

Period Mapping at Infinity *

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Abstract

Hodge theory provides a basic invariant of complex algebraic varieties. For algebraic families of smooth varieties the *global* study of the Hodge structure on the cohomology of the varieties (period mapping) is a much studied and rich subject. When one completes a family to include singular varieties the *local* study of how the Hodge structures degenerate to limiting mixed Hodge structures is also much studied and very rich. However, the *global* study of the period mapping at infinity has not been similarly developed. This has now been at least partially done and will be the topic of this talk. Sample applications include

- ▶ new global invariants of limiting mixed Hodge structures
- ▶ a generic local Torelli assumption implies that moduli spaces are log canonical (not just log general type); and
- ▶ extension data and asymptotics of the Ricci curvature
- ▶ a proposed construction of the toroidal compactification of the image of period mapping.

The key point is that the extension data associated to a *limiting* mixed Hodge structure has a rich geometric structure and this provides a new tool for the study of families of singular varieties in the boundary of families of smooth varieties.

Outline

I. Extension data

- ▶ Geometric properties of extension data of *limiting* mixed Hodge structures

II. Basic results

- ▶ Removable singularities for level 1
- ▶ Level 2 determines all the extension data
- ▶ Fundamental formula

III. Applications

- ▶ Ampleness of $K_{\overline{B}} + Z$
- ▶ Freeness of $kL_H - \ell_0 Z$
- ▶ asymptotics of the Ricci curvature
- ▶ Speculative toroidal completion of period mappings

References

I. Extension data

- ▶ Hodge structure of weight n (V, F) ,
 $F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}}$

$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 0 \leq p \leq n.$$

Equivalent to Hodge decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \quad \overline{V}^{p,q} = V^{q,p}$$
$$\left(V^{p,q} = F^p \cap \overline{F}^q, \quad F^p = \bigoplus_{p' \geq p} V^{p',q} \right).$$

- ▶ Mixed Hodge structure (V, W, F)
 - ▶ $W_0 \subset W_1 \subset \dots \subset W_n = V$;
 - ▶ $(\text{Gr}_k^W(V), F_k)$ is a Hodge structure of weight k ; here

$$F_k^p = F^p \cap W_{k,\mathbb{C}} / W_{k-1,\mathbb{C}}.$$

- ▶ *Extension data:* $\mathcal{E} = \{\text{MHS's with fixed } \text{Gr}(V, W, F)\}$
 - ▶ $\text{Gr}(V, W, F) = \{H^0, H^1, \dots, H^n\}$
 - ▶ $\mathcal{E}_k = \left\{ \begin{array}{l} \text{MHS's that are at most } k\text{-fold} \\ \text{iterated extensions of } H^0, \dots, H^n \end{array} \right\}$
 - ▶ $\mathcal{E}_1 = \bigoplus_{k=1}^n \text{Ext}_{\text{MHS}}^1(H^k, H^{k-1}) \cong \bigoplus_{k=1}^n \frac{\text{Hom}(H^k, H^{k-1})}{F^0 \text{Hom} + \text{Hom}_{\mathbb{Z}}} := J$
 - ▶ $\text{Ext}_{\text{MHS}}^q(*, *) = 0$ for $q \geq 2$
 - ▶ $\mathcal{E}_{k+1} \rightarrow \mathcal{E}_k$ has fibre $\bigoplus^{\ell} \text{Ext}_{\text{MHS}}^1(H^{k+\ell+1}, H^{\ell})$
- which is a \mathbb{C}^m/Λ where Λ is discrete (thus T 's, \mathbb{C}^* 's, \mathbb{C} 's)

- ▶ *Limiting mixed Hodge structure* $(V, W(N), F)$
 - ▶ $N \in \text{End}(V)$ nilpotent with $N^{m+1} = 0$ gives unique $W_0(N) \subset \dots \subset W_{2m}(N)$ with

$$\begin{cases} N : W_k(N) \rightarrow W_{k-2}(N) \\ N^k, W_{m+k}(N) \xrightarrow{\sim} W_{m-k}(N) \end{cases}$$

- ▶ $N : F^p \rightarrow F^{p-1}$

► *Polarization and cones*

► $Q : V \otimes V \rightarrow \mathbb{C}, N \in \text{End}(V, Q)$

$$W_\ell(N)^\perp \cong W_{2m-\ell-1}(N)$$

$$\left\{ \begin{array}{l} Q_k : \text{Gr}_{m+k}^{W(N)}(V) \otimes \text{Gr}_{m-k}^{W(N)}(V) \rightarrow \mathbb{Q} \\ Q_k(u, v) = Q(N^k u, v) \end{array} \right.$$

$$\implies \text{Gr}^{W(N)}(V, W(N), F) = \oplus (\text{polarized HS's})$$

- will have a cone $\sigma = \text{span}\{N_1, \dots, N_r\}$ where $[N_i, N_j] = 0$ and where

$$W(N) \text{ is the same for all } N = \sum \lambda_i N_i, \lambda_i > 0, \text{ in } \sigma$$

► *Special structure for extension data associated to LMHS's*

► $(T_e J) \otimes \mathbb{C}$ is a HS of weight -1 and Hodge decomposition

$$(k-1, -k) \otimes \cdots \otimes \underbrace{(-1, 0) \oplus (0, -1)} \otimes \cdots \otimes (-k, k-1)$$

► $(T_e J_{ab}) \otimes \mathbb{C}$ is the maximal sub-HS in \smile

► $\text{Gr}_{-2}^{W(\sigma)} \text{End}(V_{\mathbb{Z}}, G) \supset \Lambda^2 H_1(J, \mathbb{Z}) = H^2(J, \mathbb{Z})^*$

Proposition:

$$\left\{ \begin{array}{c} \text{integral} \\ (1,1) \text{ classes} \\ \cup \\ \check{\sigma} \otimes \mathbb{Z} \end{array} \right\} \subset H^2(J, \mathbb{Z}) \subset \text{Gr}_{+2}^{W(\sigma)} \text{End}(V_{\mathbb{Z}}, Q) \text{ and } A \in \check{\sigma}$$

gives an **ample** line bundle $L_A \rightarrow J_{ab}$

Question: What do these ample line bundles have to do with the geometry of the LMHS's along a fibre of Φ_e ?



II. Basic results

- ▶ $B =$ smooth variety with smooth completion \overline{B} and where $Z = \overline{B} \setminus B$ is a reduced normal crossing divisor $\cup Z_i$ with Z_i irreducible
- ▶ $\left\{ \begin{array}{l} \text{Variation of Hodge} \\ \text{structure } (\mathcal{V}, \mathcal{F}, \nabla, B) \\ \text{with monodromy group } \Gamma \end{array} \right\} \iff \left\{ \begin{array}{l} \text{Period mapping} \\ \Phi: B \rightarrow \Gamma \backslash D \text{ with} \\ \text{image } P = \Phi(B) \end{array} \right\}$

Here $\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_B^1$ is the Gauss-Manin connection with $\nabla^2 = 0$ and $\mathbb{V} := \ker \nabla$ is a local system. The *Hodge line bundle*

$$L_H = \bigotimes^p \det \mathcal{F}^p$$

- ▶ May assume monodromy T_i around Z_i is unipotent with logarithm N_i — a neighborhood of a point of $Z_I = \bigcap_{i \in I} Z_i$ looks like $\Delta^{*r} \times \Delta^s$ and $\sigma_I = \text{span}_{\mathbb{Z}^+} \{N_1, \dots, N_r\}$

- ▶ Geometric case $\mathcal{X} \rightarrow B$ with $\overline{\mathcal{X}} \rightarrow \overline{B}$ having the Abramowich-Karu et al. form of semi-stable-reduction over Z
 - ▶ $\overline{\mathcal{M}}_g$; essentially smooth
 - ▶ KSBA $\overline{\mathcal{M}}$; $\partial\mathcal{M}$ highly singular.
- ▶ Cattani-Kaplan-Schmid: For $b_0 \in Z_I$

$$\lim_{b \rightarrow b_0} \Phi(b) = \left\{ \begin{array}{l} \text{equivalence class of} \\ \text{limiting mixed Hodge structures} \\ \text{with monodromy cone } \sigma_I \end{array} \right\}$$

$$\rightsquigarrow B \xrightarrow{\Phi} P \subset \Gamma \setminus D$$

$$\begin{array}{ccc} \cap & & \cap \\ \overline{B} & \xrightarrow{\Phi_e} & \overline{P} \end{array}$$

$$\text{where } \Phi_e(b_0) = \text{Gr} \left(\lim_{b \rightarrow b_0} \Phi(b) \right)$$

Conjecture: \bar{P} is an analytic variety on which $L_{H,e} \rightarrow \bar{P}$ is ample.

Using model theory (0-minimal structures) assuming Γ is arithmetic Bakker-Brunebarbe-Tsimmerman proved that P is an algebraic variety and $L_H \rightarrow P$ is ample.

- ▶ What has been missing is the *global* analysis of Φ_e along a fibre B_p of Φ_e — known that B_p is a *complete* subvariety of some minimal Z_I — along B_p the $\text{Gr}(\text{LMHS's})$ are (locally) constant — what is varying is the extension data
- ▶ $B_p^* = B_p \setminus \{\text{intersection of } B_p \text{ with } Z_j\text{'s, } j \notin I\}$
 $\Phi_1 := \text{map } B_p^* \rightarrow \{\text{level 1 extension data } J\}$

Theorem A: Φ_1 extends to a map

$$\Phi_1 : B_p \rightarrow J_{ab} \subset J.$$

This is a global result using mixed Hodge theory

$\Phi_m := \text{map to extension data of level } m \text{ on fibres of } \Phi_{m-1}$

Theorem B: The Φ_m are determined by Φ_1 and Φ_2 .

Thus $\Phi_e = \Phi_0, \Phi_1, \Phi_2$ constant \implies $\left\{ \begin{array}{l} \text{extension data of LMHS's} \\ \text{is constant along } B_p \end{array} \right\}$.

This is a local result using the IPR.

- ▶ The main result is

$$\text{Theorem C: } \Phi_1^*(L_A) = - \sum_k \langle A, N_k \rangle [Z_k] \Big|_{B_p}.$$

This result relates the behavior of the LMHS *along* a fibre of Φ_e to the *normal* behavior to Z of Φ_e along that fibre.

Reflects subtle global behavior of Φ_e ,

- ▶ We note that Theorems A, B, C really are results about the behavior of the period mapping at infinity; they only use Φ_e on a neighborhood \mathcal{U} of Z in \overline{B} .

III. Applications

To illustrate a simple application of Theorem C we make the following

(*) Assumption: Φ_1 does not have any positive dimensional fibres.

- ▶ If B_p does not meet any lower dimensional strata of Z , then

$$\begin{cases} -[Z_i]|_{B_p} \rightarrow B_p & \text{is ample} \\ \iff N_{Z/\bar{B}}^*|_{B_p} \rightarrow B_p & \text{is ample} \end{cases}$$

- ▶ If (*) is not satisfied or if B_p does meet Z_j 's, $j \notin I$, Theorem C can still be used; e.g.,
If $\dim = 2$ and $\Phi_e(Z_i) = \text{point}$, then the intersection matrix

$$M = \|Z_i \cdot Z_j\| < 0.$$

Thus Z can be contracted in the \mathcal{U} above.

- ▶ There exist $a_i > 0$ such that for $m \gg 0$

$$L_{H,e} - \sum a_i [Z_i]$$

is ample (cannot choose $a_i = 1$; they depend on the maximal eigenvalue of M).

- ▶ In general uses of the main result are somewhat subtle and still being worked out — for $A \in \check{\sigma}$ and $C \subset B_p$ a curve

$$0 < \deg_C(L_A) = \sum_{i \in I} \langle A, N_i \rangle \deg_C(N_{Z_i/\bar{B}}^*) - \sum_{j \in J} \langle A, N_j \rangle (Z_j \cdot C)$$

$\begin{matrix} \vee \\ 0 \end{matrix}$
 $\begin{matrix} \wedge \\ 0 \end{matrix}$

and these inequalities must be played off against one another (e.g., the $Z_i^2 < 0$ and $Z_i \cdot Z_j \geq 0$, $i \neq j$, in the $\dim B = 2$ result).

- ▶ Among the properties a line bundle $L \rightarrow X$ over a smooth variety can have are

— L is nef	}	numerical
— L is big		
— L is free	}	geometric
— L is ample		

The Hodge theory literature abounds with results of the first two types,[†] but those of the second type are more scarce. One reason seems to be the lack of global information about $\Phi_e|_Z$. The following is an illustration of what can be done using Theorem C.

First we recall the Higgs bundle construction

- $E^p = \mathcal{F}^p / \mathcal{F}^{p+1}$, $E = \bigoplus E^p$
- $\theta^p : E^p \rightarrow E^{p-1} \oplus \Omega_B^1$ induced by ∇ , $\theta = \bigoplus \theta^p$
- $\nabla^2 = 0$ is equivalent to $\theta \wedge \theta = 0$
- $\delta : TB \rightarrow F^{-1} \text{End}(E)$ induced by θ .

On (\bar{B}, Z) we have

$$\delta_e : T_{\bar{B}}(-\log Z) \rightarrow F^{-1} \text{End}(E_e)$$

[†]e.g., certain moduli spaces are of log general type or are hyperbolic.

(LT) Local Torelli assumption: δ_e is injective.

Theorem: *LT implies*

(a) (*) above is satisfied

(b) $K_{\bar{B}} + Z$ is free

(c) $K_{\bar{B}} + Z$ is ample \iff $\left\{ \begin{array}{l} \text{the Gauss map } G(\Phi_1) \text{ has} \\ \text{no positive dimensional fibres} \end{array} \right\}$

For any map $\varphi : W \rightarrow J$ from a k -dimensional variety W to a complex torus the Gauss map is

$$G(\varphi) : W \rightarrow \text{Gr}(k, T_e J)$$

sending $w \in W$ to $\varphi_*(T_w W) \subset T_e J$.[‡]

[‡] $G(\varphi)$ is a finite map $\iff K_W$ is ample.

- ▶ Example of case (b): $\overline{B} = \overline{\mathcal{A}}_g^{\text{Tor}}$ and fibres of $\overline{\mathcal{A}}_g^{\text{Tor}} \xrightarrow{\pi} \overline{\mathcal{A}}_g^{\text{SBB}} = \Phi_e(\overline{B})$ are abelian varieties and $\Phi :=$ identity, then $K_{\overline{B}} + Z = \pi^* \mathcal{O}(2)$.
- ▶ $\omega_e :=$ Chern form of $L_{H,e} \rightarrow \overline{B}$; gives complete Kähler metric on B — then

$$c_1 \omega_e \leq -\text{Ric} \omega_e \leq c_2 \omega_e + \sigma$$

where $\sigma \geq 0$ is bounded and $\sigma > 0$ if $G(\Phi_1)$ is a finite mapping.

- ▶ With some details still to be checked, another result is
 - ▶ If LT is satisfied at a general point, then there is an ℓ_0 and a $k_0(\ell_0)$ such that for $k \geq k_0$

$$kL_e - \ell_0 Z \text{ is free.}$$

If the details are completed this would give a sharpened version of the BBT result without the assumptions that Γ is arithmetic.

► Finally we give some speculation on the

Question: *What are the natural completions of images of period mappings?*

Given $\Phi : B \rightarrow \Gamma \backslash D$ and \bar{B} as above at the set level one may define maps

$$\begin{array}{c} \Phi_e \\ \parallel \\ \Phi_0, \Phi_1, \Phi_2, \Phi_3, \dots \end{array}$$

Theorem B states that the Φ_m for $m \geq 3$ are determined by Φ_0, Φ_1, Φ_2 . The word “determined” means “determined up to constants,” like integration constants in calculus. The geometric/arithmetical meaning of these constants is yet to be worked out.

In the classical case when the period domain D is Hermitian symmetric we have only Φ_0, Φ_1, Φ_2 , so this is not an issue. That being said, provisionally we propose

$$\overline{P}_{\text{SBB}} = \overline{P} = \text{Image } \Phi_0 \quad (\text{minimal})$$

$$\overline{P}_{\text{Tor}} = \text{Image}\{\Phi_0, \Phi_1, \Phi_2\} \quad (\text{maximal})$$

for the completions of $\Phi(B) = P$. The reason for the “Tor” is that Φ_2 is only defined on Zariski open sets W^* in the fibres W of $\{\Phi_0, \Phi_1\}$, and

$$\Phi_2 : W^* \rightarrow \frac{\text{span}_{\mathbb{Z}}(\sigma_{I \cup J})}{\text{span}_{\mathbb{Z}}(\sigma_I)} \otimes \mathbb{C}^*;$$

thus Φ_2 maps to essentially a product of \mathbb{C}^* 's. Defining Φ_2 on the complete fibres of Φ_1 will necessitate at least partially completing the products of \mathbb{C}^* 's.

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