.3.1. *There exists m_o so that for any curve $C \subset \overline{B}$, we have*

$$\deg\left(L_m|_C\right) \;=\; \sum n_i \deg\left(L_m|_{C_i}\right) \;>0$$

when $m \geq m_o$.

Incomplete proof. Given Lemma 5.4.20 it suffices to prove the lemma in the case that C is an irreducible curve and $m_o = m_o(C)$.

Case 1: $C \cap B$ is a Zariski open subset of C. The desired positivity

$$\deg\left(L_m\big|_C\right) \;>\; 0$$

follows from (1.3.7) and the hypothesis that $\Phi : B \to \wp$ is locally one-to-one.

Case 2: $C \subset Z$ and $\Phi_{\mathrm{e}}(C)$ is not a point. Let Z_I^* be the open strata with the property that $C \cap Z_I^*$ is Zariski open in C. Then the argument of Case 1 applies here with Φ_I in place of Φ.

Case 3: $\Phi_{\mathrm{e}}(C)$ is a point. In this case the hypothesis that $\Phi : B \to \wp$ is locally one-to-one implies $C \subset Z$. Since C is contained in a fibre $\mathcal{F} \subset \overline{B}$ of $\Phi^{\mathrm{c}}_{\mathrm{e}}$, we have

$$L_m \cdot C \;=\; -Z \cdot C \;=\; N^*_{Z/\overline{B}} \cdot C\,.$$

So we need to show that there exists k so that

$$\deg\left.\left(N^*_{Z/\overline{B}}\right)^k\right|_C \;>\; 0\,. \tag{3.3.2}$$

Set $s = \Phi^{\mathrm{c}}_{\mathrm{e}}(\mathcal{F}) \in \overline{\mathcal{S}}$. As we are assuming that Conjecture 1.2.4 holds, we may speak of the local ring $\mathcal{O}_{\overline{\mathcal{S}},s}$ and its maximal ideal $\mathfrak{m}_s$. Given $f \in \mathfrak{m}_s$, the function $f \circ \Phi^{\mathrm{c}}_{\mathrm{e}}$ is defined in a neighborhood of C and vanishes along C; let $\mathrm{ord}_C(f \circ \Phi^{\mathrm{c}}_{\mathrm{e}}) > 0$ denote the order of vanishing.

Given a point $x \in C \subset \mathcal{F}$ and a normal disc Δ to Z at x, there exists $f \in \mathfrak{m}_s \subset \mathcal{O}_{\overline{\mathcal{S}},s}$ such that $\tilde{f} := f \circ \Phi_{\mathrm{e}}\big|_\Delta \not\equiv 0$. Let

$$k_o \;:=\; \min\left\{\mathrm{ord}_C(f \circ \Phi^{\mathrm{c}}_{\mathrm{e}}) \;\middle|\; f \in \mathfrak{m}_s\,,\;\; f \circ \Phi_{\mathrm{e}}\big|_\Delta \not\equiv 0\right\} \;>\; 0\,.$$

Then some $f \circ \Phi^{\mathrm{c}}_{\mathrm{e}}$ gives a nonzero section ν of $\left(N^*_{Z/\overline{B}}\right)^{k_o}$ along C.

At this point in order to deduce that (3.3.2) holds for $k = k_o$, and complete the proof of Conjecture 1.5.1 (with the hypothesis that $\mathrm{Eff}_1(\overline{B})$ is finitely generated) we need to rule out the case that the section ν is nowhere zero (which would imply that $\left(N^*_{Z/\overline{B}}\right)^{k_o}$ is trivial). We anticipate that this will follow from a good description of the desired separating functions $\mathcal{A}$ (Remark 1.2.5). □

To have $L_m \cdot C > 0$ for a fixed m and all $C \in \mathrm{Eff}(\overline{B})$ requires a bound

$$m \deg\left.\hat{\Lambda}_{\mathrm{e}}\right|_C \;>\; Z \cdot C\,. \tag{3.3.3}$$

If C is not contained in Z, then the right-hand side of (3.3.3) is roughly the number of singular fibres in the VHS over C. Note that the desired inequality of (3.3.3) is essentially the reverse of that given by the Arakelov inequalities. It may be that this "reverse Arakelov inequality" does not hold for general VHS, but does hold for geometric VHS, where one has the Grothendieck–Riemann–Roch, cf. [CH88].

4. Curvature properties of the extended Hodge bundle: the surface case

Theorem 1.4.1 is a central ingredient in the proof of Theorem 1.3.10. We give two proofs of Theorem 1.4.1. The first, given in this section, will be inductive on the singular strata of the boundary divisor. Moreover, it will be restricted to the geometric case arising from a family of varieties, one of the points being that in this situation the singularities of the Hodge norms are localizable and visible analytically in a way that is suggestive of the general case. The second argument is given in §5; it provides a proof of the general result. (Finally, Theorem 1.3.10 is proved in §6.)

Throughout this section we assume that $n = 2$; in particular, the Hodge bundle and the augmented Hodge bundle coincide ($\Lambda = \hat{\Lambda}$).

4.1. **Currents.**

We begin by discussing two general properties of currents that will arise.[4] On an n-dimensional complex manifold Y, we denote by $A_c^{p,q}(Y)$ the compactly supported smooth (p, q) forms. A current T of type (p, q) gives a linear function

$$A_c^{n-p,n-q}(Y) \to \mathbb{C}.$$

The currents we shall encounter will be differential (p, q) forms ψ with coefficients in the space of locally L^1 functions, and the corresponding current T_ψ is given by

$$T_\psi(\alpha) = \int_Y \psi \wedge \alpha.$$

The differential $\partial T_\psi(\alpha)$ is defined as usual by

$$\partial T_\psi(\beta) = \pm \int_Y \psi \wedge \partial\beta\,,$$

where the sign is determined by the condition that $\partial T_\psi = T_{\partial\psi}$ when ψ is smooth. Similarly we may define $\overline{\partial} T_\psi$ and $\overline{\partial}\partial T_\psi$.

For the ψ's we shall use, we will also be able to define $\partial\psi$ by applying the formal rules of calculus to the coefficient functions of ψ. The equality

$$\partial T_\psi = T_{\partial\psi} \tag{4.1.1}$$

shall mean: first the coefficients $\partial\psi$ computed formally are locally L^1 functions; and secondly that the currents satisfy (4.1.1). Similar notions hold for $\overline{\partial}\psi$ and $\overline{\partial}\partial\psi$.

Definition 4.1.2. We shall say that the current represented by a locally L^1 differential form ψ has the *property NR* if $\partial\psi$, $\overline{\partial}\psi$, $\overline{\partial}\partial\psi$ computed formally have L^1 coefficients, and if (4.1.1) holds for $\partial\psi$, $\overline{\partial}\psi$ and $\overline{\partial}\partial\psi$. The term "NR" is meant to suggest "no residues."

Remark 4.1.3. The property NR implies that the currents defined by $\psi, \partial\psi, \partial\overline{\partial}\psi$ have vanishing Lelong numbers (cf. [Dem12]).

[4] Cf. [Dem12] for a general account and references to the literature.

Example 4.1.4. In $\mathbb{C}$, we have $\partial\overline{\partial}\log|z| = 0$ formally, while up to a constant the equation of currents

$$\partial\overline{\partial}T_{\log|z|} = \delta_0 dz \wedge \mathrm{d}\bar{z}$$

holds. On the other hand, again up to a constant,

$$\partial\overline{\partial}\log(-\log|z|) = \frac{dz \wedge \mathrm{d}\bar{z}}{|z|^2(\log|z|)^2}$$

holds both formally and in the sense of currents, so $\log(-\log|z|)$ has the property NR while $\log|z|$ does not.

In both these examples the coefficients of the derivatives computed formally are in L^1; the difference is that for $\log|z|$ we pick up a residue term in $\partial\overline{\partial}T_{\log|z|}$, while no such term arises in $\partial\overline{\partial}T_{\log(-\log|z|)}$.[5]

For the second property we first recall that a current T on Y has a singular support $\operatorname{sing} T \subset Y$, defined to be the smallest closed subset such that on the complement $Y \backslash \operatorname{sing} T$, the current T is represented by a smooth differential form.

Remark 4.1.5. In this work we will want to restrict singular differential forms to submanifolds. Our approach here is motived by the notion the *wave front set* $\mathrm{WF}(T) \subset T^*Y$. If $W \subset Y$ is a submanifold, then in general the restriction to W of a distribution or current T given on Y is not defined.[6] However if $W \subset Y$ is a submanifold whose tangent spaces are transverse to the wave front set in the sense that

$$TW \subset \mathrm{WF}(T)^{\perp} \tag{4.1.6}$$

then the restriction $T\big|_W$ is valid. The singular differential forms that we work with will satisfy an analogous (and essential) restriction property.

Example 4.1.7. As an illustration of what will occur, we note as above that the currents we shall be interested in will be constructed from locally L^1-functions. It may or may not be possible to simply restrict such a function in the usual sense and obtain a well-defined function. As a simple example of what will be done below, on $\Delta \times \Delta$ with coordinates (t, w), the current given by $1/\log\frac{1}{|t|} + f(w)$ where $f(w)$ is smooth may be restricted to $\{0\} \times \Delta$ to give $f(w)$.

4.2. **Singularity structure.**

Definition 4.2.1. A positive function h defined in $\mathcal{U} \cong \Delta^{*k} \times \Delta^{\ell}$ is said to have *logarithmic singularities* if it is of the form

$$h \;=\; P\left(\log|t_1|^{-1}, \ldots, \log|t_k|^{-1}\right) + R\left(\log|t_1|^{-1}, \ldots, \log|t_k|^{-1}\right).^7$$

[5]Note that "$\partial\log|z|$ computed formally in in L^1" means that $\partial\log|z| \wedge \alpha$ is in L^1 for any C^∞ form α.

[6]A good discussion of this with illustrative examples and references may be found on Wikipedia.

[7]To be precise, the notation $\log|t|^a$ indicates $\log(|t|^a)$; we drop the parentheses to streamline notation.

Here $P(x_1,\dots,x_k)$ a homogeneous polynomial whose coefficients are real, take value in $C^\infty(\overline{\mathcal{U}})$, and are positive in the sense that

$$P(x_1,\dots,x_k) > 0 \text{ if all } x_i > 0\,.$$

The polynomial R is real, has $C^\infty(\overline{\mathcal{U}})$ coefficients, and is of lower order than P in the sense to be explained below. Finally h satisfies the following conditions:

(i) $\log h$ has the property NR;

(ii) the current $\Omega_h := (i/2)\overline{\partial}\partial \log h$ is positive and has the property that the restriction to $\Delta_I^* \times \Delta^\ell$ is well-defined. (Note that the last is the analog of (4.1.6) that we require.)

Because of (i) the current Ω_h is defined on $\Delta^k \times \Delta^\ell$ so that (ii) makes sense.

(4.2.2) *For the remainder of this section, and for all of §4.3, we will restrict to the case $k = 1$, so that $\mathcal{U} \cong \Delta^* \times \Delta^\ell$.*

Remark 4.2.3. This is essentially the case of 1-parameter degenerations with dependence on holomorphic parameters. In fact, for notational simplicity, we shall also assume that $\ell = 1$, so that we are working in $\Delta^* \times \Delta$ with coordinates (t, w).

The functions h we shall consider will be of the form

$$h = A(t,w)\left(\log|t|^{-1}\right)^m\left(1 + \frac{B_1(t,w)}{\log|t|^{-1}} + \cdots + \frac{B_m(t,w)}{(\log|t|^{-1})^m}\right) \tag{4.2.4}$$

where $A(t,w)$ and the $B_i(t,w)$ are C^∞ functions on $\Delta \times \Delta$ and $A(0,w) > 0$. We note that the expression (4.2.4) is invariant under holomorphic coordinate changes

$$\begin{cases} t' = tf(t,w) & f(t,w) \neq 0 \\ w' = g(t,w) & g_w(0,u) \neq 0. \end{cases} \tag{4.2.5}$$

As will be seen below, the motivation for considering functions of this form arises from the periods of holomorphic differentials in a degenerating family of algebraic varieties.

Proposition 4.2.6. *The function* (4.2.4) *has logarithmic singularities.*

Proof. Denoting by C the term in parentheses, since $\log h = \log A + \log(m \log|t|^{-1}) + \log C$ the only issue concerns the $\log C$ term. In

$$\partial\overline{\partial}\log C \;=\; \frac{\partial C}{C} \wedge \frac{\overline{\partial} C}{C} \;-\; \frac{\partial\overline{\partial} C}{C^2}$$

we shall separately examine the singularities in each term. For the first the most singular terms arise from:

- $\partial\left[\frac{1}{(\log|t|^{-1})^a}\right] \wedge \overline{\partial}\left[\frac{1}{(\log|t|^{-1})^b}\right]$, with $a,b>0$. This is of the order $\frac{\mathrm{d}t\wedge\mathrm{d}\bar{t}}{|t|^2\,(\log|t|^{-1})^c}$, with $c\geq 4$, and hence is $o(\mathsf{PM})$, where

$$\mathsf{PM} \;:=\; \frac{\mathrm{d}t\wedge\mathrm{d}\bar{t}}{|t|^2\,(\log|t|^{-1})^2} \tag{4.2.7}$$

  is the Poincaré metric.
- $\partial\left[\frac{1}{(\log|t|^{-1})^a}\right]\wedge\alpha$, with $a>0$ and α smooth (C^∞). This is of the order $\frac{\mathrm{d}t}{|t|\,(\log|t|^{-1})^c}\wedge\beta$, with $c\geq 2$ and β smooth, and is again $o(\mathsf{PM})$.

The terms $\partial\log C$ and $\overline{\partial}\log C$ may be estimated by those above. For $\partial\overline{\partial}C/C^2$, the most singular terms are of the order $\partial\overline{\partial}\left[\frac{1}{(\log|t|^{-1})^a}\right] \sim \frac{\mathrm{d}t\wedge\mathrm{d}\bar{t}}{|t|^2\,(\log|t|^{-1})^{a+2}}$, with $a\geq 1$, which is again $o(\mathsf{PM})$.

Note that the estimates in this argument have no room to spare. □

4.3. **Proof of Theorem 1.4.1 in the weight $n=2$ case.** We denote by

$$\Omega_h \;=\; \tfrac{\sqrt{-1}}{2}\,\partial\overline{\partial}\log h$$

the curvature form associated to the function h in (4.2.4). Then

$$\Omega_h \;=\; m\tfrac{\sqrt{-1}}{2}\,A(v,w)\frac{\mathrm{d}t\wedge\mathrm{d}\bar{t}}{|t|^2(\log|t|)^2} \;+\; o(\mathsf{PM}) \tag{4.3.1}$$

and, assuming that $m>0$, it is positive with

$$\operatorname{sing}\Omega_h \;=\; \{0\}\times\Delta.$$

It defines a closed, positive $(1,1)$ current on $\Delta\times\Delta$ (cf. [CKS86] and [Kol87]). As for $\mathrm{WF}(\Omega_h)$, the terms in Ω_h not containing a $\mathrm{d}t$ or $\mathrm{d}\bar{t}$ are of the form $\gamma(\log|t|)^{-a}$, where γ is a smooth $(1,1)$ form and $a>0$. Thus although it is *not* the case that $\mathrm{WF}(\Omega_h)=N^*_{\{0\}\times\Delta/\mathfrak{U}}$ is the co-normal bundle of $\{0\}\times\Delta$ in $\Delta\times\Delta$ in the usual sense, the restriction $\Omega_h\big|_{\{0\}\times\Delta}$ is a well-defined smooth $(1,1)$ form. Indeed, the above calculation shows that to define restriction we may use the prescription:

- In the formula for $\partial\overline{\partial}\log h$ first set $\mathrm{d}t=\mathrm{d}\bar{t}=0$.
- Then the limit as $t\to 0$ of the remaining terms exists (i.e., set $1/\log|t|^{-1}=0$).

The calculation in the proof of Proposition 4.2.6 gives

$$\Omega_e\big|_{\{0\}\times\Delta} \;=\; \tfrac{\sqrt{-1}}{2}\partial\overline{\partial}\log A(0,w)\,. \tag{4.3.2}$$

The point in (ii) is that in what remains after (i), the term $\log|t|^{-1}$ only appears in the denominator and with positive powers. We note that the above prescription is invariant under the coordinate changes (4.2.5).

We now apply the above to a weight $n=2$ variation of Hodge structure over $\Delta^*\times\Delta$. Denote the canonically extended Hodge bundle by $F_\mathrm{e}\to\Delta\times\Delta$ and let $\sigma(t,w)$ be a nowhere vanishing holomorphic section of this bundle. We assume that m is maximal with $\sigma \in W_{n+m}(N) \cap F_{\mathrm{e}}$, and denote by $\sigma_m(w)$ the projection of $\sigma(0,w)$ in $\mathrm{Gr}^{W(N)}_{n+m} F_{\mathrm{e}}$. Then $\sigma_m(w)$ is a non-zero section of $\mathrm{Gr}^{W(N)}_{n+m} V \cap F^n_e$ over $\{0\} \times \Delta$.

Proposition 4.3.3. *The Hodge norm* $\|\sigma(t,w)\|^2$ *is of the form* (4.2.4)*, and*

$$\partial\bar\partial \log \|\sigma(t,w)\|^2\Big|_{\{0\}\times\Delta} \;=\; \partial\bar\partial \log \|\sigma_m(w)\|^2 .$$

A consequence of the proposition is the following special case of Theorem 1.4.1.

Corollary 4.3.4. *If* Ω_{e} *is the Chern form of the extended Hodge line bundle* $\Lambda_{\mathrm{e}} \to \Delta \times \Delta$*, and if* $\Omega_{\{0\}\times\Delta}$ *is the Chern form of the gradeds to the associated variation of mixed Hodge structure along* $\{0\} \times \Delta$*, then the restriction* $\Omega_e|_{\{0\}\times\Delta}$ *is defined agrees with* $\Omega_{\{0\}\times\Delta}$*.*

Proof of Proposition 4.3.3. We shall prove the proposition in the weight $n = 2$ geometric case of a family $\mathcal{X}^* \xrightarrow{\pi} \Delta^* \times \Delta$ of smooth surfaces where $\sigma(t,w)$ is a section of $\pi_*\omega_{\mathcal{X}/\Delta^*\times\Delta}$ given by a family

$$\psi(t,w) \;\in\; H^0(\Omega^2_{X_{(t,w)}})$$

of holomorphic 2-forms along the smooth fibers $X_{(t,w)} = \pi^{-1}(t,w)$. By base change and semi-stable reduction we may assume that we have a smooth completion $\mathcal{X} \xrightarrow{\pi} \Delta \times \Delta$ of the family where the singular fibers $X_{(0,w)}$ have normal crossings. The local models are

(a) $X_{(0,w)}$ is smooth and the mapping π is locally given by $(x_1, x_2, x_3, w) \to (x_1, w)$; i.e., $t = x_1$;

(b) $X_{(0,w)}$ has a smooth double curve and the mapping π is given by $(x_1, x_2, x_3, w) \to (x_1x_2, w)$; i.e., $t = x_1x_2$;

(c) $X_{(0,w)}$ has a double curve with triple points and the mapping π is locally given by $(x_1, x_2, x_3, w) \to (x_1x_2x_3, w)$; i.e., $t = x_1x_2x_3$.

By a standard property of the canonical extension, the 2-forms giving sections of $\pi_*\omega_{\mathcal{X}/\Delta\times\Delta}$ are locally Poincaré residues

$$\psi(t,w) \;=\; \mathrm{Res}\left[\frac{g(x_1,x_2,x_3,w)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{f(x_1,x_2,x_3,w)}\right]$$

where g is holomorphic and f is given by

(a) $f = x_1 - t$,

(b) $f = x_1x_2 - t$,

(c) $f = x_1x_2x_3 - t$

in the three cases listed above. The properties of the extension $\psi(0,w)$ to a section of $F_{\mathrm{e}} \to \{0\} \times \Delta$ relative to the weight fibration are, for each of the cases above:

(a) The double and single residues of $\psi(0,w)$ are zero. Then $\psi(0,w)$ induces a non-zero section of $\mathrm{Gr}^{W(N)}_2$ and $\psi(0,w)$ is a holomorphic 2-form on the desingularization $\widetilde{X}_{(0,w)}$ of $X_{(0,w)}$.

(b) The double residues of $\Psi(0,w)$ are zero. Then $\psi(0,w) \in W_3(N)$ and $\psi(0,w)$ induces a non-zero section in $\mathrm{Gr}_3^{W(N)}$ if the single residues of $\psi(0,w)$ along the double curve are non-zero; ie., if $g(0,0,0,w)=0$ but $g(x_1,0,x_3,w)\neq 0$.

(c) The form $\psi(0,w)$ induces a non-zero section in $\mathrm{Gr}_4^{W(N)}$ if, and only if, the double residues of $\psi(0,w)$ at the triple points are not all zero; ie., if $g(0,0,0,w)\neq 0$.

The Hodge norm is, up to a constant, the L^2-norm

$$\|\psi(t,w)\|^2 \;=\; \int_{X_{(t,w)}} \psi(t,w)\wedge\overline{\psi(t,w)}$$

of the holomorphic 2-forms $\psi(t,w)$. Then $\|\psi(t,w)\|^2$ has an expansion in terms of powers of $\log|t|^{-1}$, and the local contributions to the expansion in each of the above cases are respectively

(a) $\|\psi(t,w)\|^2 = \displaystyle\int |g(0,x_2,x_3,w)|^2 \mathrm{d}x_2\wedge \mathrm{d}\bar{x}_2\wedge \mathrm{d}x_3\wedge \mathrm{d}\bar{x}_3$,

(b) $\|\psi(t,w)\|^2 = \left(\displaystyle\int |g(x_1,0,0,w)|^2\mathrm{d}x_1\wedge \mathrm{d}\bar{x}_1\right)\log|t|^{-1} \;+\; C(t,w)$,

(c) $\|\psi(t,w)\|^2 = |g(0,0,0,w)|^2(\log|t|^{-1})^2 \;+\; B_1(t,w)\log|t|^{-1} \;+\; B_2(t,w)$,

where B_1,B_2,C are smooth functions. This establishes the first part of the proposition: namely, that the Hodge norms are of the form (4.2.4).

For the second part we will discuss the above three cases. In case (a) the 2-form $\psi(0,w)$ is holomorphic on the desingularization $\widetilde{X}_{(0,w)}$ and the polarizing form is just the usual one given by $\int_{\widetilde{X}_{(0,w)}} \psi(0,w)\wedge\overline{\psi(0,w)}$.

In case (b) $\sigma_3(w)$ is a section of $\mathrm{Gr}_3^{W(N)}(\mathrm{LMHS})$, which is a Tate twist of a variation of Hodge structure of weight one. Geometrically, the double residues of $\psi(0,w)$ are zero and the single residues induce holomorphic 1-forms $\operatorname{Res}\psi(0,w)$ on the normalization $\widetilde{D}_w$ of the double curve of $X_{(0,w)}$. In this case there are two potential polarizing forms

(i) $Q(Nu,v)$ on $\mathrm{Gr}_3^{W(N)}(\mathrm{LMHS})$ (Hodge-theoretic one);

(ii) $\int_{\widetilde{D}_w}\operatorname{Res}\psi(0,w)\wedge\overline{\operatorname{Res}\psi(0,w)}$ (algebro-geometric one).

Up to a constant these polarizations agree; in §4.4 we will prove

Proposition 4.3.5. *On* $\mathrm{Gr}_3^{W(N)}(\mathrm{LMHS})$ *the polarizing form arising from the limiting mixed Hodge structure coincides with the natural polarizing form on sub-Hodge structures of* $H^{1,0}(\widetilde{D}_w)$. (This result holds in full generality for $\pi_*\omega_{\mathcal{X}/(\Delta^*)^k\times\Delta^\ell}$.)

In case (c), $\sigma_4(w)$ is a section of $\mathrm{Gr}_4^{W(N)}(\mathrm{LMHS})$, which is a family of polarized Hodge-Tate structures along $\{0\}\times\Delta$. The period domain is 0-dimensional and its curvature form $\frac{\sqrt{-1}}{2}\bar\partial\partial\log A(0,w)$, where $A(0,w)=|h(w)|^2$ with $h(w)$ holomorphic, is zero.[8] However, it is

[8] More precisely, one has a family of Hodge metrics on a single Hodge structure (this one being Hodge-Tate). This defines a Hermitian line bundle on the parameter space, and the associated curvature form is zero.

of interest to observe that the polarizing form on $\mathrm{Gr}_4^{W(N)}(\mathrm{LMHS})$ is by definition $Q(N^2u,\bar{v})$. On the other hand

$$h(w) \;=\; \sum \text{ double residues of } \psi(0,w)\,,$$

where the sum is over a subset of the double residues at the triple points of $X_{(0,w)}$. The identifications of the polarizing form on $\mathrm{Gr}_4^{W(N)}(\mathrm{LMHS})$ with $|h(w)|^2$ will be discussed in §4.4. □

Proof of Corollary 4.3.4. We take a section

$$\sigma(t,w) \;=\; \psi_1(t,w)\wedge\cdots\wedge\psi_{p_g}(t,w)$$

of $\det F_{\mathrm{e}}$ where the $\psi_i(t,w)$ give a framing of the canonically extended Hodge vector bundle $F_a \to \Delta\times\Delta$ that is adapted to the weight filtration $W(N)\cap F_{\mathrm{e}}$. As previously noted, that means that we filter the sections of $F_{\mathrm{e}}\to\Delta\times\Delta$ by their logarithmic growth along $\{0\}\times\Delta$. Setting $h^0=\dim I^{0,0}$ and $h^{1,0}=\dim I^{1,0}$, where we recall the $I^{p,q}$ are the Hodge decomposition of Gr(LMHS) along $\{0\}\times\Delta$, the calculation in the proof of the proposition gives that up to a constant

$$\Omega_e \;=\; (2h^0+h^{1,0})\mathsf{PM} \;+\; \mathsf{LOT}$$

where LOT are lower order terms in the sense that that the ratio LOT/PM tends to zero as $t\to 0$. Moreover the restriction $\Omega_e|_{\{0\}\times\Delta}$ of the current Ω_e is defined and there it coincides with the Chern form of the Hodge line bundle for the VHS over $\{0\}\times\Delta$ given by the associated graded to the LMHS defined there.[9] □

Remark 4.3.6. As noted in Remark 4.2.3, the assumption $\ell=1$ was made only for notational convenience. It is straightforward to see that both Proposition 4.3.3 and Corollary 4.3.4 hold for the general case $w\in\Delta^\ell$.

At this point we may complete the argument for Theorem 1.4.1 in the introduction in the special case where we consider only the weight $n=2$ case, and we restrict to the geometric situation where the period mapping (1.1.2) arises from a projective family $\mathcal{X}^*\to\Delta^{*k}\times\Delta^\ell$ of smooth algebraic surfaces.

Proof of Theorem 1.4.1 in the weight $n=2$ case. Given Remark 4.2.3, Corollary 4.3.4 establishes the result for $k=1$. To complete the argument we now consider the case of a period mapping (1.1.2) for arbitrary k and ℓ. It suffices to prove

Claim 4.3.7. *The general case may be reduced to the case $k=1$ by a succession of 1–parameter degenerations.*

The claim is a consequence of the several-variable SL(2)-orbit theorem [CKS86]. We will prove the claim in the case that $k=2$, the argument extends in a straightforward fashion.

Recall (§2.4) that the nilpotent orbit approximating the degeneration $\lim_{t_1\to 0}\Phi(t_1,t_2;w)$ induces a variation of polarized Hodge structure (VPHS) $\Phi_1(t_2,w)$ over $\Delta^*\times\Delta^\ell$. Likewise

[9]This required the *non*-vanishing of $A(0,w)$ in (4.2.4).

the degeneration $\lim_{t_1,t_2\to 0}\Phi_1(t_2,w)$ induces a VPHS $\Phi_{12}(w)$ over Δ^ℓ. Similarly, the degeneration $\lim_{t_2\to 0}\Phi_1(t_2,w)$ induces a VPHS $\check{\Phi}(w)$ over Δ^ℓ. It is a consequence of the SL(2) orbit theorem that $\check{\Phi}(w)=\Phi_{12}(w)$, and this establishes the claim. □

4.4. **Proof of Proposition 4.3.5.** We will describe the limiting mixed Hodge structure and its polarization for a family of surfaces $\mathcal{X}\xrightarrow{\pi}\Delta$ with central fiber $X=\cup_{i\in I}X_i$, with I and ordered index set, a reduced normal crossing divisor in a smooth 3-fold $\mathcal{X}$.[10] The usual notations

$$X^{[1]} = \coprod_i X_i\,,\quad X^{[2]} = \coprod_{i<j} X_i\cap X_j\,,\quad X^{[3]} = \coprod_{i<j<k} X_i\cap X_j\cap X_k$$

will be used for the desingularized strata of X.

4.4.1. *The limiting mixed Hodge structure.* The groups that appear in the complex whose cohomology gives the associated graded to the LMHS are $H^a(X^{[b]})(-c)$, $0\le c\le b-1$. The

$$I_\ell = \mathrm{Gr}_\ell^{W(N)}(\mathrm{LMHS}) = \bigoplus_{p+q=\ell} I^{p,q}\,,\qquad 0\le \ell\le 4.$$

are the E_2-terms of a spectral sequence, where the E_1-terms and differential $d_1, E_1\to E_1$ will now be described in dual pairs.

Letting G and R denote the Gysin and restriction maps, respectively, for $\mathrm{Gr}_4^{W(N)}$ and $\mathrm{Gr}_0^{W(N)}$ we have the dual complexes

(4.4.1a) $$H^0(X^{[3]})(-2)\xrightarrow{G} H^2(X^{[2]})(-1)\xrightarrow{G} H^4(X^{[1]})$$

(4.4.1b) $$H^0(X^{[1]})\xrightarrow{R} H^0(X^{[2]})\xrightarrow{R} H^0(X^{[3]})\,.$$

The initial and terminal cohomology groups are

(4.4.2a) $$I_4 = I^{2,2} = \ker\{G: H^0(X^{[3]})(-2)\to H^2(X^{[2]})(-1)\}$$

(4.4.2b) $$I_0 = I^{0,0} = \mathrm{coker}\{R: H^0(X^{[2]})\to H^0(X^{[3]})\}\,.$$

Here $N^2: I^{2,2}\to I^{0,0}$ is the "identity" under the composition

$$\ker G\to H^0(X^{[3]})(-2)\to H^0(X^{[3]})\to \mathrm{coker}\,R\,,$$

where "identity" means the usual identity mapping that ignores Tate twists.

[10] A general reference for this discussion is Chapter 11 in [PS08]. Here we will use the setting and notations developed in [GG16].

Next $\mathrm{Gr}_2^{W(N)}$ is the cohomology in the middle of the complex

$$\begin{array}{ccccc} & & H^2(X^{[1]}) & & \\ & \overset{G'}{\nearrow} & & \overset{R'}{\searrow} & \\ H^0(X^{[2]})(-1) & & & & H^2(X^{[2]}) \\ & \underset{R}{\searrow} & & \underset{G}{\nearrow} & \\ & & H^0(X^{[3]})(-1) & & \end{array} \tag{4.4.3}$$

As noted in [GG16], it is a consequence of the Friedman condition for smoothability [Fri83] that the above is actually a complex; i.e., that the composition $(R' \oplus G) \circ (G' \oplus R) = 0$. We will explain this in more detail §4.4.3.

The monodromy maps are induced by

$$\begin{array}{ccccc} \ker G & \subset & H^0(X^{[3]})(-2) & & \text{cf. } (4.4.1a) \\ \downarrow N & & \downarrow \mathrm{id} & & \\ \text{coker } R \cap \ker G & \subset & H^0(X^{[3]})(-1) & & \text{cf. } (4.4.3) \\ \downarrow N & & \downarrow \mathrm{id} & & \\ \text{coker } R & \subset & H^0(X^{[3]}) & & \text{cf. } (4.4.1b)\,, \end{array}$$

and the iteration N^2 is (4.4.2).

For the odd weights for $\mathrm{Gr}^{W(N)}(\mathrm{LMHS})$ the analogue of (4.4.1) is the pair of dual complexes

$$H^1(X^{[2]})(-1) \xrightarrow{G} H^3(X^{[1]}) \tag{4.4.4a}$$

$$H^1(X^{[2]}) \xrightarrow{R} H^1(X^{[2]})\,, \tag{4.4.4b}$$

and we have

$$I_3 = \ker(4.4.4a) \quad \text{and} \quad I_1 = \text{coker}(4.4.4b)\,.$$

Monodromy is given by

$$\ker G \subset H^1(X^{[2]})(-1) \xrightarrow{\text{“identity”}} H^1(X^{[2]}) \to \text{coker}\, R\,.$$

Replacing X by X_w we then have the descriptions

- $F_a^2 \cap \mathrm{Gr}_4^{W(N)} = I^{2,2} \subset H^1(X_w^{[2]})(-1)$ *is represented by the double residues of forms* $\psi(0,w)$*;*
- $F_a^2 \cap \mathrm{Gr}_3^{W(N)} = I^{2,1} \subset H^0(\Omega^1_{X_w^{[2]}})(-1)$ *is represented by single residues of forms* $\psi(0,w)$ *whose double residues are zero;*

- $F_a^2 \cap \mathrm{Gr}_2^{W(N)} = I^{2,0} \subset H^0(\Omega^2_{X^{[1]}})$ *is represented by the holomorphic 2-forms* $\psi(0,w)$ *both whose double and single residues vanish.*

4.4.2. *Polarizations.* We now turn to the issue of polarizations. There are two polarizing forms on the groups

$$I^{2,k} = F_a^2 \cap \mathrm{Gr}_{2+k}^{W(N)}(\mathrm{LMHS}), \qquad k = 2, 1.$$

One is the Hodge-theoretic one arising from

$$\widetilde{Q}(u, \bar{v}) = Q(N^k u, \bar{v}).$$

The other is the geometric one obtained by:

- First taking limits, we realize the elements in $I^{2,k}$ as singular differential forms on $X_{(0,w)}$.
- Then by taking sequential residues of these forms we obtain holomorphic differentials on the desingularized strata $X_w^{[1+k]}$ of $X_{(0,w)}$.
- Finally we take the usual polarizing forms $\int \alpha \wedge \bar{\beta}$ of holomorphic forms on smooth varieties.[11]

Proposition 4.3.5 asserts: *The Hodge-theoretic and geometric polarizing forms coincide.*

Proof of Proposition 4.3.5. We shall give the argument for this in the critical case $k = 1$. The situation is this:

- We have a family X_t of smooth surfaces specializing to a singular surface X_0 that has a double curve $D_0 \subset X_0$.
- The ψ_t are holomorphic 2-forms in $H^0(\Omega^2_{X_t})$ that specialize to $\psi_0 \in H^0(\Omega^2_{\widetilde{X}_0}(\widetilde{D}_0))$, which is a 2-form on the normalization $\widetilde{X}_0$ of X_0 having a log pole on the inverse image $\widetilde{D}_0$ of the double curve on X_0.

As we have seen (§4.3), there is an expansion $\displaystyle\int_{X_t} \psi_t \wedge \overline{\psi}_t \;=\; C \log \frac{1}{|t|} + \mathsf{LOT}$. On the other hand we have the 1-form $\mathrm{Res}(\widetilde{\psi}_0) =: \psi_0 \in H^0\left(\Omega^1_{\widetilde{D}_0}\right)$, and the assertion is that up to a universal constant

$$C \;=\; \int_{\widetilde{D}_0} \mathrm{Res}\,\psi_0 \wedge \overline{\mathrm{Res}\,\psi_0}\,.$$

By localizing along $\widetilde{D}_0$ and iterating the integral, this essentially amounts to the following 1-variable result: In $\mathbb{C}^2$ we consider the analytic curve C_t given by $xy = t$. On C_t we take the Poincaré residue

$$\varphi_t \;=\; \mathrm{Res}\left[\frac{g(x,y)\mathrm{d}x \wedge dy}{xy - t}\right].$$

Then locally

$$\int_{C_t} \varphi_t \wedge \bar{\varphi}_t \;=\; |g(0,0)|^2 \log |t|^{-1} \;+\; \mathsf{LOT}\,.$$

□

[11] For 0-dimensional varieties this is just the usual product of complex numbers.

4.4.3. *Friedman condition for smoothability.* We conclude this section with a brief discussion of some of how parts of [Fri83] apply to complexes constructed from an abstract normal crossing divisor $X = \cup X_i$ to give conditions on complexes constructed from the cohomology group $H^a(X^{[b]})(-c)$ to be the E_1-term of a spectral sequence whose abutment is a limiting mixed Hodge structure. If $D = \amalg_{i<j} D_{ij}$ is the double locus of X, then as in [Fri83] in terms of X above there is defined the *infinitesimal normal bundle* $\mathcal{O}_D(X)$, and a necessary condition for the smoothability of X is

$$\mathcal{O}_D(X) \cong \mathcal{O}_D. \tag{4.4.5}$$

If X is smoothable to be the central fiber in $\mathcal{X} \to \Delta$, then $\mathcal{O}_D(X) = \mathcal{O}_D \otimes \mathcal{O}_{\mathcal{X}}(X)$. The cohomological implications of (4.4.5) then give conditions that diagrams such as (4.4.3) actually be complexes whose cohomology is then the associated graded to a limiting mixed Hodge structure.[12] In other words, the condition (4.4.5) is sufficient to construct as in [PS08] the spectral sequence that would arise from $\mathcal{X} \to \Delta$. To keep the notation as simple as possible we shall do the case where $X = X_1 \cup X_2 \cup X_3$ where X_i is locally given by $x_i = 0$ in $\mathbb{C}^3$. If X is smoothable so that along the double locus the smoothing is given by $x_1 x_2 = t$, then the relation $\mathrm{d}t = x_2 \mathrm{d}x_1 + x_1 \mathrm{d}x_2$ translates away from the triple points into

$$\mathcal{O}_{D_{12}}(X_1) \otimes \mathcal{O}_{D_{12}}(X_2) \;\cong\; \mathcal{O}_{D_{12}}\,.$$

For a smoothable triple point p given by $x_1 x_2 x_3 = t$ we have $\mathrm{d}t = x_2 x_3 \mathrm{d}x_1 + x_1 x_3 \mathrm{d}x_2 + x_1 x_2 \mathrm{d}x_3$, which at $x_1 x_2 = 0$, $x_3 = 0$ gives

$$\mathcal{O}_{D_{12}}(X_2) \otimes \mathcal{O}_{D_{12}}(X_2) \otimes \mathcal{O}_{D_{12}}(p) \;\cong\; \mathcal{O}_{D_{12}}\,.$$

From this we obtain the *triple point formula*

$$D_{12}^2\big|_{X_1} + D_{12}^2\big|_{X_2} + 1 \;=\; 0\,, \tag{4.4.6}$$

where $D_{12}^2\big|_{X_i}$ is the self intersection of D_{12} in X_i.

We now explain how (4.4.6) enters into (4.4.3). In

$$\begin{array}{ccc} & & H^2(X^{[1]}) \\ & \overset{G'}{\nearrow} & \\ H^0(X^{[2]})(-1) & & \\ & \underset{R}{\searrow} & \\ & & H^0(X^{[3]})(-1) \end{array}$$

[12]This discussion may be extended to the case when X is locally a product of normal crossing divisors (such as arise from stable nodal curves), and also to the several parameter case where X is locally a product of normal crossing divisors such as arise in the semi-stable reduction constructed in [AK00]. The details and applications of this will appear elsewhere.

the map is

$$
\begin{array}{ccc}
 & & [D_{12}]\big|_{X_1} - [D_{12}]\big|_{X_2} \\
 & \nearrow & \\
1_{D_{12}} & & \\
 & \searrow & \\
 & & 1_p\,,
\end{array}
$$

where $[D_{12}]\big|_{X_i}$ is the class of D_{12} in $H^2(X_i)$. For

$$
\begin{array}{ccl}
H^2(X^{[1]}) & & \\
 & \overset{R'}{\searrow} & \\
 & & H^2(X^{[2]}) = H^2(X_{12}) \oplus H^2(X_{13}) \oplus H^2(X_{23}) \\
 & \underset{G}{\nearrow} & \\
H^0(X^{[3]})(-1) & &
\end{array}
$$

the maps are induced by

$$
[D_{12}]\big|_{X_1} - [D_{12}]\big|_{X_2} \to \left(D_{12}^2\big|_{X_2} + D_{21}^2\big|_{X_1}\right)[X_{12}] \oplus (-[X_{13}]) \oplus (-[X_{23}]) \tag{4.4.7}
$$

where as above $D_{12}^2\big|_{X_2}$ is the self-intersection of D_{12} in X_2 and similarly for $D_{21}^2\big|_{X_1}$, and where $[X_{ij}]$ is the fundamental class of X_{ij}. The points here are:

(a) If C is a smooth, irreducible curve on a surface Y, then the restriction $H^2(Y) \to H^2(C)$ maps the class $[C] \in H^2(Y)$ of C to the self-intersection number C^2 times the generator of $H^2(C)$; this accounts for the first term in (4.4.7).

(b) If C, C' are smooth, irreducible curves in Y meeting a point, then $H^2(Y) \to H^2(C')$ maps $[C]$ to a generator of $H^2(C')$; this accounts for the last two terms in (4.4.7).

Using the above to compute the maps G', R', G, R in (4.4.3) we may draw the conclusion

> *The triple point formula for each pair of components of X implies that* (4.4.3) *is a complex.*

4.5. **Appendix to §4: Extension of the geometric argument to the general case.** In this section we discuss some of the issues are that arise in trying to extend the above geometric argument to the case of an arbitrary VPHS.

The setting is a projective family $\mathfrak{X}^* \xrightarrow{\pi} \Delta^{*k} \times \Delta^\ell$ of smooth varieties $X_{(t,w)} = \pi^{-1}(t,w)$ where $(t,w) = (t_1, \dots, t_k; w_1, \dots, w_\ell)$ are coordinates in $\Delta^{*k} \times \Delta^\ell$. According to Abramovich–Karu [AK00], after successive modifications and base changes the above family may be completed to $\mathfrak{X} \xrightarrow{\pi} \Delta^k \otimes \Delta^\ell$, where $\mathfrak{X}$ is smooth and the singular fibers $X_w = \pi^{-1}(0,w)$ are locally a product of reduced normal crossing varieties. For the purposes of illustration we take the case $k = 2, \ell = 1$ of a degenerating family of surfaces. The strata of X_w together with local coordinates on $\mathcal{X}$ and the mapping π are

$$\begin{array}{rcl} X_w^{[1]}(x_1,x_2,x_3,x_4) & \to & (t_1 = x_3, t_2 = x_4), \\ X_w^{[2,1]}(x_1,x_2,x_3,x_4) & \to & (t_1 = x_1x_2, t_2 = x_4), \\ X_w^{[3,1]}(x_1,x_2,x_3,x_4) & \to & (t_1 = x_1x_2x_3, t_2 = x_4), \\ X_w^{[2,2]}(x_1,x_2,x_3,x_4) & \to & (t_1 = x_1x_2, t_2 = x_3x_4)\,, \end{array}$$

and similarly for $X_w^{[1,2]}$ and $X_w^{[1,3]}$. The sections $\psi(t,w)$ of the direct image of the relative dualizing sheaf are locally double Poincaré residues of 4-forms where the two functions in the denominator are the defining equations of the graph of π. For example, for $X_w^{[2,2]}$

$$\psi(t,w) \;=\; \mathrm{Res}\,\mathrm{Res}\left[\frac{f(x_1,x_2,x_3,x_4)\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4}{(x_1x_2-t_1)(x_3x_4-t_2)}\right].$$

The highest order terms in the expansion of the Hodge norm

$$\|\psi(t,w)\|^2 \;=\; \int_{X_{(t,w)}} \psi(t,w)\wedge\overline{\psi(t,w)}$$

are of the form

$$A_1(w)(\log|t_1|^{-1})^2 \;+\; B(w)\log|t_1|^{-1}\log|t_2|^{-1} \;+\; A_2(w)(\log|t_2|^{-1})^2\,.$$

The lower order terms are of the form

$$C_1(w)\log|t_1|^{-1} \;+\; C_2(w)\log|t_2|^{-1} \;+\; D(w)\,.$$

When we compute $\partial\overline{\partial}\log\|\psi(t,w)\|^2$ and set $\mathrm{d}t_1 = \mathrm{d}\bar{t}_1 = \mathrm{d}t_2 = \mathrm{d}\bar{t}_2 = 0$ it is possible that we could be left with a term like $\dfrac{\log|t_1|^{-1} + \log|t_2|^{-1}}{\log|t_1|^{-1}\log\;|t_2|^{-1}}$, which does not have a limit as $t_1, t_2 \to 0$.[13] Consequently we need some control of what can appear in $\partial\overline{\partial}\log\|\psi(t,w)\|^2$. As we will see in §5, the several-variable SL(2) orbit theorem gives us this control.

5. Asymptotic behavior of Chern forms

In this section we give a proof of Theorem 1.4.1 for an arbitrary variation of Hodge structure. The theorem is a consequence of a more general result on the Chern forms of the graded quotient Hodge bundles (Theorem 5.1.2).

5.1. **The general statement.** Recall the graded quotient Hodge bundles $\mathrm{Gr}^p\mathcal{F}$ (Definition 1.3.3). Let $\mathrm{Gr}^p_I\mathcal{F} \to Z_I^*$ denote the pulled-back graded quotient Hodge vector bundle under the period map $\Phi_I : Z_I^* \to \Gamma_I\backslash D_I$. Associated to $\mathrm{Gr}^p_I\mathcal{F}$ is a principle bundle with fiber group

$$\mathcal{H}_I^p \;=\; \prod_{q=0}^{p}\mathrm{Aut}(H_I^{p-q,0}(-q)) \;\times\; \prod_{q=p+1}^{n}\mathrm{Aut}(H_I^{0,q-p}(-p)) \;=\; \prod_{q=0}^{n}\mathrm{GL}(h_I^{p,q},\mathbb{C})\,.$$

[13]The term $\log|t_1|^{-1}$ corresponds to N_1, the term $\log|t_2|^{-1}$ to N_2, and $\log|t_1|^{-1}+\log|t_2|^{-1}$ to N_1+N_2.

Let

$$\mathfrak{h}_I^p \;=\; \bigoplus_{q=0}^{p} \mathrm{End}(H_I^{p-q,0}(-q)) \;\oplus\; \bigoplus_{q=p+1}^{n} \mathrm{End}(H_I^{0,q-p}(-p)) \;=\; \bigoplus_{q=0}^{n} \mathrm{Mat}(h_I^{p,q},\mathbb{C})$$

denote the Lie algebra of $\mathcal{H}_I^p$; here $\mathrm{Mat}(h,\mathbb{C})$ is the algebra of $h \times h$ complex matrices. Let $\mathbb{C}[\mathfrak{h}_I^p]$ be the algebra of $\mathbb{C}$–polynomials on $\mathfrak{h}_I^p$, and let

$$\mathfrak{P}_I^p \;=\; \mathbb{C}[\mathfrak{h}_I^p]^{\mathcal{H}_I^p} \;:=\; \left\{ P \in \mathbb{C}[\mathfrak{h}_I^p] \;:\; P(X) = P(\mathrm{Ad}_g(X)) \;\forall\; X \in \mathfrak{h}_I^p\,,\; g \in \mathcal{H}_I^p \right\}$$

be the subalgebra of polynomials that are invariant under the induced action of $\mathrm{Ad}(\mathcal{H}_I^p)$. Let h_I^p be the induced Hodge metric on $\mathrm{Gr}_I^p\mathcal{F} \to Z_I^*$, and let Υ_I^p denote the associated curvature form. We have the Chern–Weil homomorphism $\mathfrak{P}_I^p \to H^\bullet(Z_I^*,\mathbb{C})$ mapping $P \mapsto P(\Upsilon_I^p)$.

The extended Hodge bundles $\mathcal{F}_{\mathrm{e}}^p \to \overline{B}$ induce

$$\mathcal{H}_J^p \;\hookrightarrow\; \mathcal{H}_I^p \quad \text{for all} \quad I \subset J\,, \tag{5.1.1}$$

(cf. §5.2.3) so that $\mathfrak{P}_I^p \subset \mathfrak{P}_J^p$ for all $I \subset J$. Keeping in mind that $B = Z_\emptyset^*$, we write $\mathrm{Gr}^pF = \mathrm{Gr}_\emptyset^pF$, $\mathfrak{P}^p = \mathfrak{P}_\emptyset^p$ and $\Upsilon = \Upsilon_\emptyset$. With this notation we have

$$\mathfrak{P}^p \;\subset\; \mathfrak{P}_I^p \quad \text{for all } I\,.$$

Fix a point $b_o \in Z_I^*$ and a coordinate chart $(t,w) \in \Delta^k \times \Delta^\ell \simeq \overline{\mathcal{U}} \subset \overline{B}$ centered at b_o so that $\mathcal{U} = \overline{\mathcal{U}} \cap B \simeq \Delta^{*k} \times \Delta^\ell$, with $k = |I|$. Without loss of generality $I = \{1,\ldots,k\}$. Note that $Z_I^* \cap \overline{\mathcal{U}} = \{0\} \times \Delta^\ell$.

Theorem 5.1.2. *Fix $P \in \mathfrak{P}^p$. Then*

$$\lim_{t\to 0}\,[P(\Upsilon^p(t,w)) \;\mathit{mod}\; \{\mathrm{d}t_i, \mathrm{d}\bar{t}_i \mid 1 \le i \le k\}] \;\equiv\; P(\Upsilon_I^p(w))\,.$$

We will now deduce Theorem 1.4.1 from Theorem 5.1.2. Then the remainder of §5 will be occupied with the proof of Theorem 5.1.2; the argument is outlined in §5.1.2.

5.1.1. *Proof of Theorem 1.4.1.* Theorem 1.4.1 may be reformulated as

Theorem 5.1.3. *The curvature forms of the augmented Hodge line bundles satisfy*

$$\lim_{t\to 0}\left[\hat{\Omega}(t,w) \;\mathit{mod}\; \{\mathrm{d}t_i, \mathrm{d}\bar{t}_i \mid 1 \le i \le k\}\right] \;\equiv\; \hat{\Omega}_I(w)\,.$$

Theorem 5.1.3 is a corollary of

Theorem 5.1.4. *Let Ω_p and $\Omega_{p,I}$ be the Chern forms of the line bundles* $\det \mathrm{Gr}^p\mathcal{F}$ *and* $\det \mathrm{Gr}_I^p\mathcal{F}$*, respectively. Then*

$$\lim_{t\to 0}\,[\Omega_p(t,w) \;\mathit{mod}\; \{\mathrm{d}t_i, \mathrm{d}\bar{t}_i \mid 1 \le i \le k\}] \;\equiv\; \Omega_{p,I}(w)\,.$$

Proof of Theorem 5.1.3. This follows directly from Theorem 5.1.4 and the fact that $\hat{\Lambda}$ is expressed as a tensor product of the powers of the $\det \mathrm{Gr}^p\mathcal{F}$'s (Definition 1.3.4). □

Proof of Theorem 5.1.4. This is Theorem 5.1.2 in the case that $P = \mathrm{trace}$. □

5.1.2. *Outline of the proof of Theorem 5.1.2.* Given a nondegenerate Hermitian matrix $h = (h_{ab}(t,w))$, define $h^{ab}(t,w)$ by $h^{ac}h_{bc} = \delta^a_b$. The associated curvature matrix (modulo $\mathrm{d}t, \mathrm{d}\bar{t}$) is given by $\Upsilon[h] = (\Upsilon[h]^a_b)$ with

$$\Upsilon[h]^a_b \;:=\; -\sum_{i,j} \partial_{\overline{w}_j}\left(h^{ac}\cdot\partial_{w_i}h_{bc}\right)\mathrm{d}w_i\wedge\mathrm{d}\overline{w}_j\,.$$

The Hodge metric induces nondegenerate Hermitian forms h^p on $\mathrm{Gr}^p\mathcal{F}\to\mathcal{U}$ and h^p_I on $\mathrm{Gr}^p_I\mathcal{F}\to\Delta^\ell$.[14] When these forms are expressed as matrices (relative to a holomorphic framing), we have $\Upsilon[h^p]\equiv\Upsilon^p \bmod \mathrm{d}t_i,\, \mathrm{d}\bar{t}_i$, and $\Upsilon^p_I = \Upsilon[h^p_I]$. We wish to show that

$$\lim_{t\to 0} P\big(\Upsilon[h^p]\big)(t,w) \;=\; P\big(\Upsilon[h^p_I]\big)(w)\,. \tag{5.1.5}$$

This will be a consequence of the analysis of asymptotics utilized by Cattani–Kaplan–Schmid to establish their estimates for the Hodge metric [CKS86, §5]. The hypothesis $P\in\mathfrak{P}^p$ enters as follows: If $A(t) = (A^a_b(t))$ and $B(t) = (B^a_b(t))$ are invertible matrices, and

$$\tilde{h}_{ab}(t,w) \;=\; A^c_a(t)\,h_{cd}(t,w)\,\overline{B^d_b(t)}\,, \tag{5.1.6a}$$

then $\Upsilon[\tilde{h}]^a_b = (A^{-1})^a_c\,\Upsilon[h]^c_d\,A^d_b$, so that

$$P\big(\Upsilon[h]\big) \;=\; P\big(\Upsilon[\tilde{h}]\big) \quad \text{for all} \quad P\in\mathfrak{P}^p \tag{5.1.6b}$$

by definition of $\mathfrak{P}^p$. This is important because the metric h blows-up as we approach the divisor Z at infinity. The several-variable SL(2)–orbit theorem provides a method for replacing h with an $\tilde{h}$ that is bounded at infinity (§5.4.1). One may then argue, via additional applications of (5.1.6), that (5.1.5) holds (§5.4.3). More precisely, we fix a sequence $t_\mu = (t_{\mu 1},\ldots,t_{\mu k})\in\Delta^{*k}$ converging to 0. Writing $\ell(t_{\mu j}) = z_{\mu j} = x_{\mu j} + \sqrt{-1}y_{\mu j}$, we have $y_{\mu j} = -\frac{1}{\pi}\log|t_{\mu j}|$. Restricting to a subsequence if necessary (and dropping the subscript $_\mu$), we may assume without loss of generality that either $y_i/y_j\to 0$, $y_i/y_j\to\infty$, or y_i/y_j is bounded (away from both 0 and ∞). Reordering indices if necessary, we may assume that there exists $K = \{k_1,\ldots,k_\rho\}\subset\{1,\ldots,k\} = I$ so that $y_{k_\alpha}/y_{k_{\alpha+1}}\to\infty$ and y_j/y_{k_α} is bounded for all $k_{\alpha-1} < j < k_\alpha$. In §5.3.1 we will employ a collection of commuting SL(2)'s that is well-suited to studying the asymptotic behavior of $\Upsilon[h^p](t,w)$ under such a sequence. The key tool here is a semisimple automorphism $\varepsilon(t,w)$ associated with the SL(2)'s, and it is the asymptotic behavior of $\mathrm{Ad}_{\varepsilon(t,w)}$, and its eigenvalues, that yields the bounded $\tilde{h}$, cf. Lemma 5.3.14.

The remainder of §5 is occupied with the proof of Theorem 5.1.2. After recalling the local coordinate expressions for the metrics on the Hodge bundles (§5.2), we review the Cattani–Kaplan–Schmid asymptotics (§5.3). Equation (5.1.5) is proved in §5.4.

5.2. **Review of the Hodge bundles and their curvature.** We begin by reviewing the bundles $\mathrm{Gr}^p_I\mathcal{F}$ and their curvature forms Υ^p_I.

[14]When $p = n$, the form h^n_I is positive definite. In general, h^p_I will have mixed signature.

5.2.1. *Deligne's $\mathbb{R}$–split PMHS.* Set

$$N \;:=\; N_1 + \cdots + N_k \quad \text{and} \quad W \;:=\; W(N)\,.$$

Recall the notation (2.1.2) for the local variation of Hodge structure $\Phi(t,w) = \exp\left(\sum_{i\in I}\ell(t_i)N_i\right)\cdot F(t,w)$. In general, the LMHS $(W, F(0,w))$ will not be $\mathbb{R}$–split. It will be convenient to work with Deligne's associated $\mathbb{R}$–split MHS $(W(N), \tilde F_w)$, cf. [CKS86, (2.20)]; here $\tilde F_w = \exp(-\sqrt{-1}\,\delta_w)\cdot F(0,w)$. The element $\delta_w \in \mathfrak{g}_\mathbb{R}$ commutes with the N_i, and $w \mapsto \delta_w$ is a real analytic map $\Delta^\ell \to \mathfrak{g}_\mathbb{R}$. Let

$$V_\mathbb{C} \;=\; \bigoplus \tilde I^{p,q}_w \quad \text{and} \quad \mathfrak{g}_\mathbb{C} \;=\; \bigoplus \tilde{\mathfrak{g}}^{p,q}_w \tag{5.2.1}$$

be the associated Deligne splittings. Set

$$\tilde{\mathfrak{n}}_w \;:=\; \bigoplus_{\substack{p<0\\ q}} \tilde{\mathfrak{g}}^{p,q}_w \;=\; \tilde{\mathfrak{g}}^{-,\bullet}_w\,.$$

Note that

$$\mathfrak{g}_\mathbb{C} \;=\; \tilde{\mathfrak{n}}_w \oplus \tilde{\mathfrak{p}}_w\,, \tag{5.2.2}$$

where $\tilde{\mathfrak{p}}_w$ is the Lie algebra of $\mathrm{Stab}_{G_\mathbb{C}}(\tilde F_w)$. There exists a unique holomorphic map

$$X : \overline{\mathcal{U}} \;\to\; \tilde{\mathfrak{n}}_0$$

so that $X(0,0) = 0$ and

$$\Phi(t,w) \;=\; \exp\left(\sqrt{-1}\,\delta_0 \;+\; \textstyle\sum_{i=1}^k \ell(t_i)N_i\right)\zeta(t,w)\cdot \tilde F_0\,, \tag{5.2.3a}$$

where

$$\zeta(t,w) \;:=\; \exp X(t,w)\,. \tag{5.2.3b}$$

See [CKS86, §5] for details. Set

$$\eta(t) \;:=\; \exp\left(\sqrt{-1}\,\delta_0 \;+\; \textstyle\sum_{i=1}^k \ell(t_i)N_i\right)\,.$$

5.2.2. *The metric on the quotient Hodge vector bundle* $\mathrm{Gr}^p\mathcal{F}$*.* Fix a set $\{\mathsf{v}_a\} \subset \tilde F^p_0$ of linearly independent vectors with the property that $\{\mathsf{v}_a \bmod \tilde F^{p+1}_0\}$ forms a basis of $\mathrm{Gr}^p\tilde F_0 = \tilde F^p_0/\tilde F^{p+1}_0$. Then

$$v_a(t,w) \;:=\; \eta(t)\zeta(t,w)\,\mathsf{v}_a$$

is naturally identified with a local holomorphic framing of the quotient Hodge vector bundle $\mathrm{Gr}^p\mathcal{F} \to \mathcal{U}$. Let $\{v^a(t,w)\}$ denote the dual coframing. Then the Hermitian metric

$$h(t,w) \;=\; h_{ab}(t,w)\,v^a \otimes \bar v^b \tag{5.2.4a}$$

on $\mathrm{Gr}^p\mathcal{F}$ is given by

$$h_{ab}(t,w) \;:=\; (\sqrt{-1})^n Q\left(v_a(t,w)\,,\, \overline{v_b(t,w)}\right)\,. \tag{5.2.4b}$$

Define $h^{ab}(t,w)$ by

$$h^{ab}(t,w)\,h_{ac}(t,w) \;:=\; \delta^b_c\,.$$

The curvature form on the quotient Hodge vector bundle $\mathrm{Gr}^p\mathcal{F} \to \mathcal{U}$ is

$$\Upsilon^p \ = \ \bar\partial\,(h^{ac}\cdot\partial h_{bc})\ v_a\otimes v^b\,.$$

5.2.3. *The metric the quotient Hodge vector bundle* $\mathrm{Gr}^p_I\mathcal{F}$. Let $X_0^{p,q}$ denote the component of X taking value in $\tilde{\mathfrak{g}}_0^{p,q}$. Define $X_I(w) = \oplus\, X_0^{-p,p}(0,w)$, and set

$$\zeta_I(w)\ :=\ \exp\, X_I(w)\,. \tag{5.2.5}$$

Recall that $\tilde F_0^p = \oplus_{r\ge p}\tilde I_0^{r,\bullet}$, so that $\mathrm{Gr}^p\tilde F_0 \simeq \tilde I_0^{p,\bullet} = \oplus_q \tilde I^{p,q}$. Refine the set $\{\mathsf{v}_a\}$ so that $\mathsf{v}_a\in\tilde I_0^{p,q}$ for some $q = q(a)$. (It is this refined basis that gives us the inclusion (5.1.1).) Then

$$v_a^I(w)\ :=\ \zeta_I(w)\,\mathsf{v}_a$$

defines a local holomorphic framing of the quotient Hodge vector bundle $\mathrm{Gr}^p_I\mathcal{F}\to\Delta^\ell$. Let $v_I^a(w)$ denote the dual coframing. Then the Hermitian metric

$$h^I(w)\ =\ h^I_{ab}(w)\,v_I^a\otimes v_I^b \tag{5.2.6a}$$

on $\mathrm{Gr}^p_I\mathcal{F}$ is given by

$$h^I_{ab}(w)\ =\ \begin{cases} (\sqrt{-1})^{n-q}\,Q\left(\zeta_I(w)\mathsf{v}_a\,,\ N^q\,\overline{\zeta_I(w)\mathsf{v}_a}\right), & q=q(a)=q(b)\,,\\ 0 & q(a)\neq q(b)\,.\end{cases} \tag{5.2.6b}$$

Defining $h_I^{ab}(w)$ by

$$h_I^{ab}(w)\,h^I_{ac}(w)\ =\ \delta^b_c\,,$$

the curvature form of $\mathrm{Gr}^p_I\mathcal{F}\to\Delta^\ell$ is

$$\Upsilon^p_I\ :=\ \bar\partial\left(h_I^{ac}\cdot\partial h^I_{bc}\right)\ v^I_a\otimes v_I^b\,.$$

5.2.4. *Horizontality.* Recall (§2.3) that $X(0,w)$ lies in the Lie algebra

$$\mathfrak{z}_\mathbb{C}\ :=\ \{Z\in\mathfrak{g}_\mathbb{C}\mid [Z,N_j]=0\,,\ \forall j\}\ =\ \bigcap_{j=1}^k \ker\,(\mathrm{ad}\,N_j)$$

of the centralizer

$$\mathscr{Z}_\mathbb{C}\ :=\ \{g\in G_\mathbb{C}\mid \mathrm{Ad}_g N_i = N_i\,,\ \forall i\}\,.$$

Notice that $\mathscr{Z}_\mathbb{C}$ is defined over $\mathbb{Q}$, preserves the weight filtration $W(N)$, and that the Lie algebra inherits a decomposition

$$\mathfrak{z}_\mathbb{C}\ =\ \bigoplus_{\ell\le 0}\mathfrak{z}_\ell(w)\,,\quad \text{with}\quad \mathfrak{z}_\ell(w)\ :=\ \mathfrak{z}_\mathbb{C}\cap\oplus_{p+q=\ell}\,\tilde{\mathfrak{g}}_w^{p,q}$$

from the Deligne splitting. The function $X_I(w)$ of §5.2.3 is the component of $X(0,w)$ taking value in $\tilde{\mathfrak{g}}_\ell(0)$. In particular, both $X_I(0,w)$ and $\zeta_I(w)$ commute with the $\{N_j\}_{j=1}^k$.

5.2.5. *Relationship between the $\mathbb{R}$–split $\tilde{F}_w$.* The Hodge filtrations $\tilde{F}_w$ are all congruent to $\tilde{F}_0$ under the action of $\mathcal{Z}_\mathbb{R}$ [KP16]. In particular, we may choose a real analytic function $\psi : \Delta^\ell \to \exp(\mathfrak{z}_\mathbb{R}) \subset \mathcal{Z}_\mathbb{R}$ so that $\psi(0)$ is the identity and

$$\psi(w) \cdot \tilde{F}_w \;=\; \tilde{F}_0 \,. \tag{5.2.7}$$[15]

5.3. **The Cattani–Kaplan–Schmid asymptotics.** Here we briefly review the necessary results from [CKS86, §5].[16]

5.3.1. *The CKS coordinates.* Fix $K = \{k_1, \ldots, k_\rho\} \subset \{1, \ldots, k\}$ so that $1 \le k_1 < \cdots < k_\rho = k$. Define

$$\begin{aligned} s_\alpha \;&:=\; y_{k_\alpha}/y_{k_{\alpha+1}}\,, \quad \text{for } \alpha < \rho\,, \quad \text{and} \quad s_\rho \;:=\; y_k\,;\\ u^j_\alpha \;&:=\; y_j/y_{k_\alpha}\,, \quad \text{for } \; k_{\alpha-1} < j < k_\alpha\,. \end{aligned}$$

Define

$$\begin{aligned} \mathbb{R}^k_+ \;&:=\; \{y = (y_j) \in \mathbb{R}^k \mid y_j > 0\}\\ \mathbb{R}^\rho_+ \;&:=\; \{s = (s_\alpha) \in \mathbb{R}^\rho \mid s_\alpha > 0\}\\ \mathbb{R}^{k-\rho}_+ \;&:=\; \{u = (u^j_\alpha) \in \mathbb{R}^{k-\rho} \mid u^j_\alpha > 0\}\,. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{A} \;&:=\; \text{(real) analytic functions of } (u,w) \in \mathbb{R}^{k-\rho}_+ \times \Delta^\ell\,,\\ \mathcal{L} \;&:=\; \text{Laurent polys. in } \{s_\alpha^{1/2}\} \text{ with coef. in } \mathcal{A}\,,\\ \mathcal{O} \;&:=\; \text{pullback to } \mathcal{H}^k \times \Delta^\ell \text{ of the ring of holo. germs at } 0 \in \Delta^r = \Delta^k \times \Delta^\ell\\ &\;\; \text{via } \mathcal{H}^k \to (\Delta^*)^k \hookrightarrow \Delta^k\,,\\ \mathcal{L}^\flat \;&:=\; \text{polys. in } s_\alpha^{-1/2} \text{ with coef. in } \mathcal{A}\,,\\ (\mathcal{O} \otimes \mathcal{L})^\flat \;&:=\; \text{subring of } \mathcal{O} \otimes \mathcal{L} \text{ gen. by } \mathcal{O},\, \mathcal{L}^\flat, \text{ and all monomials of the form}\\ &\;\; t_j s_1^{m_1/2} \cdots s_\rho^{m_\rho/2} \text{ with } m_\alpha \in \mathbb{Z} \text{ and } m_\alpha \neq 0 \text{ only if } j \le i_\alpha. \end{aligned}$$

---

[15]This choice of $\psi(w)$ is not unique. There is a unique choice of $\psi_\mathbb{C} : \Delta^\ell \to \exp(\mathfrak{z}_\mathbb{C} \cap \tilde{\mathfrak{n}}_0) \subset \mathcal{Z}_\mathbb{C}$ so that $\psi_\mathbb{C}(0)$ is the identity and $\psi_\mathbb{C}(w) \cdot \tilde{F}_w = \tilde{F}_0$. Nonetheless it is better to work with the $\mathcal{Z}_\mathbb{R}$–value $\psi(w)$, because the Hermitian metric h_{F^n} is $G_\mathbb{R}$–invariant, but not $G_\mathbb{C}$–invariant. And ultimately the argument and result are independent of our choice.

[16]The arguments of [CKS86, §5] assume that $\ell = 0$, so that the holomorphic parameter w does not play a role. However, the proofs there (up to and including that of [CKS86, (5.14)]) all apply, in a straightforward manner, in our more general setting to yield the assertions below.

We identify $\mathbb{R}^k_+$ with $\mathbb{R}^\rho_+ \times \mathbb{R}^{k-\rho}_+$ by $y \mapsto (s,u)$. Recall that $\mathcal{H} \subset \mathbb{C}$ denotes the upper-half plane. Given $c > 0$ define

$$\begin{aligned}
(\mathbb{R}^k_+)^K_c &:= \left\{ y \in \mathbb{R}^k_+ \mid s_\alpha > c\,,\ 1/c \le u^j_\alpha \le c \right\} \\
(\mathcal{H}^k_+)^K_c &:= \left\{ z \in \mathcal{H}^k \mid z = x + \sqrt{-1}y\,,\ y \in (\mathbb{R}^k_+)^K_c \right\} \\
(\Delta^{*k})^K_c &:= \left\{ t \in (\Delta^*)^k \mid \ell(t_j) \in (\mathcal{H}^k)^K_c \right\}.
\end{aligned}$$

Lemma 5.3.1 ([CKS86, (5.7)]). (a) *For any $c > 1$, the regions $(\Delta^{*k})^K_c$ corresponding to the various permutations of the variables and choices of $K \subset \{1, \ldots, k\}$ cover the intersection of $(\Delta^*)^k$ with a neighborhood of $0 \in \Delta^k$.* (b) *The set $(\mathcal{O} \otimes \mathcal{L})^\flat$ consists of precisely those elements in $\mathcal{O} \otimes \mathcal{L}$ that are bounded on $(\mathcal{H}^k)^K_c$ for some (any) c.*

Remark 5.3.2. The coordinates $y = (s,u)$ are well-adapted to study the asymptotic behavior of the Hodge metric and Chern form for the sequence t_μ (cf. the outline of the proof of Theorem 5.1.2 in §5.1.2). Throughout the remainder of §5, the notation

$$f(x,s,u;w) \ \hat{\longrightarrow}\ g(x,u;w) \quad (\text{or } f(t,w) \ \hat{\longrightarrow}\ g(x,u;w))$$

will indicate that *$f(x,s,u;w)$ converges to $g(x,u;w)$ as $s_1, \ldots, s_\rho \to \infty$, and that this convergence is uniform on compact subsets of $\{x \in \mathbb{R}^k\} \times \{u^j_\alpha \in \mathbb{R}^{k-\rho}_+\} \times \{w \in \Delta^\ell\}$.*

5.3.2. *Commuting* SL(2)*'s.* Define

$$\mathcal{N}_\alpha(u) \ := \ N_{k_\alpha} \ + \sum_{k_{\alpha-1}<j<k_\alpha} u^j_\alpha N_j \ = \ \frac{1}{y_{k_\alpha}} \sum_{k_{\alpha-1}<j\le k_\alpha} y_j N_j \,.$$

Note that

$$\sum_{j=1}^k y_j N_j \ = \ \sum_{\alpha=1}^\rho (s_\alpha s_{\alpha+1} \cdots s_\rho) \mathcal{N}_\alpha(u)\,. \tag{5.3.3}$$

Since each $\exp(\sum z_j N_j) \cdot \tilde{F}_w$ is a nilpotent orbit, it follows that $\exp(\sum_\alpha z_{k_\alpha} \mathcal{N}_\alpha(u)) \cdot \tilde{F}_w$ is also a nilpotent orbit. The several-variable SL(2)–orbit theorem [CKS86, (4.20)] associates to this nilpotent orbit a collection $\{\nu_\alpha : \mathrm{SL}(2,\mathbb{C}) \to G_\mathbb{C}\}^\rho_{\alpha=1}$ of commuting horizontal SL(2)'s. Let $\{\hat{\mathcal{N}}_\alpha(u,w)\,,\, \hat{\mathcal{Y}}_\alpha(u,w)\,,\, \hat{\mathcal{N}}^+_\alpha(u,w)\}^\rho_{\alpha=1}$ denote the ν_α–images of the standard generators of $\mathfrak{sl}(2,\mathbb{R})$. Each of $\hat{\mathcal{N}}_\alpha$, $\hat{\mathcal{Y}}_\alpha$ and $\hat{\mathcal{N}}^+_\alpha$ is a $\mathfrak{g}_\mathbb{R}$–valued member of $\mathcal{A}$. Furthermore,

$$\left\{ \hat{\mathcal{N}}_\alpha(u,w)\,,\, \hat{\mathcal{Y}}_\alpha(u,w)\,,\, \hat{\mathcal{N}}^+_\alpha(u,w) \right\} \ = \ \mathrm{Ad}^{-1}_{\psi(w)} \left\{ \hat{\mathcal{N}}_\alpha(u,0)\,,\, \hat{\mathcal{Y}}_\alpha(u,0)\,,\, \hat{\mathcal{N}}^+_\alpha(u,0) \right\}, \tag{5.3.4}$$

for all $1 \le \alpha \le \rho$.

Proof of (5.3.4). This is a consequence of (5.2.7) and [CKS86, (4.75)]: note that the functions $T = T(W,F)$ and $\Phi = \Phi(Y,F)$ of [CKS86, p. 506] are $G_\mathbb{R}$–equivariant. □

Define

$$\mathbf{Y}^\alpha(u,w) \;:=\; \sum_{\beta=1}^{\alpha} \hat{\mathcal{Y}}_\beta(u,w) \;\stackrel{(5.3.4)}{=}\; \mathrm{Ad}^{-1}_{\psi(w)}\mathbf{Y}^\alpha(u,0)\,. \tag{5.3.5}$$

It follows directly from the CKS–construction that

$$\mathbf{Y}^\rho(u,w) \quad \text{is the element of } \mathfrak{g}_\mathbb{R} \text{ acting on } \tilde{\mathfrak{g}}_{w,\ell} \text{ by } \ell \in \mathbb{Z}; \tag{5.3.6}$$

in particular, $\mathbf{Y}^\rho_w := \mathbf{Y}^\rho(u,w)$ is independent of u.

5.3.3. *Eigenspace decompositions.* Set

$$\varepsilon(y,w) \;=\; \varepsilon(s,u;w) \;:=\; \exp\left(\tfrac{1}{2}\sum_\alpha \log s_\alpha \mathbf{Y}^\alpha(u,w)\right) \;=\; \mathrm{Ad}^{-1}_{\psi(w)}\varepsilon(s,u;0)\,. \tag{5.3.7}$$

Recall that the eigenvalues of $\mathbf{Y}^\alpha(u,w)$ are integers. Since $\mathbf{Y}^\alpha$ depends real-analytically on (u,w), both the eigenvalues and their multiplicities are independent of (u,w), and the eigenspaces depend real-analytically on (u,w).

$$\begin{array}{l}\text{If } \mathbf{Y}^\alpha(u,w) \text{ acts by the eigenvalue } e_\alpha \in \mathbb{Z},\\ \text{then } \varepsilon(s,u;w) \text{ acts by the eigenvalue } \prod_\alpha s_\alpha^{e_\alpha/2}.\end{array} \tag{5.3.8}$$

So $\varepsilon(s,u;w)$ is a $G_\mathbb{R}$–valued function in $\mathcal{L}$. Additionally $\mathbf{Y}^\alpha(u,w) \in \tilde{\mathfrak{g}}^{0,0}_{w,\mathbb{R}}$, so that $\mathbf{Y}^\alpha(u,w)$ preserves the Deligne splittings (5.2.1). Consequently,

$$\varepsilon(s,u;w) \;\in\; G_\mathbb{R}^{\mathrm{DS}_w} \;:=\; \{g \in G_\mathbb{R} \mid g(\tilde{I}^{p,q}_w) = \tilde{I}^{p,q}_w\,,\ \forall\ p,q\}\,. \tag{5.3.9}$$

This has two consequences: first,

$$\varepsilon(s,u;w)\cdot \tilde{F}_w \;=\; \tilde{F}_w \;=\; \varepsilon(s,u;w)^{-1}\cdot\tilde{F}_w\,. \tag{5.3.10}$$

Second, since $\varepsilon(s,u;w)$ is semisimple,

$$\tilde{I}^{p,q}_w \text{ decomposes into a direct sum of } \varepsilon(s,u;w)\text{–eigenspaces.} \tag{5.3.11}$$

Since the $\{\mathbf{Y}_\alpha(u,w)\}_{\alpha=1}^\rho$ are commuting semisimple endomorphisms, the Lie algebra admits a simultaneous eigenspace decomposition

$$\mathfrak{g}_\mathbb{R} \;=\; \bigoplus_{e_1,\ldots,e_\rho\in\mathbb{Z}} \mathfrak{g}_{e_1,\ldots,e_\rho}(u,w)\,,$$

with $\mathrm{ad}\,\mathbf{Y}^\alpha$ acting on $\mathfrak{g}_{e_1,\ldots,e_\rho}$ by the eigenvalue $e_\alpha \in \mathbb{Z}$. In particular,

$$\mathrm{Ad}_{\varepsilon(s,u;w)} \text{ acts on } \mathfrak{g}_{e_1,\ldots,e_\rho}(u,w) \text{ by the eigenvalue } \prod_\alpha s_\alpha^{e_\alpha/2}\,. \tag{5.3.12}$$

The eigenspaces $\mathfrak{g}_{e_1,\ldots,e_\rho}(u,w)$ depend real-analytically on (u,w), and (5.3.5) implies

$$\mathfrak{g}_{e_1,\ldots,e_\rho}(u,w) \;=\; \mathrm{Ad}^{-1}_{\psi(w)}\mathfrak{g}_{e_1,\ldots,e_\rho}(u,0)\,.$$

Recollect that the common intersection of the weight filtrations

$$\mathfrak{w} \;:=\; \bigcap_{\alpha=1}^{\rho} W_0\left(\mathrm{ad}(\mathcal{N}_1+\cdots+\mathcal{N}_\alpha)\right) \;=\; \bigoplus_{e_1,\ldots,e_\rho\geq 0} \mathfrak{g}_{-e_1,\ldots,-e_\rho}(u,w) \tag{5.3.13}$$

is the direct sum of the eigenspaces for the nonpositive eigenvalues. The following is the *key lemma* in our analysis of the asymptotic behavior of the curvature matrix.

Lemma 5.3.14. *Suppose that $U(u,w)\in\mathfrak{w}$ depends continuously on (u,w). Let $U'(u,w)$ denote the component of $U(u,w)$ taking value in*

$$\mathfrak{g}_{0,\ldots,0}(u,w) \;=\; \{X\in\mathfrak{g} \mid [\mathbf{Y}^\alpha(u,w),X]=0\,,\ \forall\ \alpha\} \;=\; \bigcap_\alpha \ker(\mathrm{ad}\,\mathbf{Y}^\alpha(u,w))\,,$$

with respect to the decomposition (5.3.13)*. Then* (5.3.12) *yields*

$$\varepsilon(s,u;w)\,\exp(U(u,w))\,\varepsilon(s,u;w)^{-1} \;\hat{\longrightarrow}\; \exp(U'(u,w))\,,$$

cf. Remark 5.3.2.

Proof. This is an immediate consequence of (5.3.12). □

5.3.4. *Asymptotic behavior.* We will find it useful to write

$$\eta(t)\,\zeta(t,w) \;=\; \psi(w)\exp(\textstyle\sum x_jN_j)\nu_w(y)\xi(t,w)\psi(w)^{-1}\,, \tag{5.3.15a}$$

(cf. §5.2.1), where

$$\begin{aligned} \nu_w(y) &\;:=\; \exp\sqrt{-1}\left(\mathrm{Ad}^{-1}_{\psi(w)}\delta_0 \;+\; \textstyle\sum_{i=1}^k y_iN_i\right)\\ \xi(t,w) &\;:=\; \psi(w)^{-1}\zeta(t,w)\psi(w) \;=\; \exp\mathrm{Ad}^{-1}_{\psi(w)}X(t,w)\,. \end{aligned} \tag{5.3.15b}$$

This allows us to rewrite (5.2.3) as

$$\Phi(t,w) \;=\; \psi(w)\exp(\textstyle\sum x_jN_j)\nu_w(y)\xi(t,w)\cdot\tilde{F}_w\,. \tag{5.3.15c}$$

Define

$$\mu(s,u;w) \;:=\; \varepsilon(s,u;w)\xi(0,w)\varepsilon(s,u;w)^{-1}\,.$$

Recall that $X(0,w)=\log\zeta(0,w)$ takes value in the centralizer $\mathfrak{z}=\cap_j\ker(\mathrm{ad}N_j)$ of the $\{N_j\}_{j=1}^k$ (§2.3). Notice that

$$\mathfrak{z} \;\subset\; \bigcap_{\alpha=1}^{\rho}\ker\left(\mathrm{ad}(\mathcal{N}_1+\cdots+\mathcal{N}_\alpha)\right) \;\subset\; \bigcap_{\alpha=1}^{\rho} W_0\left(\mathrm{ad}(\mathcal{N}_1+\cdots+\mathcal{N}_\alpha)\right) \;=\; \mathfrak{w}\,. \tag{5.3.16}$$

Let $X_u(w)$ denote the component of $X(0,w)$ taking value in $\mathfrak{g}_{0,\ldots,0}(u,0)$ with respect to the decomposition $\mathfrak{g}_\mathbb{C}=\oplus\,\mathfrak{g}_{e_1,\ldots,e_\rho}(u,0)$, and set $\zeta_u(w):=\exp(X_u(w))$. Then (5.3.5) implies $U'(u,w)=\mathrm{Ad}^{-1}_{\psi(w)}X_u(w)$.

Lemma 5.3.17. *Both μ and μ^{-1} belong to $\mathcal{L}^\flat$, and*

$$\mu(s,u;w) \;\hat{\longrightarrow}\; \exp\mathrm{Ad}^{-1}_{\psi(w)}X_u(w) \;=\; \psi(w)^{-1}\zeta_u(w)\psi(w)\,.$$

Proof. Write $\xi(0,w) = \exp(U(w))$ with $U(w) = \mathrm{Ad}^{-1}_{\psi(w)} X(0,w)$. The result now follows from Lemma 5.3.14 □

Define $\lambda = \lambda_1 \cdot \lambda_2$ by

$$\begin{aligned} \lambda_1(t,w) &:= \varepsilon(s,u;w)\,\nu_w(y)\,\varepsilon(s,u;w)^{-1} \\ \lambda_2(t,w) &:= \varepsilon(s,u;w)\,\xi(t,w)\,\varepsilon(s,u;w)^{-1}\,, \end{aligned}$$

so that

$$\begin{aligned} \eta(t)\,\zeta(t,w) &= \psi(w)\exp\left(\textstyle\sum x_i N_i\right)\varepsilon(t,w)^{-1}\lambda(t,w)\varepsilon(t,w)\psi(w)^{-1}\,, \\ \Phi(t,w) &= \psi(w)\exp\left(\textstyle\sum x_i N_i\right)\varepsilon(t,w)^{-1}\lambda(t,w)\varepsilon(t,w)\cdot\tilde{F}_w\,. \end{aligned} \tag{5.3.18}$$

Set

$$\mathbf{N}(u,w) \;:=\; \sum_\alpha \hat{\mathcal{N}}_\alpha(u,w) \;\overset{(5.3.4)}{=}\; \mathrm{Ad}^{-1}_{\psi(w)}\mathbf{N}(u,0)\,. \tag{5.3.19}$$

To simplify notation we set

$$\mathbf{N}(u) \;:=\; \mathbf{N}(u,0)\,.$$

Lemma 5.3.20 ([CKS86, §5])**.** *Both* λ *and* λ^{-1} *belong to* $(\mathcal{O}\otimes\mathcal{L})^\flat$*, and*

$$\lambda(t,w) \stackrel{\hat{}}{\longrightarrow} \psi(w)^{-1}\exp\left(\sqrt{-1}\,\mathbf{N}(u,0)\right)\zeta_u(w)\psi(w)\,.$$

When $\ell = 0$, the lemma is proved by Cattani–Kaplan–Schmid in [CKS86, pp. 511–512]. Their argument extends to the general case with only minor modification; we sketch the proof here for completeness.

Proof. The proof of [CKS86, (5.12)] applies here to yield

$$\begin{aligned} \mathrm{Ad}_{\varepsilon(s,u;w)}\mathrm{Ad}^{-1}_{\psi(w)}\delta_0 &\stackrel{\hat{}}{\longrightarrow} 0\,, \\ \mathrm{Ad}_{\varepsilon(s,u;w)}\textstyle\sum_{j=1}^k y_j N_j &\stackrel{\hat{}}{\longrightarrow} \mathbf{N}(u,w)\,, \end{aligned} \tag{5.3.21}$$

so that

$$\lambda_1(t,w) \stackrel{\hat{}}{\longrightarrow} \exp(\sqrt{-1}\,\mathbf{N}(u,w))\,. \tag{5.3.22}$$

Briefly, (5.3.21) is a consequence of Lemma 5.3.14, and the facts (a) that

$$\mathrm{Ad}^{-1}_{\psi(w)}\delta_0 \;\in\; \mathfrak{z} \cap \bigoplus_{p,q\le -1} \tilde{\mathfrak{g}}^{p,q}_w \;\subset\; \bigoplus_{\substack{e_1,\cdots,e_{\rho-1}\ge 0\\ e_\rho \ge 2}} \mathfrak{g}_{-e_1,\ldots,-e_\rho}(u,w)$$

and (b) the observation (5.3.3) and the fact [CKS86, (4.20.iii)] that $\hat{\mathcal{N}}_\alpha(u,w)$ is the component of $\mathcal{N}_a(u)$ taking value in

$$\mathfrak{z} \cap \bigcap_{\beta=1}^{\alpha-1}\ker(\mathrm{ad}\,\mathbf{Y}^\beta(u,w)) \;\subset\; \bigoplus_{e_\alpha,\ldots,e_\rho\ge 0}\mathfrak{g}_{0,\ldots,0,-e_\alpha,\ldots,-e_\rho}(u,w)\,.$$

From (5.2.3) and (5.3.15) we see that the nilpotent orbit asymptotically approximating $\Phi(t,w)$ is

$$\begin{aligned}\theta(t,w) &= \eta(t)\zeta(0,w)\cdot\tilde F_0\\ &= \psi(w)\exp(\textstyle\sum x_jN_j)\nu_w(y)\xi(0,w)\cdot\tilde F_w\\ &= \psi(w)\exp(\textstyle\sum x_jN_j)\,\varepsilon(s,u;w)^{-1}\lambda_1(s,u;w)\varepsilon(s,u;w)\,\xi(0,w)\cdot\tilde F_w\,.\end{aligned}$$

Fix a $G_{\mathbb{R}}$–invariant distance d on D. Keeping (5.3.10) and (5.3.18) in mind, the nilpotent orbit theorem [CKS86, (1.15.iii)] implies

$$d\,(\Phi(t,w)\,,\,\theta(t,w)) \;=\; d\left(\lambda(t,w)\cdot\tilde F_w\,,\,\lambda_1(t,w)\,\mu(s,u;w)\cdot\tilde F_w\right)\;\hat{\longrightarrow}\;0\,.$$

It then follows from Lemma 5.3.17 and (5.3.22) that

$$\lambda(t,w)\cdot\tilde F_w\;\hat{\longrightarrow}\;\exp(\sqrt{-1}\,\mathbf{N}(u,w))\cdot\psi(w)^{-1}\zeta_u(w)\psi(w)\cdot\tilde F_w\,.$$

Therefore

$$\lambda_2(t,w)\cdot\tilde F_w\;\hat{\longrightarrow}\;\psi(w)^{-1}\zeta_u(w)\psi(w)\cdot\tilde F_w\,.$$

Since both $\lambda_2(t,w)$ and $\psi(w)^{-1}\zeta_u(w)\psi(w)$ take value in $\exp(\tilde{\mathfrak{n}}_w)$, it follows from (5.2.2) that

$$\lambda_2(t,w)\;\hat{\longrightarrow}\;\psi(w)^{-1}\zeta_u(w)\psi(w)\,. \tag{5.3.23}$$

The lemma now follows from (5.3.19), (5.3.22) and (5.3.23). □

5.4. **Proof of Theorem 5.1.2.** First we show that the limit on the left-hand side of (5.1.5) exists (Lemma 5.4.4), and then we show that equality holds (5.4.21).

5.4.1. *Step 1: the limit exists.* From (5.3.9) we see that there exists an invertible matrix $A(u) = (A_a^b(u))$, depending real-analytically on u so that $A_a^b(u) = 0$ if $q(a)\neq q(b)$, and $\sum_b A_a^b(u)\mathsf{v}_b$ is an eigenvector of $\mathbf{Y}^\alpha(u,0)$ with eigenvalue $e_{\alpha a}$, for all α. Then, as noted in (5.3.8), $\varepsilon(s,u;0)$ acts on $A_a^b(u)\mathsf{v}_b$ by the eigenvalue

$$e_a(s)\;:=\;\prod_\alpha s_\alpha^{e_{\alpha a}/2}\,.$$

Let $A(u)^{-1} = (B_a^b(u))$ denote the inverse matrix. Then

$$\varepsilon(s,u;0)\,\mathsf{v}_a\;=\;\sum_{b,c}B_a^b(u)e_b(s)A_b^c(u)\,\mathsf{v}_c\,. \tag{5.4.1a}$$

Notice that $E(s,u) = (E_a^c(s,u))$,

$$E_a^c(s,u)\;:=\;\sum_b B_a^b(u)e_b(s)A_b^c(u)\,, \tag{5.4.1b}$$

is the matrix representing $\varepsilon(s,u;0)$ with respect to the basis $\{\mathsf{v}_a\}$, and we have

$$\det E(s,u)\;=\;\prod_c e_c(s)\,. \tag{5.4.2}$$

Set

$$\tilde{\mathsf{v}}_a(w)\;:=\;\psi(w)^{-1}\,\mathsf{v}_a\,. \tag{5.4.3}$$

Then (5.3.4) implies $\varepsilon(s,u;w)$ acts on $A_a^b(u)\tilde{\mathsf{v}}_b(w)$ by the eigenvalue $e_a(s)$, and

$$\varepsilon(s,u;w)\,\tilde{\mathsf{v}}_a(w) \;=\; \sum_c E_a^c(s,u)\tilde{\mathsf{v}}_c(w)\,.$$

Define $h^1(u,w) = (h^1_{ab}(u,w))$ by

$$h^1_{ab}(u,w) \;=\; Q\left(\exp(2\sqrt{-1}\,\mathbf{N}(u))\zeta_u(w)\,\mathsf{v}_a\,,\,\overline{\zeta_u(w)\,\mathsf{v}_b}\right)$$

Lemma 5.4.4. *Suppose that $t \in (\Delta^{*k})^K_c$. Then*

$$P\big(\Upsilon[h](t,w)\big) \;\hat{\longrightarrow}\; P\big(\Upsilon[h^1](u,w)\big)\,.$$

Proof. We have

$$\begin{aligned} v_a(t,w) &= \eta(t)\zeta(t,w)\,\mathsf{v}_a \\ &= \psi(w)\exp(\textstyle\sum x_jN_j)\varepsilon(s,u;w)^{-1}\lambda(t,w)\varepsilon(s,u;w)\,\tilde{\mathsf{v}}_a(w) \\ &= \sum_c E_a^c(s,u)\,\psi(w)\exp(\textstyle\sum x_jN_j)\varepsilon(s,u;w)^{-1}\lambda(t,w)\,\tilde{\mathsf{v}}_c(w)\,. \end{aligned}$$

Define $\tilde{v}_c(t,w) := \psi(w)\exp(\sum x_jN_j)\varepsilon(s,u;w)^{-1}\lambda(t,w)\,\tilde{\mathsf{v}}_c(w)$, so that $v_a(t,w) = E_a^c(s,u)\,\tilde{v}_c(t,w)$. Then

$$\begin{aligned} h_{ab}(t,w) &= (\sqrt{-1})^n\,E_a^c(s,u)\overline{E_b^d(s,u)}\,Q\left(\tilde{v}_c(t,w)\,,\,\overline{\tilde{v}_d(t,w)}\right) \\ &= (\sqrt{-1})^n\,E_a^c(s,u)\overline{E_b^d(s,u)}\,\tilde{h}_{cd}(t,w)\,, \end{aligned} \tag{5.4.5}$$

where

$$\tilde{h}_{cd}(t,w) \;:=\; Q\left(\tilde{v}_c(t,w)\,,\,\overline{\tilde{v}_d(t,w)}\right) \;=\; Q\left(\lambda(t,w)\,\tilde{\mathsf{v}}_c(w)\,,\,\overline{\lambda(t,w)\,\tilde{\mathsf{v}}_d(w)}\right)\,.$$

Then (5.1.6) yields

$$P(\Upsilon[h]) \;=\; P(\Upsilon[\tilde{h}])\,. \tag{5.4.6}$$

From Lemma 5.3.20 and (5.4.3) we see that

$$\tilde{h}(t,w) \;\hat{\longrightarrow}\; h^1(u,w)\,. \tag{5.4.7}$$

Therefore $P(\Upsilon[\tilde{h}])\hat{\longrightarrow}P(\Upsilon[h^1])$. The lemma now follows from (5.4.6). □

5.4.2. *Interlude: the Chern form of the Hodge line bundle.* We pause in the proof of Theorem 5.1.2 to recover a result of Kollár's (Proposition 5.4.12). Consider the case that $p = n$, so that $\mathrm{Gr}^p\mathcal{F} = \mathcal{F}^n$ is the Hodge vector bundle. Then $\det h(t,w)$ is the metric on the Hodge line bundle $\Lambda = \det\mathcal{F}^n \to B$. The asymptotic relationship of the Chern form

$$\Omega \;:=\; \overline{\partial}\partial\log\det h(t,w)$$

to the Poincaré metric (4.2.7) is given by (5.4.10). From (5.4.2), (5.4.5) and (5.4.7) we see that

$$\det h(t,w) \;=\; \sqrt{-1}^m\det\tilde{h}(t,w)\,(\det E(s,u))^2 \;=\; \det\tilde{h}(t,w)\prod_a e_a(s)^2\,,$$

with $m = n \operatorname{rank} F^n$, and

$$\det \tilde{h}(t,w) \stackrel{\hat{}}{\longrightarrow} \det h^1(u,w)\,.$$

So (dropping the $\sqrt{-1}^m$)

$$\begin{aligned}\log\det h(t,w) &= \log\det\tilde{h}(t,w) \;+\; \sum_{\alpha,a} e_{\alpha a}\log s_\alpha \\ &= \log\det\tilde{h}(t,w) \;+\; \sum_{\alpha,a} e_{\alpha a}\log\left(\frac{\log|t_{k_\alpha}|}{\log|t_{k_{\alpha+1}}|}\right)\,,\end{aligned} \tag{5.4.8}$$

and

$$\log\det\tilde{h}(t,w) \stackrel{\hat{}}{\longrightarrow} \log\det h^1(u,w)\,. \tag{5.4.9}$$

Differentiating yields

$$\begin{aligned}\partial_{t_{k_\beta}}\log\det h(t,w) &\stackrel{\hat{}}{\longrightarrow} \sum_a (e_{\beta,a}-e_{\beta-1,a})\,\frac{\mathrm{d}t_{k_\beta}}{t_{k_\beta}\log|t_{k_\beta}|^2} \\ \partial_{\bar{t}_{k_\beta}}\partial_{t_{k_\beta}}\log\det h(t,w) &\stackrel{\hat{}}{\longrightarrow} \sum_a (e_{\beta,a}-e_{\beta-1,a})\,\frac{\mathrm{d}t_{k_\beta}\wedge\mathrm{d}\bar{t}_{k_\beta}}{|t_{k_\beta}|^2\left(\log|t_{k_\beta}|^2\right)^2}\,.\end{aligned} \tag{5.4.10}$$

Remark 5.4.11. We claim that the coefficients $e_{\beta,a} - e_{\beta-1,a}$ are all *nonnegative integers.* The way to see this is to recall that $e_{\beta,a}$ is an eigenvector of $\mathbf{Y}^\beta$. So $e_{\beta,a} - e_{\beta-1,a}$ is an eigenvector of $\mathbf{Y}^\beta - \mathbf{Y}^{\beta-1} = \hat{\mathcal{Y}}_\beta$. The $\mathsf{v}_a \in H_I^{n-q,0}(-q)$ are all highest weight vectors for the SL(2)'s; that is, $\hat{\mathcal{N}}^+_\alpha(u,0)\mathsf{v}_a = 0$. The claim follows from standard SL(2)–representation theory.

Proposition 5.4.12 (Kollár [Kol87])**.** *The integral* $\int_B \Omega^{k+\ell} < \infty$ *is finite.*

Proof. It suffices to prove

$$\int_{\mathcal{U}} \Omega^{k+\ell} < \infty\,. \tag{5.4.13}$$

Fix $c > 1$. As noted in Lemma 5.3.1, a neighborhood of 0 in $\mathcal{U}$ is covered by (a *finite* number of) sets of the form

$$\mathcal{U}_c^K \;:=\; \left\{(t,w)\in(\Delta^*)^K_c\times\Delta^\ell \mid 1/c \le u^j_\alpha \le c\,,\; s_\alpha > c\right\}\,.$$

Consequently, we see that to prove (5.4.13) it suffices to show that

$$\int_{\mathcal{U}_c^K} \Omega^{k+\ell} \;<\; \infty\,. \tag{5.4.14}$$

From (5.4.8) we see that $\Omega = \eta + \tau$, where $\eta = \overline{\partial}\partial\log\det\tilde{h}(t,w)$ and

$$\tau \;:=\; \overline{\partial}\partial\sum_{\alpha,a} e_{\alpha a}\log\left(\frac{\log|t_{k_\alpha}|}{\log|t_{k_{\alpha+1}}|}\right) \;=\; \sum_a (e_{\beta,a}-e_{\beta-1,a})\,\frac{\mathrm{d}t_{k_\beta}\wedge\mathrm{d}\bar{t}_{k_\beta}}{|t_{k_\beta}|^2\left(\log|t_{k_\beta}|^2\right)^2}\,.$$

So to prove (5.4.14), it suffices to show that

$$\int_{\mathcal{U}_c^K} \eta^a \wedge \tau^b \ < \ \infty\,, \tag{5.4.15}$$

for every $0 \le a, b \in \mathbb{Z}$ such that $a + b = k + \ell$. We will prove (5.4.15) by induction on $|K| = \rho$.

Set $\eta_1 := \overline{\partial}\partial \log\det h^1(u,w)$. Then $\int_{\mathcal{U}_c^K} \eta_1^a \wedge \tau^b < \infty$. It follows from (5.4.9) that there exists c' (depending on $c > 1$) so that

$$\int_{\mathcal{V}_{c,c'}^K} \eta^a \wedge \tau^b \ < \ \infty\,, \tag{5.4.16}$$

where

$$\mathcal{V}_{c,c'}^K \ = \ \left\{ (t,w) \in \mathcal{U}_c^K \mid s_\alpha > c'\,,\ \forall\, \alpha \right\} .$$

Notice that

$$\mathcal{U}_c^K \setminus \mathcal{V}_{c,c'}^K \ = \ \left\{ (t,w) \in \mathcal{U}_c^K \mid \exists\, \alpha \text{ s.t. } c < s_\alpha \le c' \right\} .$$

In particular,

$$\mathcal{U}_c^K \setminus \mathcal{V}_{c,c'}^K \ \subset \ \bigcup_{K' \subsetneq K} \mathcal{U}_{c'}^{K'}\,. \tag{5.4.17}$$

If $|K| = 1$, then $s = (s_1, \ldots, s_\rho) = (s_1)$. So that

$$\mathcal{U}_c^K \setminus \mathcal{V}_{c,c'}^K \ = \ \left\{ (t,w) \in \mathcal{U}_c^K \mid c < s_1 \le c' \right\}$$

has compact closure in $\mathcal{U}$. The desired (5.4.15) then follows from (5.4.16).

For $|K| > 1$, the desired (5.4.15) now follows from (5.4.17) by induction. □

As Kollár [Kol87] observes, the proposition yields

Corollary 5.4.18. $\displaystyle\int_B \Omega^{k+\ell} \ = \ (-2\pi\sqrt{-1})^{k+\ell} c_1(\mathcal{F}_{\mathrm{e}}^n)^{k+\ell}.$

Remark 5.4.19. Proposition 5.4.12 and Corollary 5.4.18 also hold in the case that ω is the Chern form $\hat{\Omega}$ of the augmented Hodge line bundle, essentially by the same argument. There is one subtlety here regarding Remark 5.4.11: the eigenvalues $e_{\beta,a} - e_{\beta-1,a}$ appearing in this case need not be non-negative. However, the powers f_p in Definition 1.3.4 ensure that any negative eigenvalues are dominated by positive eigenvalues. More precisely, in the case of the augmented Hodge line bundle, the right-hand side of (5.4.10) becomes

$$\sum_{p=n}^{\lceil (n+1)/2 \rceil} \sum_{\mathsf{v}_a \in I_0^{p,\bullet}} f_p\left(e_{\beta,a} - e_{\beta-1,a}\right) \frac{\mathrm{d}t_{k_\beta} \wedge \mathrm{d}\bar{t}_{k_\beta}}{|t_{k_\beta}|^2 \left(\log|t_{k_\beta}|^2\right)^2}\,.$$

Standard SL(2)–theory ensures that the sum

$$\sum_{p=n}^{\lceil (n+1)/2 \rceil} \sum_{\mathsf{v}_a \in I_0^{p,\bullet}} f_p\left(e_{\beta,a} - e_{\beta-1,a}\right)$$

is a non-negative integer.

Finally we would like to close this section by noting that the analysis above (and the compactness of $\overline{B}$) imply

Lemma 5.4.20. *If $\psi \in \mathcal{A}^{1,1}_{\overline{B}}$ is a smooth $(1,1)$-form on $\overline{B}$, then there exists $\epsilon = \epsilon(\psi) > 0$ so that $\int_C \hat\Omega \geq \epsilon \int_C \psi$ for all curves $C \subset \overline{B}$.*

5.4.3. *Step 2: equality.* In order to prove (5.1.5), and establish Theorem 5.1.2, it remains to show that

$$P\big(\Upsilon[h^1](u,w)\big) \;=\; P\big(\Upsilon[h_I](w)\big)\,. \tag{5.4.21}$$

Proof of (5.4.21). First recall that $\sum y_j N_j$ commutes with $\xi(0,w)$. Consequently, $\mathrm{Ad}_{\varepsilon(s,u;w)} \sum y_j N_j$ commutes with $\mu(s,u;w)$. It follows from Lemma 5.3.17 and (5.3.21) that $\mathbf{N}(u,w)$ and $\psi(w)^{-1}\zeta_u(w)\psi(w)$ commute. Then (5.3.19) implies $\mathbf{N}(u) = \mathbf{N}(u,0)$ and $\zeta_u(w)$ commute. Finally, we note that (5.3.6) implies that $\zeta_u(w)$ preserves the $\tilde I_{0,\ell} = \oplus_{p+q=\ell} \tilde I_0^{p,q}$, for all ℓ. These observations, along with the fact that $\mathbf{N}(u)$ polarizes the MHS $(W, \tilde F_0)$, implies

$$h^1_{ab}(u,w) \;:=\; \begin{cases} (2\sqrt{-1})^q\, Q\left(\zeta_u(w)\mathbf{N}(u)^q\,\mathsf{v}_a\,,\, \overline{\zeta_u(w)\,\mathsf{v}_b}\right)\,, & q = q(a) = q(b)\,; \\ 0\,, & \text{otherwise.} \end{cases}$$

The observation (5.1.6) implies that the Hermitian matrix $h^2(u,w) = (h^2_{ab}(u,w))$ given by

$$h^2_{ab}(u,w) \;:=\; \begin{cases} Q\left(\zeta_u(w)\mathbf{N}(u)^q\,\mathsf{v}_a\,,\, \overline{\zeta_u(w)\,\mathsf{v}_b}\right)\,, & q = q(a) = q(b) \\ 0\,, & q(a) \neq q(b)\,, \end{cases}$$

satisfies

$$P\big(\Upsilon[h^1](u,w)\big) \;=\; P\big(\Upsilon[h^2](u,w)\big)\,. \tag{5.4.22}$$

Each of the cones

$$\sigma \;:=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_1,\ldots,N_k\} \quad\text{and}\quad \hat\sigma_u \;:=\; \mathrm{span}_{\mathbb{R}_{>0}}\{\hat{\mathcal{N}}_1(u,0),\ldots,\hat{\mathcal{N}}_\rho(u,0)\}$$

is contained in an $\mathrm{Ad}(G^{\mathrm{DS}_0}_{\mathbb{R}})$–orbit (cf. (5.3.9)), [Rob16, Corollary 4.9]. Additionally, [CKS86, (4.20.vi)] implies they lie in the same orbit. In particular, there exists $g(u) \in G^{\mathrm{DS}_0}_{\mathbb{R}}$ so that $\mathbf{N}(u) = \mathrm{Ad}_{g(u)} N$. Therefore $\mathbf{N}(u)^q \mathsf{v}_a = g(u) N^q g(u)^{-1}\mathsf{v}_a$. Since $g(u)$ preserves both $\tilde I^{n,q}$ and $N^q(\tilde I^{n,q}) = \tilde I^{n-q,0}$, there exist functions $g(u)^a_b$, $q = q(a) = q(b)$, so that

$$\mathbf{N}(u)^q \mathsf{v}_a \;=\; g(u) N^q g(u)^{-1} \mathsf{v}_a \;=\; g(u)^b_a\, N^q \mathsf{v}_b\,. \tag{5.4.23}$$

So we have

$$h^2_{ab}(u,w) \;:=\; \begin{cases} g(u)^c_a\, h^3_{cb}(u,w)\,, & q = q(a) = q(b)\,, \\ 0\,, & q(a) \neq q(b)\,, \end{cases}$$

where

$$h^3_{cd}(u,w) \;:=\; \begin{cases} Q\left(\zeta_u(w)N^q\,\mathsf{v}_c\,,\,\overline{\zeta_u(w)\,\mathsf{v}_d}\right)\,, & q=q(a)=q(b)\,,\\ 0\,, & q(a)\neq q(b)\,. \end{cases} \tag{5.4.24}$$

So (5.1.6) yields

$$P\big(\Upsilon[h^2](u,w)\big) \;=\; P\big(\Upsilon[h^3](u,w)\big)\,. \tag{5.4.25}$$

Recall that $\zeta_I(w)=\exp(X_I(w))$, with $X_I(w)$ taking value in

$$\mathfrak{z}\;\subset\;\mathfrak{w}\;=\;\bigoplus_{e_1,\ldots,e_\rho\geq 0}\mathfrak{g}_{-e_1,\ldots,-e_\rho}(u,0)\,.$$

(cf. §2.3, (5.2.5), (5.3.13) and (5.3.16)). In fact, (5.3.6) implies that $X_I(w)$ is the component of $X(0,w)\in\mathfrak{z}$ taking value in

$$\mathfrak{z}\;\cap\;\ker\mathbf{Y}^\rho_0\;=\;\mathfrak{z}\;\cap\bigoplus_{e_1,\ldots,e_{\rho-1}\geq 0}\mathfrak{g}_{-e_1,\ldots,-e_{\rho-1},0}(u,0)\,,$$

and that this intersection is independent of u. Consequently, $\zeta_u(w)=\exp X_u(w)$, with $X_u(w)$ the component of $X_I(w)$ taking value in $\mathfrak{g}_{0,\ldots,0}(u,0)$. It follows from Lemma 5.3.14 that

$$\varepsilon(s,u;0)\zeta_I(w)\varepsilon(s,u;0)^{-1}\;=\;\exp\mathrm{Ad}_{\varepsilon(s,u;0)}X_I(w)\;\hat{\longrightarrow}\;\exp X_u(w)\;=\;\zeta_u(w)\,. \tag{5.4.26}$$

Since Q is G–invariant, and $\varepsilon(s,u;0)$ is $G_\mathbb{R}$–valued, we have

$$\begin{aligned} &Q\left(\varepsilon(s,u;0)\zeta_I(w)\varepsilon(s,u;0)^{-1}N^q\,\mathsf{v}_a\,,\,\overline{\varepsilon(s,u;0)\zeta_I(w)\varepsilon(s,u;0)^{-1}\,\mathsf{v}_b}\right)\\ &\qquad\qquad=\;Q\left(\zeta_I(w)\varepsilon(s,u;0)^{-1}N^q\,\mathsf{v}_a\,,\,\overline{\zeta_I(w)\varepsilon(s,u;0)^{-1}\,\mathsf{v}_b}\right)\,. \end{aligned} \tag{5.4.27}$$

Setting

$$h'_{ab}(s,u;w)\;:=\;\begin{cases} Q\left(\zeta_I(w)\varepsilon(s,u;0)^{-1}N^q\,\mathsf{v}_a\,,\,\overline{\zeta_I(w)\varepsilon(s,u;0)^{-1}\,\mathsf{v}_b}\right)\,, & q=q(a)=q(b)\\ 0 & q(a)\neq q(b)\,, \end{cases}$$

(5.1.6), (5.4.24), (5.4.26) and (5.4.27) yield

$$P\big(\Upsilon[h'](s,u;w)\big)\;\hat{\longrightarrow}\;P\big(\Upsilon[h^3](u,w)\big)\,. \tag{5.4.28}$$

On the other hand, by (5.4.1), we have $\varepsilon(s,u;0)^{-1}\,\mathsf{v}_a = E^{-1}(s,u)^b_a\,\mathsf{v}_b$. An analogous argument yields $\varepsilon(s,u;0)^{-1}\,N^q\mathsf{v}_a = D^{-1}(s,u)^b_a\,N^q\mathsf{v}_b$, for some invertible matrix $D(s,u)$. On the other hand, as noted after (5.3.11), $\varepsilon(s,u;0)$ preserves both $\tilde I^{n,q}_0=\mathrm{span}\{\mathsf{v}_a\}_{q=q(a)}$ and $\tilde I^{n-q,0}_0=\mathrm{span}\{N^q\mathsf{v}_a\}_{q=q(a)}$. In particular, there exist invertible matrices $A(s,u)^c_a$ and

$B(s,u)_b^d$ so that $\varepsilon(s,u;0)^{-1}N^q\,\mathsf{v}_c = A(s,u)_a^c\,N^q\,\mathsf{v}_a$ and $\varepsilon(s,u;0)^{-1}\,\mathsf{v}_b = B(s,u)_b^d\,\mathsf{v}_d$. Consequently,

$$\begin{aligned} h'_{ab}(s,u;w) &= \sum_{q=q(c)=q(d)} D^{-1}(s,u)_a^c\, Q\left(\zeta_I(w)N^q\,\mathsf{v}_c\,,\,\overline{\zeta_I(w)\,\mathsf{v}_d}\right)\,\overline{E^{-1}(s,u)_b^d} \\ &= \sum_{q=q(c)=q(d)} D^{-1}(s,u)_a^c\, h''_{cd}(w)\,\overline{E^{-1}(s,u)_b^d}\,, \end{aligned} \tag{5.4.29}$$

where the Hermitian metric $h''(w) = (h''_{ab}(w))$ is defined by

$$h'_{ab}(w) \;:=\; \begin{cases} Q\left(N^q\zeta_I(w)\,\mathsf{v}_a\,,\,\overline{\zeta_I(w)\,\mathsf{v}_b}\right)\,, & \text{if } q = q(a) = q(b)\,, \\ 0 & q(a) \neq q(b)\,. \end{cases}$$

So (5.1.6) yields

$$P(\Upsilon[h_I]) \;=\; P(\Upsilon[h']) \;=\; P(\Upsilon[\tilde h''])\,. \tag{5.4.30}$$

The desired (5.4.21) now follows from (5.4.22), (5.4.25), (5.4.28) and (5.4.30). □

5.5. **Proof of Corollary 1.4.2.** Let $\hat\Omega_{\mathrm{e},\overline{B}}$ denote the Chern form of $\hat\Lambda_\mathrm{e} \to \overline{B}$. It is clear from (1.3.7) that $\hat\Omega_{\mathrm{e},\overline{B}}$ descents to a current $\hat\Omega_{\mathrm{e},\overline{\mathbb{S}}}$ that represents the Chern form of $\hat\Lambda_\mathrm{e} \to \overline{\mathbb{S}}$ (as made precise by Theorem 1.4.1). It remains only to show that $\hat\Omega_{\mathrm{e},\overline{\mathbb{S}}}$ is positive on the Zariski tangent spaces of $\overline{\mathbb{S}}$.

Let $\widetilde C \subset \overline{B}$ be any curve that is transverse to the fibers of $\Phi_\mathrm{e}^\mathrm{c} : \overline{B} \to \overline{\mathbb{S}}$. Then for some index set I the intersection $\widetilde C^* =: \widetilde C \cap Z_I^* \subset Z_W^\star$ will be a Zariski open set in $\widetilde C$, and Theorem 1.4.1 implies that $\hat\Omega_{\mathrm{e},\overline{B}}\Big|_{\widetilde C^*}$ is well-defined. When applied to Φ_I, (1.3.7) yields $\hat\Omega_{\mathrm{e},\overline{B}}(\xi) = \|\Phi^\star_{W,*}(\xi)\|^2$. Whence we have

$$\deg\left(\hat\Lambda_\mathrm{e}\big|_{\widetilde C}\right) \;=\; \int_{\widetilde C} \hat\Omega_{\mathrm{e},\overline{B}} \;>\; 0\,. \tag{5.5.1}$$

(In particular, the integral is defined.)

Now $\Phi_\mathrm{e}^*(\hat\Omega_{\mathrm{e},\overline{\mathbb{S}}}) = \hat\Omega_{\mathrm{e},\overline{B}}$ and (5.5.1) yield

$$\deg\left(\hat\Lambda_\mathrm{e}\big|_{C}\right) \;=\; \int_{C} \hat\Omega_{\mathrm{e},\overline{\mathbb{S}}} \;>\; 0 \tag{5.5.2}$$

for any curve $C \subset \overline{\mathbb{S}}$. Likewise, the discussion of §5.4.2 (cf. especially Remark 5.4.19) implies that

$$\int_{\overline{\mathbb{S}}} \hat\Omega^d_{\mathrm{e},\overline{\mathbb{S}}} \;>\; 0\,, \tag{5.5.3}$$

where $\dim\overline{\mathbb{S}} = d$. The positivity of $\hat\Omega_{\mathrm{e},\overline{\mathbb{S}}}$ now follows from (5.5.2) and (5.5.3).

6. Proof of ampleness of extended augmented Hodge line bundle

In this section we will prove Theorem 1.3.10: *the extended augmented Hodge bundle* $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$ *is ample.* We proceed as follows. After recalling some definitions and results (§6.1), we give a proof of the classical Kodaira embedding theorem (§6.2), that we will then generalize to establish Theorem 1.3.10 (§6.3). The general strategy is outlined at the beginning of §6.3.

6.1. **Preliminaries.** Our working definition of "ample" in §6 is nonstandard:

Definition 6.1.1. Let X be a complex analytic variety of dimension d, possibly singular or non-reduced. A line bundle $L \to X$ is:

- *ample* if for any coherent sheaf $\mathcal{F} \to X$, there exists $m_o(\mathcal{F})$ so that $H^q(X, \mathcal{F} \otimes L^m) = 0$ for all $q > 0$ and $m \geq m_o(\mathcal{F})$;
- *free* (or *semi-ample*) if there exists $0 < m \in \mathbb{Z}$ such that $H^0(X, L^m) \to L^m_x \to 0$ is exact for all $x \in X$;
- *big* if $h^0(X, L^m) = Cm^d + \cdots$ for some $C > 0$;
- *strictly nef* if for any analytic curve $C \subset X$ we have $L \cdot C = \deg\left(L\big|_{C_{\mathrm{red}}}\right) > 0$.

Recall the solution to the Grauert–Riemenschneider conjecture:

Theorem 6.1.2 (Demailly [Dem85], Siu [Siu84, Siu85])**.** *Let $L \to X$ be a line bundle on a compact complex manifold of dimension d. Suppose L admits a smooth Hermitian metric with the property that the Chern form ω is non-negative ($\omega \geq 0$) everywhere, and there exists a point $x \in X$ at which $\omega^d > 0$. Then L is big.*

6.2. **The classical Kodaira Embedding Theorem.** The goal here is to give a proof of the classical Kodaira embedding theorem that we will be able to generalize to our setting. We assume throughout this section that X is a compact complex manifold. (This assumption will be relaxed in §6.3.)

6.2.1. *Relationship between ampleness and embeddings.*

Theorem 6.2.1 (Kodaira embedding)**.** *Let X be a compact complex manifold. A line bundle $L \to X$ is ample* (in the sense of Definition 6.1.1) *if and only if there exists an embedding $f : X \to \mathbb{P}^{\mathsf{N}}$ with $f^*\mathcal{O}_{\mathbb{P}^{\mathsf{N}}}(1) = L^m$.*

Proof. Assume L is ample. Let $\mathcal{I}_x$ be the ideal sheaf of $x \in X$. Then taking $\mathcal{F}$ to be $\mathcal{I}_x^2$ and $\mathcal{I}_x \otimes \mathcal{I}_y$, one shows in the usual way [GH94, pp. 180–181] that the embedding exists.

Conversely, if we have such an embedding, then Serre's FAC [Ser55, n°66] and GAGA imply that L is ample. □

6.2.2. *Free to ample, with positivity.* The point of this section is to prove

Lemma 6.2.2. *Let X be a compact, complex manifold and $L \to X$ is a line bundle equipped with a smooth Hermitian metric such that the corresponding Chern form ω is positive. If L is free, then L is ample.*

Proof of Lemma 6.2.2. Since $L \to X$ is free, we have a regular map

$$f : X \;\to\; \mathbb{P}^{\mathsf{N}} \;=\; \mathbb{P}H^0(X, L^m)^\vee .$$

Let $\mathcal{F}$ be a coherent sheaf on X. The Leray spectral sequence [GH94, §3.5] for $X \to f(X)$ has

$$E_2^{p,q} \;=\; H^q(f(X), R^p_f(\mathcal{F} \otimes L^m)) , \tag{6.2.3}$$

and abuts to $H^{p+q}(X, \mathcal{F} \otimes L^m)$. It follows from Lemma 6.2.4 that $E_2^{p,q} = 0$ for any $p > 0$. Whence $H^{p+q}(X, \mathcal{F} \otimes L^m) = 0$ for any $p+q > 0$. It follows that L is ample (in the sense of Definition 6.1.1). □

Lemma 6.2.4. *Let X be a compact, complex manifold and $L \to X$ a line bundle equipped with a smooth Hermitian metric such that the corresponding Chern form ω is positive. If L is free, then the map $f : X \to \mathbb{P}^{\mathsf{N}} = \mathbb{P}H^0(X, L^m)^\vee$ has no positive dimensional fibres.* (Note that $f^*\mathcal{O}_{\mathbb{P}^{\mathsf{N}}(1)} = L^m$.)

Proof. Suppose that $Y \subset X$ were a fibre of dimension $k > 0$. Then $\int_Y \omega^k = c_1(L)^k[Y] = 0$, contradicting the positivity of ω. Therefore f is a finite-to-one map. □

6.3. **Generalizing Kodaira to $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$.** Theorem 1.3.10 is a consequence of those properties of the Chern form $\hat{\Omega}_{\mathrm{e}}$ of $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$ established in Theorems 1.4.1 and 5.1.3, Corollary 1.4.2 and §5.5. Following the argument of §6.2, we will first show that $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$ is free (§6.3.1–6.3.3), and then deduce from this that $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$ is ample (§6.3.4). The arguments are by induction on the dimension of $\overline{\mathcal{S}}$. We first assume that $\overline{\mathcal{S}}$ is smooth (§6.3.1); then relax to singular, but reduced (§6.3.2); and finally treat the case that $\overline{\mathcal{S}}$ is singular and non-reduced (§6.3.3). To simplify notation we write $L \to X$ and Ω in place of $\hat{\Lambda}_{\mathrm{e}} \to \overline{\mathcal{S}}$ and $\hat{\Omega}_{\mathrm{e}}$. Keep in mind throughout that Ω is positive (Corollary 1.4.2).

6.3.1. *Proof of Theorem 1.3.10: positive to free, assuming smoothness.* Assume for the moment that X is smooth. The Hirzebruch–Riemann–Roch Theorem implies $\chi(X, L^k) = Ck^d + \cdots$, with $C > 0$. Corollary 1.4.2 implies that the domain $X(L, h, q)$ of [Dem12, (8.1)] is empty when $q > 0$; then Demailly's holomorphic Morse inequalities give $h^q(X, L^k) = o(k^d)$ for $q > 0$ [Dem12, (8.2.a)]. Therefore,

$$h^0(X, L^k) \;\sim\; Ck^d \,+ \cdots$$

with $C > 0$. It follows that $L^k \to X$ has a section s when $k \gg 0$. Let $Y = (s) \in |L^k|$ be the associated divisor.

Suppose that Y is smooth. By induction on dimension we have that $L^k \to Y$ is ample. Consider the long exact sequence in cohomology determined by the short exact sequence

$$0 \;\to\; \mathcal{F} \otimes L^{km} \;\xrightarrow{\;s\;}\; \mathcal{F} \otimes L^{k(m+1)} \;\longrightarrow\; \mathcal{F} \otimes L^{k(m+1)}\Big|_Y \;\to\; 0 .$$

It follows from $h^q(Y, \mathcal{F} \otimes L^{k(m+1)}) = 0$ for $q > 0$ and $m \gg 0$, that

$$H^1(X, \mathcal{F} \otimes L^{km}) \;\to\; H^1(X, \mathcal{F} \otimes L^{k(m+1)}) \tag{6.3.1}$$

is a surjection when $m \gg 0$. Therefore the $h^1(X, \mathcal{F} \otimes L^{km})$ are non-increasing in m when $m \gg 0$. So the maps (6.3.1) must be isomorphisms when $m \gg 0$. It follows that $H^0(X, \mathcal{F} \otimes L^{k(m+1)}) \to H^0(Y, \mathcal{F} \otimes L^{k(m+1)})$ is surjective. Since $s(x) \neq 0$ for $x \in X \backslash Y$, we may conclude that $L^k \to X$ is free.

It may happen that there exists no smooth and/or reduced $Y \in |L^k|$. So in order to push this inductive argument through, we have to allow X to be (i) singular, and (ii) non-reduced. These two arguments are found in §6.3.2 and §6.3.3, respectively.

6.3.2. *Proof of Theorem 1.3.10: positive to free, allowing singular.* Let us first consider the case (i) that X is singular.

Lemma 6.3.2. *The line bundle $L \to X$ is big.*

It follows from Lemma 6.3.2 that $h^0(X, L^m) \neq 0$ for $m \gg 0$. This gives the divisor Y, and if Y is reduced, then we may proceed by induction (as in §6.3.1) to conclude that $L \to X$ is free.

Proof. We wish to show that $H^0(X, L^m) \neq 0$ for some m by applying Theorem 6.1.2. This theorem requires a smooth base, so consider a desingularization $\pi : \widetilde{X} \to X$. Set $\tilde{L} = \pi^*(L)$ and $\widetilde{\Omega} = \pi^*(\Omega)$. Then $\widetilde{\Omega} \geq 0$, and $\widetilde{\Omega}^d > 0$ on an open subset of $\tilde{X}$. Theorem 6.1.2 implies

$$\tilde{L} \quad \text{is big.} \tag{6.3.3a}$$

Lemma 6.3.6 and Demailly's [Dem12, (8.4)] give

$$h^q(\widetilde{X}, \tilde{L}^m) \ = \ o(m^d) \quad \text{when} \quad q > 0\,. \tag{6.3.3b}$$

The Leray spectral sequence [GH94, §3.5] yields

$$H^0(\widetilde{X}, \tilde{L}^m) \ \simeq \ H^0(X, \pi_* \tilde{L}^m) \ = \ H^0(X, L^m \otimes \pi_* \mathcal{O}_{\widetilde{X}})\,. \tag{6.3.3c}$$

We have a short exact sequence

$$0 \ \to \ \mathcal{O}_X \ \to \ \pi_* \mathcal{O}_{\widetilde{X}} \ \to \ \mathcal{Q} \ \to \ 0 \tag{6.3.4}$$

with $\mathcal{Q}$ supported on a subvariety of dimension $\leq d-1$. So $\mathcal{O}_X \simeq \pi_* \mathcal{O}_{\widetilde{X}}$ away from a proper subvariety; therefore,

$$h^0(X, L^m \otimes \mathcal{Q}) \ = \ c m^{d-1} + \cdots \tag{6.3.5a}$$

Tensor (6.3.4) with L^m, and consider the resulting long exact sequence in cohomology

$$0 \ \to \ H^0(X, L^m) \ \to \ H^0(X, \pi_* \tilde{L}^m) \ \to \ H^0(X, L^m \otimes \mathcal{Q}) \ \to \ H^1(X, L^m) \ \to \cdots \tag{6.3.5b}$$

It now follows from (6.3.3) and (6.3.5) that L is big. □

6.3.3. *Proof of Theorem 1.3.10: positive to free, allowing singular and non-reduced.* Now consider the case (ii) that X is singular and non-reduced. It follows from §6.3.2 that $L_{\mathrm{red}} \to X_{\mathrm{red}}$ is free. It remains to deduce from this that $L \to X$ is free.

Lemma 6.3.6. *The line bundle $L \to X$ is strictly nef.*

Proof. Let $C \subset X$ be an analytic curve (as in Definition 6.1.1). Set $\widetilde{C} = \pi^{-1}(C) \subset \widetilde{X}$, so that

$$c_1(L)[C_{\mathrm{red}}] \;=\; c_1(\widetilde{L})[\widetilde{C}_{\mathrm{red}}]\,.$$

It is a consequence of Corollary 1.4.2 (and the computations of §5.5) that $\widetilde{\Omega} = \pi^*(\Omega)$ is a non-negative $(1,1)$-form on $\widetilde{X}$ with the property that for $\xi \in T\widetilde{X}$,

$$\widetilde{\Omega}(\xi) \;=\; 0 \quad\iff\quad \pi_*(\xi) \;=\; 0\,.$$

It follows that

$$c_1(\widetilde{L})[\widetilde{C}_{\mathrm{red}}] \;=\; \int_{\widetilde{C}_{\mathrm{red}}} \widetilde{\Omega} \;>\; 0\,.$$

□

Remark 6.3.7. The proof of Lemma 6.3.2 applies here, so that the line bundle $L \to X$ is big.

Lemma 6.3.8. *The line bundle $L \to X$ is free.*

Proof. We proceed as in §6.3.2. Replacing L by a high power, Remark 6.3.7 implies that there exists a possibly non-reduced effective divisor $Y \in |L|$. By the inductive assumption, $L \to Y$ is ample. From the cohomology sequence of

$$0 \;\to\; L_X^{m-1} \;\to\; L_X^m \;\to\; L_Y^m \;\to\; 0$$

and $h^1(L_Y^m) =$ for $m \gg 0$, we obtain

$$H^1(L_X^{m-1}) \;\to\; H^1(L_X^m) \;\to\; 0, \qquad m \geq m_0\,.$$

Thus the $h^1(L^m)$ are non-increasing for $m \geq m_0$, and for $m \geq m_1$ we will have

$$H^1(L_X^{m-1}) \;\xrightarrow{\simeq}\; H^1(L_X^m)\,.$$

This gives

$$H^0(L_X^m) \;\to\; H^0(L_Y^m) \;\to\; 0\,.$$

Then, since $L_Y \to Y$ is free, the same will be true for $L_X \to X$. □

6.3.4. *Proof of Theorem 1.3.10: free to ample.* In the case that X is reduced the linear systems $|mL|$ for $m \gg 0$ give holomorphic (not just meromorphic) maps

$$\varphi_m : X \to \mathbb{P}^{\mathsf{N}_m}$$

with

$$\varphi_m^* \mathcal{O}_{\mathbb{P}^{\mathsf{N}_m}}(1) = L^m.$$

By Lemma 6.3.6 no positive-dimensional subvariety of X is contracted by φ_m, so that φ_m is a finite map and this gives the result in this case.

To give the argument when X may not be reduced we proceed by induction on $\dim X$. If $\dim X = 1$ the result follows from

$$\deg \left(L\big|_{X_{\alpha,\mathrm{red}}} \right) > 0$$

where X_α are the irreducible components of X.

Now suppose that $\dim X = d$ is arbitrary. Assume, as we may, that X is irreducible. The exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \to \mathcal{O}_{X_{\mathrm{red}}} \to 0 \tag{6.3.9}$$

defines the sheaf $\mathcal{F}$. This sheaf admits a filtration whose associated graded sheaves $\mathrm{Gr}^\bullet \mathcal{F}$ are $\mathcal{O}_{X_{\mathrm{red}}}$–modules. Tensoring (6.3.9) with L^m and using the result $h^1(\mathcal{O}_{X_{\mathrm{red}}}, \mathrm{Gr}^\bullet \mathcal{F} \otimes L^m) = 0$ for $m \gg 0$ in the reduced case leads, by the usual spectral sequence argument [GH94, §3.5], to $h^1(X, \mathcal{F} \otimes L^m) = 0$ for $m \gg 0$.

Appendix A. The Siegel property

It is a consequence of the Schmid's nilpotent orbit theorems that the image of a local lift of a *one-variable* period map at infinity is contained in a Siegel domain (§A.1). This is an important property. For example, it is a key ingredient in Sommese's proof that the image of a (non-constant) *one-variable* period map is algebraic [Som73]. The obstacle to extending Sommese's result to *several-variable* period maps the is the absence of an analogous "Siegel property" in this setting. Recently Bakker–Klingler–Tsimerman proved that the local lift of a *several-variable* period map at infinity is contained in a *finite* union Siegel domains [BKT18], and this property plays a key role in their proof that period maps are $\mathbb{R}_{\mathrm{an,exp}}$–definable. The purpose of this appendix is to observe that Bakker–Klingler–Tsimerman's Siegel property is the best that one can hope for in general: there exist nilpotent orbits that are not contained (asymptotically) in any *single* Siegel domain (Claims A.5.8 and A.5.9. Of course, these nilpotent orbits necessarily depend nontrivially on more than one variable).

A.1. **The one–variable case: Schmid's work.** Schmid [Sch73, (5.29)] proved the following:

Theorem A.1.1 (Schmid)**.** *Let* $\tilde{\Phi} : \mathcal{H} \to D$ *be a lift of a one-variable period map* $\Phi : \Delta^* \to \Gamma\backslash D$*. Given* $c_1, c_2 > 0$*, there exists a Siegel domain* $\mathfrak{D} \subset D$ *so that* $\tilde{\Phi}(z) \in \mathfrak{D}$ *when* $|\mathrm{Re}\, z| < c_1$ *and* $\mathrm{Im}\, z > c_2$*.*

This result is a consequence of the Nilpotent Orbit Theorem and Schmid's [Sch73, (5.25)]. The latter basically asserts

Proposition A.1.2 (Schmid)**.** *Given a one-variable nilpotent orbit $\theta(z) := \theta(zN)\cdot F$, the point $\theta(\sqrt{-1}y)$ lies in a Siegel domain when $y \gg 0$.*

So to generalize Theorem A.1.1 to the several-variable case, it appears that it would suffice to generalize this result of nilpotent orbits to the several-variable case. Unfortunately, the generalization does not hold.

A.2. **Feasibility of generalizing Schmid.** The definition (§A.3) of a Siegel domain $\mathfrak{D}^{P,K}$ depends on a choice of parabolic subgroup $P \subset G$ and maximal compact subgroup $K \subset G$. Schmid's statements in [Sch73, §5] are for the case that P is a minimal (rational) parabolic subgroup P_0. These do not generalize:

> *There exist several–variable nilpotent orbits $\theta(z_1,\ldots,z_s)$ on the period domain D for Hodge numbers $\mathbf{h} = (2,2)$ for which there exist* **no** *(P_0,K)–Siegel domain so that $\theta(\sqrt{-1}y_1,\ldots,\sqrt{-1}y_s) \in \mathfrak{D}^{P_0,K}$ when $y_j \gg 0$* (Claim A.5.8).

This is the sense in which Schmid's Proposition A.1.2 does *not* generalize. However, there is another parabolic that, unlike the minimal parabolic, is canonically associated with the nilpotent orbit: if P is the (in general, non-minimal) parabolic stabilizing the weight filtration of θ, then one might imagine that the answer to the following question is "yes".

> *Given any several–variable nilpotent orbit $\theta(z_1,\ldots,z_s)$ on the period domain D for Hodge numbers $\mathbf{h} = (2,2)$, are the points $\theta(z_1,\ldots,z_s)$ contained in a (P,K)–Siegel domain when $|\mathrm{Re}\, z_j|$ is bounded and $\mathrm{Im}\, z_j \gg 0$?*

An affirmative answer to this questions seems to us to be the most natural and plausible generalization of Schmid's one-variable result. If it fails, then it seems *extremely unlikely* that any generalization of Schmid's one-variable result will hold (i.e., that it might hold for another choice of parabolic). Unfortunately, we will see that the answer is indeed "no" (Claims A.5.8 and A.5.9).

A.3. **Siegel domains.** Let D be any period domain with automorphism group G. The definition of a Siegel domain depends on a choice of parabolic subgroup $P \subset G$ and maximal compact subgroup K. Briefly: Let $P = UAM \simeq U \times A \times M$ be the *Langlands decomposition* of P; here

- U is the unipotent radical of P,
- $A = \exp(\mathfrak{a})$, with $\mathfrak{a} \subset \mathfrak{p}$ a diagonalizable (abelian) subalgebra,
- $Z(A) = A \times M$ is a Levi subgroup,
- both A and M are invariant under the Cartan involution θ_K determined by K.

Together $G = PK$ and the Langlands decomposition yield the *horospherical decomposition*

$$D \;=\; U \times A \times (MZ)$$

where $Z = K \cdot \varphi$, with $\varphi \in D$ satisfying $\mathrm{Stab}_G(\varphi) \subset K$.

Let $\Sigma(\mathfrak{u}, \mathfrak{a}) \subset \mathfrak{a}^*$ be a choice of simple roots for the action of $\mathfrak{a}$ on $\mathfrak{u}$. We may identify $\Sigma(\mathfrak{u}, \mathfrak{a})$ with characters $\Sigma(U, A)$ on A as follows: given $a = \exp(\xi) \in A$, define $a^\alpha = e^{\alpha(\xi)}$. Given $t > 0$, define

$$A_t \ := \ \{a \in A \mid a^\alpha > t\,,\ \forall\, \alpha \in \Sigma\}\,.$$

A *Siegel domain* is any set of the form

$$\mathfrak{D}^{P,K}_{\omega,t,\mu} \ := \ \omega A_t \mu \cdot Z \ \subset \ D\,,$$

where $\omega \subset U$ and $\mu \subset M$ are open pre-compact sets. Note that $\omega A_t \mu \subset UAM = P$ is open, so that $\mathfrak{D}^{P,K}_{\omega,t,\mu}$ is open in D.

A.4. **Nilpotent orbits.** Fix a several variable nilpotent orbit

$$\theta(z) \ = \ \exp(\textstyle\sum z_j N_j) \cdot F\,;$$

here $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$. Let (W, F) be the associated MHS. Let $(W, \tilde{F} = e^{\sqrt{-1}\delta} F)$ be Deligne's associated $\mathbb{R}$–split MHS, with Deligne splittings

$$V_{\mathbb{C}} \ = \ \oplus\, I^{p,q} \quad \text{and} \quad \mathfrak{g}_{\mathbb{C}} \ = \ \oplus\, \mathfrak{g}^{p,q}\,,$$

and $\mathbb{R}$–split nilpotent orbit

$$\tilde{\theta}(z) \ := \ \exp(\textstyle\sum z_j N_j) \cdot \tilde{F}\,.$$

Set

$$N \ := \ \textstyle\sum N_j\,,$$

and

$$\varphi \ := \ \exp(\sqrt{-1}N) \cdot \tilde{F} \ \in \ D\,.$$

Let $K \subset G$ be the maximal compact subgroup containing the stabilizer of φ. Let $P \subset G$ be the parabolic subgroup stabilizing the filtration W. If the $N_a \in \mathfrak{g}_{\mathbb{Q}}$, then W and P are defined over $\mathbb{Q}$. Assume this is the case.

Set

$$G^{0,0} \ = \ \{g \in G \mid g(I^{p,q}) = I^{p,q}\,,\ \forall\, p, q\}\,.$$

Let

$$\mathcal{N} \ := \ \mathrm{Ad}(G^{0,0}) \cdot N \ \subset \ \mathfrak{g}^{-1,-1}_{\mathbb{R}}$$

be the $G^{0,0}$–orbit of N. Then

Lemma A.4.1 ([BPR16])**.** *The nilpotent cone σ is contained in the orbit $\mathcal{N}$. Conversely, any nilpotent cone $\tau = \mathrm{span}_{\mathbb{R}_{>0}}\{M_1, \ldots, M_r\} \subset \mathcal{N}$ generated by commuting nilpotents M_j underlies a nilpotent orbit through $\tilde{F}$.*

A.5. **The counter-example: $\mathbf{h} = (2,2)$.** There are two types of MHS on the period domain D for $\mathbf{h} = (2,2)$; that is, there are two types of Hodge diamonds. One is Hodge–Tate. For the other, the polarizing nilpotent cones are necessarily one dimensional, so that the associated nilpotent orbits are one-variable. Here Schmid's Proposition A.1.2 applies: "eventually" the nilpotent orbit lies in a Siegel domain. We will show that the several-variable nilpotent orbits of Hodge–Tate type *fail* to "eventually" lie in Siegel domains (Claims A.5.8 and A.5.9).

A.5.1. *Set-up.* Suppose

$$V \;=\; \mathbb{Q}^4 \;=\; \mathrm{span}_{\mathbb{Q}}\{e_1, \ldots, e_4\}\,.$$

Define a skew–symmetric bilinear form Q on V by $Q(u,v) = u^t\mathbf{q}v$, where

$$\mathbf{q} \;:=\; \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let D be the period domain parameterizing Q–polarized Hodge structures on D with Hodge numbers $\mathbf{h} = (2,2)$. Fix point $\varphi \in D$ by

$$H^{1,0}_\varphi \;:=\; \mathrm{span}_{\mathbb{C}}\{\zeta_1 = e_1 + \sqrt{-1}e_4\,,\, \zeta_2 = e_2 + \sqrt{-1}e_3\}\,.$$

Note that the maximal compact subgroup of G containing the stabilizer of φ is the group K of §A.5.1. Then

$$G \;=\; \mathrm{Aut}(\mathbb{R}^4, Q) \;=\; \mathrm{Sp}(4,\mathbb{R}) \quad\text{and}\quad K \;=\; \mathrm{Stab}_G\varphi \;=\; \mathrm{U}(2)\,.$$

The Lie algebras are

$$\mathfrak{g} \;=\; \left\{ \left(\begin{array}{cc|cc} a & b & s & t\\ c & d & u & s\\ \hline x & y & -d & -b\\ z & x & -c & -a \end{array}\right) \;\middle|\; a,b,c,\ldots,z \in \mathbb{R} \right\}.$$

and

$$\begin{aligned} \mathfrak{k} \;&=\; \mathfrak{g} \cap \mathfrak{so}(4) \;=\; \{X \in \mathfrak{g} \mid X + X^t = 0\}\\ &=\; \left\{ \left(\begin{array}{cc|cc} 0 & b & s & t\\ -b & 0 & u & s\\ \hline -s & -u & 0 & -b\\ -t & -s & b & 0 \end{array}\right) \;\middle|\; b,s,t,u \in \mathbb{R} \right\}. \end{aligned}$$

A.5.2. *A minimal parabolic.* Let $\mathfrak{p}_0 \subset \mathfrak{g}$ be the minimal parabolic subalgebra of upper–triangular matrices. Then $\mathfrak{p}_0 = \mathfrak{a}_0 \oplus \mathfrak{u}_0$, where

$$\mathfrak{a}_0 \ = \ \left\{ \mathrm{diag}(a,d,-d,-a) \ := \left(\begin{array}{cc|cc} a & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ \hline 0 & 0 & -d & 0 \\ 0 & 0 & 0 & -a \end{array} \right) \ \middle| \ a,d \in \mathbb{R} \right\}$$

is the maximal abelian subalgebra of diagonal matrices, and

$$\mathfrak{u}_0 \ = \ \left\{ \left(\begin{array}{cc|cc} 0 & b & s & t \\ 0 & 0 & u & s \\ \hline 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \end{array} \right) \ \middle| \ b,s,t,u \in \mathbb{R} \right\}$$

are the strictly upper-triangular matrices. Note that the real rank of $\mathfrak{g}$ is $2 = \dim \mathfrak{a}_0$.

Remark A.5.1 (Siegel domains). In this example M_0 is the set of diagonal matrices of the form $\mathrm{diag}(\epsilon_1, \epsilon_2, -\epsilon_2, -\epsilon_1)$, with $\epsilon_j = \pm 1$. These matrices all act trivially on φ. So $U_0 A_0 M_0 \cdot \varphi = U_0 A_0 \cdot \varphi$. Therefore, the (P_0, K)–Siegel domains in D are of the form

$$\mathfrak{D}^{P_0,K}_{\omega,t} \ = \ \omega A_{0,t} \cdot \varphi \,,$$

with $\omega \subset \exp(\mathfrak{u}_0) = U_0$ an open pre-compact set, and

$$\begin{aligned} A_{0,t} \ &:= \ \{ \mathrm{diag}(e^a, e^d, e^{-d}, e^{-a}) \mid a,d \in \mathbb{R} \,, \ a-d, 2d > \ln t \} \\ &\subset \ A_0 \ := \ \exp(\mathfrak{a}_0) \,, \end{aligned} \tag{A.5.2}$$

with $t > 0$. Here we are using the simple roots $\{\varepsilon_1 - \varepsilon_2, 2\varepsilon_2\} = \Sigma(\mathfrak{a}_0, \mathfrak{u}_0)$ where $\{\varepsilon_1, \varepsilon_2\}$ is the basis of $\mathfrak{a}_0^*$ dual to $\{e_1^1 - e_4^4, e_2^2 - e_3^3\} \subset \mathfrak{a}_0$.

A.5.3. *A maximal parabolic.* Let $\mathfrak{p} \subset \mathfrak{g}$ be the maximal parabolic subalgebra stabilizing $\mathrm{span}_{\mathbb{R}}\{e_1, e_2\}$. Then $\mathfrak{p} = \mathfrak{u} \oplus \mathfrak{a} \oplus \mathfrak{m}$, where

$$\mathfrak{a} \ = \ \mathrm{span}_{\mathbb{R}}\{ Y = \mathrm{diag}(-1,-1,1,1) \} \,,$$

$$\mathfrak{m} \ = \ \left\{ \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ \hline 0 & 0 & a & -b \\ 0 & 0 & -c & -a \end{array} \right) \ \middle| \ a,b,c \in \mathbb{R} \right\} \ \simeq \ \mathfrak{sl}(2,\mathbb{R}) \,.$$

and

$$\mathfrak{u} \ = \ \left\{ \left(\begin{array}{cc|cc} 0 & 0 & s & t \\ 0 & 0 & u & s \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \ \middle| \ s,t,u \in \mathbb{R} \right\} .$$

Remark A.5.3 (Siegel domains). In this example $M = \mathrm{SL}(2,\mathbb{R})$. We have $\Sigma(\mathfrak{a},\mathfrak{u}) = \{\varepsilon_1 + \varepsilon_2 = 2\varepsilon_2\}$, so that

$$A_t \;=\; \{\mathrm{diag}(e^a, e^a, e^{-a}, e^{-a}) \mid 2a > \ln t\}\,.$$

A.5.4. *Commuting* SL(2)*'s.* Let $\{e^1, \ldots, e^4\}$ denote the dual basis of V^*, and set

$$e_i^j \;:=\; e_i \otimes e^j\,.$$

The following two commuting standard triples $\{\hat N_j^+, Y_j, \hat N_j\} \subset \mathfrak{g}$

$$\begin{array}{rclrclrcl} \hat N_1 & = & -e_1^4\,, & Y_1 & = & e_4^4 - e_1^1\,, & \hat N_1^+ & = & e_4^1\,,\\ \hat N_2 & = & -e_2^3\,, & Y_2 & = & e_3^3 - e_2^2\,, & \hat N_2^+ & = & e_3^2\,. \end{array}$$

determine horizontal SL(2)'s at the point $\varphi \in D$ given by

$$H^{2,0}_\varphi \;=\; \mathrm{span}_{\mathbb{C}}\{e_3 - \sqrt{-1}e_2\,,\, e_4 - \sqrt{-1}e_1\}\,.$$

The filtration

$$\hat W \;:=\; W(\hat N)[-1]$$

is independent of our choice of $\hat N$ in the cone

$$\hat\sigma \;:=\; \mathrm{span}_{\mathbb{R}_{>0}}\{\hat N_1, \hat N_2\}\,.$$

To be explicit

$$\hat W_0 \;=\; \hat W_1 \;=\; \mathrm{span}_{\mathbb{R}}\{e_1, e_2\} \quad\text{and}\quad \hat W_2 \;=\; V_{\mathbb{R}}\,.$$

Set

$$\hat F \;:=\; \mathrm{span}_{\mathbb{C}}\{e_3, e_4\} \;\in\; \check D\,.$$

Then $(\hat W, \hat F)$ is a MHS polarized by $\hat\sigma$.[17] Modulo the action of G we may assume that

$$(W, \tilde F) \;=\; (\hat W, \hat F)\,,$$

and $\hat\sigma \subset \mathcal{N}$. We make this assumption.

A.5.5. *Details for* $G^{0,0}$ *and* $\mathcal{N}$*.* Any element $g \in G^{0,0}$ is of the form

$$g \;=\; \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix}, \tag{A.5.4a}$$

with

$$B \;=\; \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \quad\text{and}\quad B' \;=\; \frac{1}{\det P}\begin{pmatrix} b_1 & -b_2 \\ -b_3 & b_4 \end{pmatrix} \tag{A.5.4b}$$

[17] If we define $F_j \in \check D \;\subset\; \mathrm{Gr}(2, \mathbb{C}^4)$ by

$$\begin{array}{rcl} F_1 & = & \mathrm{span}_{\mathbb{C}}\{e_4\,,\; e_2 + \sqrt{-1}e_3\}\,,\\ F_2 & = & \mathrm{span}_{\mathbb{C}}\{e_3\,,\; e_1 + \sqrt{-1}e_4\}\,, \end{array}$$

then the $(W(N_j)[-1], F_j)$ are $\mathbb{R}$–split PMHS.

invertible 2×2 matrices. We have

$$\mathrm{Ad}_g(\lambda_1 \hat{N}_1 + \lambda_2 \hat{N}_2) \;=\; \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix} ,$$

with

$$\eta \;=\; B \begin{pmatrix} 0 & -\lambda^1 \\ -\lambda^2 & 0 \end{pmatrix} (B')^{-1} \;=\; -\lambda^1 \begin{pmatrix} b_1 b_3 & b_1 b_1 \\ b_3 b_3 & b_3 b_1 \end{pmatrix} - \lambda^2 \begin{pmatrix} b_2 b_4 & b_2 b_2 \\ b_4 b_4 & b_4 b_2 \end{pmatrix} .$$

Remark A.5.5. We have

$$\mathfrak{g}_{\mathbb{R}}^{-1,-1} \;=\; \mathrm{span}_{\mathbb{R}}\{e_1^4 ,\ e_2^3 ,\ e_1^3 + e_2^4\} .$$

The boundary of $\mathcal{N}$ is the $G^{0,0}$ orbit of $\hat{N}_1 = -e_1^4$. Moreover, the orbit is convex. So our given nilpotent cone $\sigma \subset \mathcal{N}$, is contained in a nilpotent cone

$$\sigma' \;=\; \mathrm{span}_{\mathbb{R}_{>0}}\{N_1', \ldots, N_s'\} \subset \mathcal{N}$$

with $N_j' \in \partial\mathcal{N}$. Without loss of generality, we will assume that $\sigma = \sigma'$ and $N_j = N_j'$. In particular,

$$N_j \;=\; -r_j(e_1^3 + e_2^4) - p_j e_2^3 - q_j e_1^4 ,$$

with

(A.5.6a) $$p_j, q_j \;\geq\; 0 \quad \text{and} \quad r_j^2 \;=\; p_j q_j .$$

Moreover, we may assume that $\hat{N}_1 + \hat{N}_2 = -e_1^4 - e_2^3 \in \sigma$. In fact, we assume that

$$\hat{N}_1 + \hat{N}_2 \;=\; \sum N_j ;$$

equivalently,

(A.5.6b) $$\sum r_j = 0 \quad \text{and} \quad \sum p_j = 1 = \sum q_j .$$

A.5.6. *Nilpotent orbits and Siegel sets.* It will be convenient to introduce the following notation. Define

$$\|y\|^2 \;:=\; \sum y_j^2 \quad \text{and} \quad \lambda_j \;:=\; \frac{y_j}{\|y\|} ,$$

so that

(A.5.7) $$y_j \;=\; \|y\| \lambda_j .$$

Set

$$\lambda \;=\; (\lambda_1, \ldots, \lambda_s) ,$$

and

$$N_\lambda \;:=\; \sum \lambda_j N_j ,$$

so that $\tilde{\theta}(\sqrt{-1} y_1, \ldots, \sqrt{-1} y_s) = \exp(\sqrt{-1}\|y\| N_\lambda) \cdot \tilde{F}$. For fixed λ, Schmid's Proposition A.1.2 asserts that $\exp(\sqrt{-1}\|y\| N_\lambda)$ lies in a (P_0, K)–Siegel domain if $\|y\| \gg 0$. (From here it is straightforward to show that $\exp(\zeta N_\lambda)$ lies in a (P_0, K)–Siegel domain if $\mathrm{Re}\,\zeta$ is bounded and $\mathrm{Im}\,\zeta \gg 0$.) Claim A.5.8 asserts that this fails (when $s > 1$) if we allow λ to vary.

Likewise, Claim A.5.9 asserts that $\tilde\theta(z_1,\ldots,z_s)$ fails to lie in a (P,K)–Siegel domain when the $\mathrm{Re}\, z_j$ are bounded and $\mathrm{Im}\, z_j \gg 0$ (Claim A.5.9).

Claim A.5.8. *If $s>1$, then there exist no bounds $c_1,c_2>0$ so that $\tilde\theta(z_1,\ldots,z_s)$ is contained a (P_0,K)–Siegel domain when $\mathrm{Re}\, z_j$ bounded and $\mathrm{Im}\, z_j \gg 0$.*

Claim A.5.9. *If $s>1$, then there exist no bounds $c_1,c_2>0$ so that $\tilde\theta(z_1,\ldots,z_s)$ is contained a (P,K)–Siegel domain when $\mathrm{Re}\, z_j$ bounded and $\mathrm{Im}\, z_j \gg 0$.*

Proof of Claim A.5.8. We will show that, for all $T>0$, there exists no (P_0,K)–Siegel domain containing $\tilde\theta(\sqrt{-1}y_1,\ldots,\sqrt{-1}y_s)$ for all $y_j>T$. On the one hand we have

$$\begin{aligned}\tilde\theta(\sqrt{-1}y_1,\ldots,\sqrt{-1}y_s) &= \exp\sqrt{-1}(y_1N_1+\cdots+y_sN_s)\cdot\tilde F\\ &= \mathrm{span}_{\mathbb{C}}\{e_3-\sqrt{-1}(\textstyle\sum r_jy_j)e_1-\sqrt{-1}(\sum p_jy_j)e_2\,,\\ &\qquad e_4-\sqrt{-1}(\textstyle\sum q_jy_j)e_1-\sqrt{-1}(\sum r_jy_j)e_2\}\,.\end{aligned}\tag{A.5.10}$$

Set

$$r(y)\ :=\ \textstyle\sum r_jy_j\,,\quad p(y)\ :=\ \sum p_jy_j\quad\text{and}\quad q(y)\ :=\ \sum q_jy_j\,.$$

On the other hand, elements of $U_0=\exp(\mathfrak{u}_0)$ are of the form

$$\nu\ =\ \begin{pmatrix}1&\beta&u_1&u_3\\0&1&u_0&u_2\\0&0&1&-\beta\\0&0&0&1\end{pmatrix}$$

with $u_1-u_2=\beta u_0$. Elements of A_0 are of the form $\gamma=\mathrm{diag}(e^a,e^d,e^{-d},e^{-a})$ with $a,d\in\mathbb{R}$. So elements of $U_0A_0\cdot\varphi$ are of the form

$$\begin{aligned}\nu\gamma\cdot H^{1,0}_\varphi &= \nu\cdot\mathrm{span}_{\mathbb{C}}\{e_3-\sqrt{-1}e^{2d}e_2\,,\ e_4-\sqrt{-1}e^{2a}e_1\}\\ &= \mathrm{span}_{\mathbb{C}}\{e_3+(u_1-\sqrt{-1}\beta e^{2d})e_1+(u_0-\sqrt{-1}e^{2d})e_2\,,\\ &\qquad e_4+(u_3+\beta u_1-\sqrt{-1}(e^{2a}+\beta^2e^{2d}))e_1\\ &\qquad +(u_2+\beta u_0-\sqrt{-1}\beta e^{2d})e_2\}\,.\end{aligned}\tag{A.5.11}$$

Comparing (A.5.10) and (A.5.11), we see that we see that $\tilde\theta(\sqrt{-1}y_1,\ldots,\sqrt{-1}y_s)$ will lie in $U_0A_0\cdot\varphi$ if and only if it lies in $(U_0\cap G^{0,0})A_0\cdot\varphi$. (That is, $u_i=0$.) Elements of the latter are of the form

$$\nu\delta\cdot H^{1,0}_\varphi\ =\ \mathrm{span}_{\mathbb{C}}\left\{\begin{array}{c}e_3-\sqrt{-1}\beta e^{2d}e_1-\sqrt{-1}e^{2d}e_2\,,\\ e_4-\sqrt{-1}(e^{2a}+\beta^2e^{2d})e_1-\sqrt{-1}\beta e^{2d}e_2\end{array}\right\}.\tag{A.5.12}$$

Comparing (A.5.10) and (A.5.12), we see that

$$r(y)\ =\ \beta e^{2d}\,,\quad p(y)\ =\ e^{2d}\quad\text{and}\quad q(y)\ =\ e^{2a}+\beta^2e^{2d}\,,\tag{A.5.13}$$

and $\tilde\theta(\sqrt{-1}y_1,\ldots,\sqrt{-1}y_s)$ will lie in a (P_0,K)–Siegel domain when $y_j \gg 0$ if and only if β is bounded and there exists $t>0$ so that $e^{2a}/e^{2d}, e^{2d} > t$ for $y_j \gg 0$. Solving (A.5.13) yields

$$\begin{aligned} e^{2d} &= p(y) \ \to\ \infty \text{ as } y_j \to \infty \text{ (assuming some } p_i \neq 0)\,,\\ \beta &= \frac{r(y)}{p(y)} \text{ is bounded}\,,\\ e^{2a}/e^{2d} &= \frac{q(y)}{p(y)} \ -\ \left(\frac{r(y)}{p(y)}\right)^2 . \end{aligned}$$

(Note some p_i is necessarily nonzero, else $\sigma = \mathrm{span}_{\mathbb{R}_{>0}}\{\hat N_1\} \subset \partial\mathcal{N}$.) The third expression is a problem. Substituting with (A.5.7), the equations above become

$$\begin{aligned} e^{2d} &= \|y\|\, p(\lambda) \ \to\ \infty \text{ as } y_j \to \infty \text{ (some } p_i \neq 0)\,,\\ \beta &= \frac{r(\lambda)}{p(\lambda)} \text{ is bounded}\,,\\ e^{2a}/e^{2d} &= \frac{q(\lambda)}{p(\lambda)} \ -\ \left(\frac{r(\lambda)}{p(\lambda)}\right)^2 . \end{aligned}$$

To see the problem with the third equation, consider the case that $\sigma = \hat\sigma$; that is, $s=2$ and $p_1 = 1$, $p_2 = 0$, $q_1 = 0$, $q_2 = 1$, so that $r_i = 0$. We have $r(\lambda) = 0$ and $q(\lambda)/p(\lambda) = \lambda_2/\lambda_1 = y_2/y_1$, which may be arbitrarily close to zero regardless of the size of y_j. □

Proof of Claim A.5.9. In this case elements of $U = \exp(\mathfrak{u})$ are of the form

$$\nu \ =\ \begin{pmatrix} 1 & 0 & u_1 & u_2\\ 0 & 1 & u_0 & u_1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Elements of A are of the form $\gamma = \mathrm{diag}(e^a, e^a, e^{-a}, e^{-a})$ with $a \in \mathbb{R}$. And elements $g \in M$ are of the form (A.5.4) with $B \in \mathrm{SL}(2,\mathbb{R})$. So elements of $UAM \cdot \varphi$ are of the form

(A.5.14)
$$\begin{aligned} \nu\gamma g \cdot H^{1,0}_\varphi \ =\ \mathrm{span}_{\mathbb{C}}\big\{ & e_3 + u_1 e_1 + u_0 e_2 \ -\ \sqrt{-1}e^{2a}(B_1\cdot B_2\, e_1 \ +\ B_2 \cdot B_2\, e_2)\,,\\ & e_4 + u_2 e_1 + u_1 e_2 \ -\ \sqrt{-1}e^{2a}(B_1\cdot B_1\, e_1 \ +\ B_1\cdot B_2\, e_2)\big\}\,. \end{aligned}$$

Here B_1 and B_2 are the first and second rows of B, respectively, and $B_i \cdot B_j$ denotes the dot product. As in the proof of Claim A.5.8, comparing (A.5.14) with (A.5.11) we see that $\tilde\theta(\sqrt{-1}y_1,\ldots,\sqrt{-1}y_s)$ will lie in $UAM\cdot\varphi$ if and only if it lies in $AM\cdot\varphi$. (That is, $u_i = 0$.) Equivalently, if we can solve

(A.5.15) $$r(y) \ =\ e^{2a}\, B_1\cdot B_2\,, \quad p(y) \ =\ e^{2a}\, B_2\cdot B_2 \quad \text{and} \quad q(y) \ =\ e^{2a}\, B_1 \cdot B_1$$

for a and B (in terms of y). Moreover, $\tilde\theta(\sqrt{-1}y_1,\ldots,\sqrt{-1}y_s)$ will lie in a (P,K)–Siegel domain when $y_j \gg 0$ if and only if $B(y)$ is bounded (for this will then force $e^{2a(y)} \to \infty$ as $y_j \to \infty$).

To see that this will fail in general, consider the case that $\sigma = \mathrm{span}_{\mathbb{R}_{>0}}\{\hat{N}_1, \hat{N}_2\}$. That is, $p = (p_1, p_2) = (0, 1)$ and $q = (q_1, q_2) = (1, 0)$, and $r_1 = r_2 = 0$. Then $r(y) = 0$ implies $B_1(y)$ and $B_2(y)$ are orthogonal, and $\det B(y) = 1$ implies $1 = \|B_1(y)\| \cdot \|B_2(y)\|$. Then

$$\|B_1(y)\|^4 \;=\; \frac{\|B_1(y)\|^2}{\|B_2(y)\|^2} \;=\; \frac{q(y)}{p(y)} \;=\; \frac{y_1}{y_2}$$

may be arbitrarily large regardless of a lower bound on the y_j. That is, there exists no lower bound $c > 0$ so that $\theta(\sqrt{-1}y_1, \sqrt{-1}y_2)$ lies in a (P, K)–Siegel domain when $y_j > c$. This example extends to the general case that $s > 1$ and some $r_i = 0$. □

References

[AK00] D. Abramovich and K. Karu, *Weak semistable reduction in characteristic 0*, Invent. Math. **139** (2000), no. 2, 241–273.

[AMRT10] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, With the collaboration of Peter Scholze.

[BB66] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.

[BBT18] B. Bakker, Y. Brunebarbe and J. Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, arXiv:1811.12230, 2018.

[BKT18] B. Bakker, B. Klingler and J. Tsimerman, *Tame topology of arithmetic quotients and algebraicity of Hodge loci*, arXiv:1810.04801, 2018.

[BK77] E. Bedford and M. Kalka, *Foliations and complex Monge-Ampère equations*, Comm. Pure Appl. Math. **30** (1977), 543–571.

[BT14] C. Brav and H. Thomas, *Thin monodromy in Sp(4)*, Compos. Math. **150** (2014), no. 3, 333–343.

[BKT13] B. Bakker, B. Klingler and J. Tsimerman, *Symmetric differentials and the fundamental group*, Duke Math. J. **162** (2013), no. 14, 2797–2813.

[BKT18] Y. Brunebarbe, B. Klingler and B. Totaro, *Tame topology of arithmetic quotients and algebraicity of Hodge loci*, arXiv:1810.04801, 2018.

[Bor72] A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differential Geometry **6** (1972), 543–560.

[BPR16] P. Brosnan, G. Pearlstein and C. Robles, *Nilpotent cones and their representation theory*, Hodge theory and L^2-analysis, Adv. Lect. Math. (ALM), **39**, Int. Press, Somerville, MA, 2017, pp. 151–205.

[Bru16a] Y. Brunebarbe, *A strong hyperbolicity property of locally symmetric varieties*, arXiv:1606.03972, to appear in Ann. Sci. de l'Ecole Norm. Sup., 2016.

[Bru16b] ———, *Symmetric differential and variations of Hodge structures*, J. Reine Angew. Math., **743** (2018), 133–161.

[Car80] J. Carlson, *Extensions of mixed Hodge structures*, Journées de Géometrie Algébrique d'Angers, 1979, Sijthoff & Noordhoff, 1980, pp. 107–127.

[CMSP] J. Carlson, S. Müller-Stach and C. Peters, *Period mappings and period domains*, Cambridge University Press, Cambridge, 2003.

[Cat84] E. Cattani, *Mixed Hodge structures, compactifications and monodromy weight filtration*, Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), Ann. of Math. Stud., vol. 106, Princeton Univ. Press, Princeton, NJ, 1984, pp. 75–100.

[Cat14] ———, *Introduction to variations of Hodge structure*, Hodge theory, Math. Notes, vol. 49, Princeton Univ. Press, Princeton, NJ, 2014, pp. 297–332.

[CDK95] E. Cattani, P. Deligne, and A. Kaplan, *On the locus of Hodge classes*, J. Amer. Math. Soc. **8** (1995), no. 2, 483–506.

[CK14] E. Cattani and A. Kaplan, *Algebraicity of Hodge loci for variations of Hodge structure*, Hodge theory, complex geometry, and representation theory, Contemp. Math., vol. 608, Amer. Math. Soc., Providence, RI, 2014, pp. 59–83.

[CKS86] E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), no. 3, 457–535.

[CH88] M. Cornalba and J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. de l'Ecole Norm. Sup. (Ser. 4) **21**, 455–475 (1988).

[Del70] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.

[Dem85] J. P. Demailly, *Champs magnétiques et inégalités de Morse pour la d''-cohomologie*, C. R. Acad. Sci. Paris Sér. I Math. bf 301 (1985), no. 4, 119-122.

[Dem12] ———, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

[FPR15a] M. Franciosi, R. Pardini and S. Rollenske, *Log-canonical pairs and Gorenstein stable surfaces with $K_X^2 = 1$*, Compos. Math. 151 (2015), no. 8, 1529–1542.

[FPR15b] ———, *Computing invariants of semi-log-canonical surfaces*, Math. Z. 280 (2015), no. 3-4, 1107–1123.

[FPR17] ———, *Gorenstein stable surfaces with $K_X^2 = 1$ and $p_g > 0$*, Math. Nachr. 290 (2017), no. 5-6, 794–814.

[Fri83] R. Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) **118** (1983), no. 1, 75–114.

[Fuj78] T. Fujita, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978), no. 4, 779–794.

[Gra62] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[Gra83] ———, *Set theoretic complex equivalence relations*, Math. Ann. **265** (1983), 137–148.

[GR84] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Springer-Verlag, Berlin, 1984.

[GG16] M. Green and P. A. Griffiths, *Deformation theory and limiting mixed Hodge structures*, Recent advances in Hodge theory, London Math. Soc. Lecture Note Ser., vol. 427, Cambridge Univ. Press, Cambridge, 2016, pp. 88–133.

[GG20] ———, *Positivity of vector bundles and Hodge theory*, arXiv:1803.07405, to appear in Adv. Math., (2020).

[GGR20] M. Green, P. A. Griffiths and C. Robles, *Global properties of period mappings at infinity*, in preparation 2020.

[Gri69] P. A. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 185–251.

[Gri70a] ———, *Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems*, Bull. Amer. Math. Soc. **76** (1970), 228–296.

[Gri70b] ———, *Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 38, 125–180.

[Gri18] ———, *Hodge Theory and Moduli*, in proceedings of the conference "Geometry at the Frontier III", AMS Cont. Math., to appear. https://hdl.handle.net/20.500.12111/7900.

[Gri19] ———, *Using Hodge theory to detect the structure of a compactified moduli space*, November 2020 talk at IMSA conference held at IMAT at UNAM, Mexico City, https://hdl.handle.net/20.500.12111/7873.

[GH94] P. A. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons Inc., 1994.

[GKS20] P. Gallardo, M. Kerr and L. Schaffler, *Geometric interpretation of toroidal compactifications of moduli of points in the line and cubic surfaces*, arXiv:2006.0131 (2020).

[GRT14] P. A. Griffiths, C. Robles and D. Toledo, *Quotients of non-classical flag domains are not algebraic*, Algebr. Geom. **1** (2014), no. 1, 1–13.

[GS69] P. A. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253–302.

[GS75] ______, *Recent developments in Hodge theory: a discussion of techniques and results*, 31–127.

[Hor76] Hörmander, Lars, *Linear partial differential operators*, Springer Verlag, Berlin-New York, 1976.

[Kaw81] Y. Kawamata, *Characterization of abelian varieties*, Compositio Math. **43** (1981), no. 2, 253–276.

[Kaw83] ______, *Kodaira dimension of certain algebraic fiber spaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), no. 1, 1–24.

[Kaw85] ______, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46.

[KK10] J. Kollár and S. Kovács, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.

[Kol87] J. Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 361–398.

[Kol12] ______, *Quotients by finite equivalence relations*, Current developments in algebraic geometry, Math. Sci. Res. Inst. Publ., vol. 59, Cambridge Univ. Press, Cambridge, 2012, With an appendix by Claudiu Raicu, pp. 227–256.

[Kol13] ______, *Moduli of varieties of general type*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 131–157.

[KP16] M. Kerr and G. Pearlstein, *Boundary components of Mumford-Tate domains*, Duke Math. J. **165** (2016), no. 4, 661–721.

[KU09] K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Annals of Mathematics Studies, vol. 169, Princeton University Press, Princeton, NJ, 2009.

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 49, Springer-Verlag, Berlin, 2004.

[Laz16a] R. Laza, *Perspectives on the construction and compactification of moduli spaces*, Compactifying moduli spaces, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Basel, 2016, pp. 1–39.

[Laz16b] R. Laza, *The KSBA compactification for the moduli space of degree two K3 pairs*, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 2, 225279.

[MS18] H.-B. Moon, L. Schaffler, *KSBA compactification of the moduli space of K3 surfaces with purely non-symplectic automorphism of order four*, arXiv:1809.05182 (2018).

[Mum61] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math., 9 (1961), 5–22.

[Pau18] M. Păun, *Singular Hermitian metrics and positivity of direct images of pluricanonical bundles*, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 519–553.

[PS08] C. Peters and J. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 52, Springer-Verlag, Berlin, 2008.

[Rob16] C. Robles, *Classification of horizontal* SL(2)*s*, Compos. Math. **152** (2016), no. 5, 918–954.

[Sat60] I. Satake, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. of Math. (2) **72** (1960), 555–580.

[Sch73] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

[Ser55] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955) 197–278.

[Siu84] Y. T. Siu, *A vanishing theorem for semipositive line bundles over non-Kähler manifolds*, J. Diff.Geom. **19** (1984), no. 2, 431–452.

[Siu85] ———, *Some recent results in complex manifold theory related to vanishing theorems for the semipositive case*, Workshop Bonn 1984, 169–192, Lecture Notes in Math., **1111**, Springer, Berlin, 1985.

[Som59] F. Sommer, *Komplex-analytische Blitterang reeler Hyperflčahen im C^n*, Math. Ann. **137** (1959), 397–411.

[Som73] A. J. Sommese, *Some algebraic properties of the image of a period mapping,* Rice Univ. Studies **59** (1973), no. 2, 123–128.

[Som78] ———, *On the rationality of the period mapping*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), no. 4, 683–717.

[SZ85] J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure. I*, Invent. Math. **80** (1985), no. 3, 489–542.

[Uen75] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, Vol. 439, Springer-Verlag, Berlin-New York, 1975.

[Uen78] ———, *Classification of algebraic varieties. II. Algebraic threefolds of parabolic type*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, pp. 693–708.

[Vie83a] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 329–353.

[Vie83b] ———, *Weak positivity and the additivity of the Kodaira dimension. II. The local Torelli map*, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 567–589.

[Zuo00] K. Zuo, *On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications*, Asian J. Math. **4** (2000), no. 1, 279–301, Kodaira's issue.

E-mail address: mlg@math.ucla.edu

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555

E-mail address: pg@math.ias.edu

Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540

University of Miami, Department of Mathematics, 1365 Memorial Drive, Ungar 515, Coral Gables, FL 33146

E-mail address: radu.laza@stonybrook.edu

Mathematics Department, Stony Brook University, Stony Brook, NY 11794-3651

E-mail address: robles@math.duke.edu

Mathematics Department, Duke University, Box 90320, Durham, NC 27708-0320