Global properties of period mappings on the boundary

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*Working notes—not for circulation

Outline

I. Introduction

The *global* study of variations of polarized, pure Hodge structures is an extensively studied and in many ways fairly well-developed subject (cf. [CM-SP]). Thus for a period mapping

$$\Phi : B \to \Gamma \backslash D$$

where B is a smooth quasi-projective variety, if we assume that the differential Φ_* is generically injective, then it is known that both $K_{\overline{B}}(\log Z)$ and $T^*_{\overline{B}}(\log Z)$ are nef and big (cf. [Z]). Here $\overline{B}$ is any completion of B with boundary $Z = \overline{B} \backslash B$ a normal crossing divisor. It is also known that B is hyperbolic modulo a proper subvariety (cf. [LSZ]). With no assumption on Φ_* it has been recently proved that for Γ arithmetic the image $\Phi(B) \subset \Gamma \backslash D$ is a quasi-projective algebraic variety over which the augmented Hodge bundle $L \to P$ is ample ([BBT]).†

†For background and the terminology from Hodge theory used in these notes we refer to [GGLR] and [GG].

On the other hand, assuming unipotent monodromies around the irreducible branches Z_i of Z,[‡] it is well known that the VHS over B extends canonically to $\overline{B}$ and on the boundary Z one has a variation of limiting mixed Hodge structures whose *local* structure has been extensively studied through the work of Cattani-Kaplan- Schmid and others (cf. [CKS]). From multiple perspectives it has recently become clear that what is now needed is a *global* study of the variation of *limiting* mixed Hodge structures along Z.[§] That is what will be undertaken in these notes.

[‡]This assumption is made for convenience of exposition, not because it is necessary. As illustrated by the application of Hodge theory in various places, e.g., in the expository talks [G1], [G2], [G3], the case when the semi-simple part of monodromy is non-trivial is of important geometric interest.

[§]There is a global study of variations of regular mixed Hodge structures (cf. [PS] and the references there). As we will see the global study for limiting mixed Hodge structures is a somewhat different story.

We denote by $Z_I = \bigcap_{i \in I} Z_i$ the closed strata in Z with $Z_I^* \subset Z_I$ being the Zariski open smooth part of Z_I. Then taking the associated graded pure Hodge structures of the limiting mixed Hodge structures along Z_I^* gives period mappings

$$\Phi_I : Z_I^* \to \Gamma_I \backslash D_I \tag{I.1}$$

in the usual sense.

Thus the new information concerns the question

What happens globally along the fibres Y^* of (I.1)*?*

What is varying along these fibres is the *extension data* associated to the family of limiting mixed Hodge structures along Y^*. Here it is important to keep in mind is that these limiting mixed Hodge structures have locally constant associated graded pure Hodge structures. There will be global monodromy arising from the action of $\pi_1(Y^*)$ but it will preserve the weight filtration along each connected component of Y^*, and will act by a finite group on the graded quotients.

As a first step this invites the study of the geometry of the set $\mathcal{E}$ of all extension data associated to the set of mixed Hodge structures with a fixed associated graded pure Hodge structures. This is discussed in Section II, and there it is observed that the set $\mathcal{E}_k$ of at most k-fold extensions of pure Hodge structures, which we shall refer to as *extensions of level at most k*, fibres over $\mathcal{E}_{k-1}$ with typical fibre an $\mathrm{Ext}^1_{\mathrm{MHS}}(H^{m+k}, H^m)$ where H^i is a pure Hodge structure of weight i. We shall refer to these fibres as *extensions of level k*.

From [C] we have

> $\mathrm{Ext}^1_{\mathrm{MHS}}(H^{m+k}, H^m)$ *is the quotient of a complex Euclidean space by a discrete abelian subgroup.*

In fact, this Ext-group is an extension of a compact complex torus by a product of $\mathbb{C}^*$'s and $\mathbb{C}$'s, which with a somewhat abuse of language we shall call a *semi-abelian-torus*. In general other than this structural result there doesn't seem to be a lot more that one can say about the set $\mathcal{E}$ of extensions of mixed Hodge structures having a fixed associated graded.

However when we come to *limiting* mixed Hodge structures the story is richer. For such a LMHS $(V, Q, W(N), F)$ where N lies in the interior of a monodromy cone σ with dual cone $\check{\sigma}$, elements A in $\check{\sigma} \otimes \mathbb{Z}$ canonically define line bundles $L_A \to J$ over the level 1 extension data J.

Now J is a compact complex torus that looks like a Tate twist of an intermediate Jacobian of the type $H^{2m-1}_{\mathbb{C}}/F^m H^{2m-1}_{\mathbb{C}} + H^{2m-1}_{\mathbb{Z}}$, and one may define the abelian part $J_{ab} \subset J$ of J to be the largest sub-torus lying under $H^{m,m-1} \oplus H^{m-1,m}$. Then for $A \in \check{\sigma}$ the restriction

$$L_A \to J_{ab}$$

is a positive line bundle. The construction of $L_A \to J$ involves the existence of the level 2 extension data and the structure of the LMHS. The point here is that the extension data for *limiting* mixed Hodge structures has an associated geometry not present for general mixed Hodge structures.

We now turn to the Abel-Jacobi type maps that arise on a subvariety $Y^* \subset Z_I^*$ along which the associated graded to the limiting mixed Hodge structures are locally constant.

There are inductively defined maps, the first of which is[¶]

$$\Phi_1 : Y^* \to \{\text{ level 1 extension data}\}.$$

Concerning Φ_1 there are three main results:

- Φ_1 extends to the closure $Y \subset Z_I$;
- $\Phi_1 : Y \to J_{ab}$ maps to the abelian part of J;

[¶]Actually the first map should probably be thought of as

$$\Phi_e : \overline{B} \to \overline{P}$$

where $\overline{P}$ is the canonical set-theoretic completion of the image $P = \Phi(B) \subset \Gamma\backslash D$ of the original period mapping. On the boundary Z the map Φ_e associates to a limiting mixed Hodge structure the associated graded pure Hodge structure. Thus it is a minimal Satake-Baily-Borel (SSB) type completion of P (cf. [GGLR] and [GG]). The notation Φ_e is the one used in [GGLR]. In these notes we will defines three maps denoted by Φ_0, Φ_1, Φ_2 and the first one Φ_0 will be Φ_e.

and we have the equation of the line bundles over Y

- (I.2) $$\Phi_1^* L_A = -\sum_{i \in I} \langle A, N_i \rangle \, [Z_i]\big|_Y .$$

In (I.2) the pairing is between σ and the dual cone $\check{\sigma}$; the $N_i \in \sigma$ are the logarithms of monodromy around the branches Z_i of Z. We note that the formula holds when we sum over all the indices where we set $[Z_j]_Y = 0$ if $Y \cap Z_j = \emptyset$. This basic formula relates the variation of the extension data *along* $Y \subset Z$ to the normal bundle to Z in $\overline{B}$ which points *out of* Y into $\overline{B}$.

It is **the** central geometric information that arises when one considers the variation of the extension data along the fibres in the boundary at infinity of the extended period mapping Φ_e to the Satake-Baily-Borel completion of a period mapping Φ.[‖]

[‖] It is well known that given a variation of polarized Hodge structures with unipotent monodromy over the punctured disc Δ^*, at the origin one has an *equivalence class* of limiting mixed Hodge structures. To get rid of the equivalence class one uses the cotangent space $T^*_{\{0\}}\Delta$ to obtain a well-defined limiting mixed Hodge structure. The relation (I.2), together with extensions of that relation discussed below, may be thought of as an intrinsic, globalized version of using $T^*_0\Delta$.

For example a geometric consequence of (I.2) is

If Y is contained in one Z_i and doesn't meet the other Z_j's, then

$$\Phi_1^* L_A = \langle A, N_i \rangle \, N^*_{Z_i/B}\Big|_Y .$$

In particular, if $\langle A, N_i \rangle > 0$ and the differential $\Phi_{1,*}$ is injective, then $N^*_{Z/\overline{B}}\Big|_Y$ is ample.[**]

We note that the proof of the above results are Hodge theoretic. The first uses mixed Hodge theory, the second the infinitesimal period relation (IPR), and the third the structure equations of a family of degenerating Hodge structures.

[**]In these notes to isolate the essential geometric content of various results we have chosen to state special cases rather than the most general formulations. For example, this result remains valid if we only assume that $\Phi_{1,*}$ is injective at 1 point.

The next map

$$\Phi_2 : \Phi_1^{-1}(\text{point}) \to \{\text{level 2 extension data}\}$$

is defined on Zariski open sets S^* in the fibres S of Φ_1 of the fibration $\mathcal{E}_2 \to \mathcal{E}_1$. This mapping turns out to have some surprising features, properties that are of both a local and global nature. For the first, the level 2 extension data to which S^* maps is an extension of a $\mathbb{C}^m/\Lambda$ where Λ is a partial lattice by a quotient M of a product of $\mathbb{C}^*$'s. Then as a result of the IPR we have

(1) $\Phi_2 : S^* \to M$.[††]

The second is a global result which informally stated is

(2) Φ_2 is determined up to a constant by discrete data.

[††]Thus $\Phi_2(S^*)$ lies in a subspace of the level 2 extension data, a subspace that in 1^{st} approximation may be thought of as a product of $\mathbb{C}^*$'s.

This discrete data involves the monodromy arising from the fundamental group of the minimal stratum Z_I^* that contains S^*, the monodromies around the Z_j where $j \notin I$ but Z_j intersects S, and the line bundles

$$N^*_{Z_i/\overline{B}}\Big|_S.$$

The third is in response to the question

(3) *What object does all of S map to?*

The answer explained in (V.9) is that we need to interpret the mapping Φ_2 as defined on $\mathbb{P}N_{Z_I/\overline{B}}\Big|_S$ where Z_I is the minimal stratum containing S.

A fourth feature is that if we think of attaching to $\overline{P}$ the Hodge-theoretically constructed objects P_1 and P_2 given by the image of Φ_1 and Φ_2, then one may think of all of the data as given a toroidal like object $\overline{P}_T$ lying over the minimal SBB completion $\overline{P}$ of the image of a period mapping. Among other things, as will be illustrated by example, this construction suggests how one may at least partially desingularize moduli spaces of some general type surfaces (cf. [G1], [G2], [G3] for further discussion of this).[†]

For the next step we will state it informally and refer to Section IV(A) below for explanations of the terminology and a proof. The result is

> *the Abel-Jacobi maps* $\Phi_k, k \geqq 3$, *are determined up to constants by* Φ_1 *and* Φ_2.

[†] This aspect of the theory has not yet been precisely formulated and worked out.

In particular,

> *If Φ_1 and Φ_2 are constant, then the map to the extension data is constant.*

This is again a Hodge theoretic result whose proof uses the infinitesimal period relation (IPR). In the classical case where the period domain is Hermitian symmetric, there is **only** extension data of levels 1 and 2 ([R]). Thus again we see illustrated the general phenomenon that due to the IPR the general case has many structural properties in common with the classical case.

In the appendix to Sections IV and V we give two examples that illustrate the geometric interpretation of the extension data and the maps Φ_1 and Φ_2. The first example is the analysis of the moduli space $\overline{\mathcal{M}}_2$ of stable genus 2 curves.

Although $\overline{\mathcal{M}}_2$ does not quite fit the $(\overline{B}, Z)$ framework, it is easy to adapt the results about the latter to the former. One sees very clearly the mappings Φ_1 and Φ_2. The story of $\overline{\mathcal{M}}_2$ is of course quite classical and in fact pre-dates the general construction of $\overline{\mathcal{M}}_g$.

The second example is recent. In the 1980's and 1990's Kollár-Shepherd-Barron-Alexeev proved the existence of a projective moduli space for stable surfaces of general type.[‡] However only recently are examples of the construction being worked out, especially in the papers [FPR] on the structure of the moduli space $\overline{\mathcal{M}}_I$ for I-surfaces.

[‡]In fact, they constructed the moduli space for general type varieties of any dimension.

These are regular general type surfaces X with $K_X^2 = 1$ and $p_g(X) = 2$; they are in many ways the analogue of genus 2 curves.[§]

However, unlike $\overline{\mathcal{M}}_2$ the moduli space $\overline{\mathcal{M}}_I$ is highly singular along the boundary, and as discussed in the appendix it is exactly the extension data in the limiting mixed Hodge structures that helps to understand and provide a guide to how to desingularize the boundary.

[§] In addition to their papers, expository accounts of how some of their work may be organized and interpreted using Hodge theory are given in [G1], [G2], [G3].

In Section VI we will discuss what seem to be rather natural conditions for local Torelli to hold, including along the boundary. The main result is that if this version of local Torelli holds, then there is a natural completion of the image P of a period mapping that captures the complete information in the limiting mixed Hodge structures along the boundary of P. Applications of this that establish freeness and ampleness criteria for various line bundles associated to a global VHS will be given in Sections VI and VIII.[¶]

In Section VII we will give a fairly complete analysis of the global structure of variations of Hodge structure over complete algebraic surfaces.

[¶]We note that there are many results in Hodge theory concerning the numerical properties of nefness and bigness. In these notes we are primarily concerned with results involving the geometric properties of freeness and ampleness.

Summarizing that section and using the notations from [GGLR] the results include

(i) the image $\Phi_e(\overline{B})$ of the extended period mapping is either an algebraic curve or an algebraic surface;

(ii) the augmented Hodge line bundle $L_e \to \overline{B}$ is free and

$$\Phi_e(\overline{B}) = \mathrm{Proj}(L_e);$$

(iii) in case Φ_* is everywhere injective, the are integers $a_i \geqq 0$ and there is an m_0 such that for $m \geqq m_0$ the line bundle

$$mL_e - \sum a_i[Z_i]$$

is ample on $\overline{B}$; and

(iv) in general, if $\dim \Phi_e(B) = 2$, there is a structure result that will be explained in Section VII.

As will be also discussed in Section VII, assuming for simplicity of explanation that all $\Phi_e(Z_i) =$ point, one may consider the two conditions:

(a) the intersection matrix $\|Z_i \cdot Z_j\| < 0$ is negative definite;

(b) for some positive integers a_i the line bundle $\sum_i a_i[Z_i]$ restricted to Z is negative.

These will be seen to be equivalent. The first condition will be established using Hodge theory. This is what relates to (ii) above. We note that (b) is *not* equivalent to the statement that $\sum_i [Z_i]\big|_Z$ is negative.

We may summarize the above as saying that when $\dim B = 2$ there is a fairly complete qualitative global description of a variation of Hodge structure, including its behavior along the boundary.

In Section VIII we will bring in the role of the canonical bundle $K_{\overline{B}}$ in the study of the behavior of a period mapping at infinity. In addition to the numerical properties of nefness and bigness of line bundles, the geometric properties of freeness and ampleness require knowledge of the canonical bundle. Of special note turns out to be the understanding of $K_{\overline{B}}$ along the fibres of the extended period mapping. A typical result gives conditions under which the logarithmic canonical bundle $K_{\overline{B}}(Z)$ should be free or ample.

The former is always the situation in the classical case where $K_{\overline{B}}(Z)$ is the pullback to a toroidal compactification of a $\Gamma\backslash D$ of the ample "automorphic form" line bundle on the Satake-Baily-Borel completion. At the other extreme, in the general case there is a natural Gauss mapping associated to the level 1 extension data and it is the non-degeneracy of this mapping that gives the condition for ampleness of $K_{\overline{B}}(Z)$. Specifically, under assumptions (the most important of which is a non-degeneracy one) given at the beginning of Section VIII, in terms of the Gauss mapping we give necessary and sufficient conditions that $K_{\overline{B}}(Z)$ be ample.

With the notations explained below the precise result is

(I.3) *Assume the local condition that the natural map*

$$T_{\overline{B}}(-\log Z) \to F^{-1}\,\mathrm{End}(E_e)$$

is injective. Let B_p be a fibre of Φ_e and

$$G(\Phi_1) : B_p \to \mathrm{Grass}(k, T_e J_{ab})$$

the Gauss mapping associated to $G(\Phi_1)$. Then $K_{\overline{B}} + Z$ is big and nef, and it is ample if, and only if $G(\Phi_1)$ is locally injective.

Notations and terminology

- $\Phi : B \to \Gamma\backslash D$ denotes a period mapping from a smooth, quasi-projective variety B to the quotient of a period domain $D = G_{\mathbb{R}}/H$;
- this period mapping corresponds to a variation of weight n polarized Hodge structures over B; we denote by $F^p \to B$ the corresponding Hodge bundles;
- the *augmented Hodge line bundle* is defined to be

$$L = \bigotimes_{p=0}^{[(n-1)/2]} \det(\mathrm{Gr}^{n-p} F)^{n_p} = \bigotimes_{p=0}^{[(n-1)/2]} \det F^{n-p}$$

where $[n_p = (n-p+1)/2]$; it has a canonical Hodge metric with Chern form ω, and for $\xi \in T_b B$ we have

$$\text{(I.4)} \qquad \|\phi_*(\xi)\|^2 = \omega(\xi)$$

where the left-hand side is given by the $G_{\mathbb{R}}$-invariant metric on TD;

- the image $P = \Phi(B) \subset \Gamma\backslash D$ is a locally closed analytic subvariety that has a canonical algebraic structure over which $L \to P$ is ample (cf. [BBT]);
- we assume that B has a smooth projective completion $\overline{B}$ such that the divisor at infinity $Z := \overline{B}\backslash B$ is a reduced normal crossing divisor $\sum_i Z_i$;
- we assume that the local monodromies around the Z_i are unipotent with logarithm N_i, and we denote by $F_e^p \to \overline{B}$, $L_e \to \overline{B}$ etc. the canonical Deligne extensions of the bundles F^p, L_e;
- there is a canonical completion $\overline{P}$ as a compact Hausdorff space to which the period mapping extends to a proper continuous mapping

$$\Phi_0 := \Phi_e : \overline{B} \to \overline{P};$$

$\overline{P}$ is stratified by complex analytic subvarieties and Φ_e is holomorphic on the inverse images of these strata (cf. [GGLR]);

- we set
$$\Phi_e\big|_{Z_I^*} = \Phi_I : Z_I^* \to P_I \subset \Gamma_I \backslash P_I;$$
- it is conjectured that $L_e \to \overline{B}$ is free[‖] and that $\overline{P} = \mathrm{Proj}(L_e)$; in Section VII we will prove that when $\dim B = 2$, L_e is free and that at the set-theoretic level $\overline{P} = \mathrm{Proj}(L_e)$;

[‖] By the results of Satake-Baily-Borel this is true in the classical case when D is a Hermitian symmetric domain. In the non-classical case it holds when $\dim B = 2$, when $-K_{\overline{B}}$ is nef, and in a number of further special cases.

- as mentioned above we will introduce inductively defined maps $\Phi_0, \Phi_1, \Phi_2, \ldots$ where
  - $\Phi_0 = \Phi_e$,
  - Φ_1 is defined on the fibres of Φ_0 and maps to level one extension data;
  - Φ_2 is defined on the fibres of Φ_0 and Φ_1 and maps to level two extension data of level $\leqq 2$.

    $\vdots$

  We will see that the Φ_k for $k \geqq 3$ are determined up to constants by Φ_0, Φ_1, and Φ_2.

II. Extension data for a mixed Hodge structure

In these notes we assume the existence of a lattice $V_{\mathbb{Z}}$ in the $\mathbb{Q}$-vector space V. In this section we will consider extension data for mixed Hodge structures $(V, W, F^{\bullet})$. The weight filtration will be

$$\{0\} \subset W_0 \subset W_1 \subset \cdots \subset W_n = V$$

and the graded quotients are weight k pure Hodge structures

$$H^k = \mathrm{Gr}_k^W(V).$$

We will consider only those extensions for which the H^k are fixed Hodge structures, and $\mathcal{E}$ will denote the set of all such. There is a filtration

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

where $\mathcal{E}_m$ denotes the set of at most m-fold extensions in $\mathcal{E}$, which we will refer to as extension of level $\leqq m$.

For example, $\mathcal{E}_1$ is the set of extensions

$$0 \to H^k \to W_{k+1}/W_{k-1} \to H^{k+1} \to 0,$$

$\mathcal{E}_2$ is the set of these extensions plus the extensions

$$0 \to H^{k-2} \to W_k/W_{k-3} \to W_k/W_{k-2} \to 0.$$

Equivalently, the filtration of $\mathcal{E}$ by levels is the one induced on the sets of extensions by the filtration $W_\bullet \operatorname{End}(V)$. Thus the level 1 information is in $\operatorname{Gr}^W_{-1} \operatorname{End}(V)$, the level 2 information is in $W_{-2} \operatorname{End}(V)/W_0 \operatorname{End}(V)$, and so forth. One might say that the level 2 information reflects extensions of extensions from level 1.

We note that the level 1 extension data is equivalently given by

$$\bigoplus_{k=1}^{n} \mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-1}).^\dagger \tag{II.1}$$

The main points concerning the structure of $\mathcal{E}$ are

(i) $\mathcal{E}$ is a complex manifold that is an iterated fibration

$$\mathcal{E}_{k+1} \to \mathcal{E}_k; \tag{II.2}$$

(ii) the fibres of (II.2) are connected, abelian complex Lie groups that are extensions of a compact complex torus by a product of C^*'s and $\mathbb{C}$'s; we will refer to these are semi-abelian-tori;

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† In contrast, one has for any mixed Hodge structures that

$$\mathrm{Ext}^i_{\mathrm{MHS}}(A, B) = 0, \qquad i \geqq 2.$$

For reference a proof of this well-known fact will be given in the appendix to this section.

(iii) $\mathcal{E}_1$ given by (II.1) is a sum of compact, complex tori

$$J_k := \mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-1}) \cong \frac{\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1})}{F^0\,\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1}) + \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})}.$$

Here $\mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1}) := \mathrm{Hom}(H^k_{\mathbb{Z}}, H^{k-1}_{\mathbb{Z}})$ where we use the integral structures induced by $V_{\mathbb{Z}}$; $\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1}) = \mathrm{Hom}(H^k_{\mathbb{C}}, H^{k-1}_{\mathbb{C}})$.

(iv) In J_k there is a compact sub-torus

$$J_{k,ab} \subset J_k$$

given by intersecting $\mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})$ with the $(0,-1) \oplus (-1,0)$ part of the weight -1 Hodge structure whose complexification is $\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1})$. We set

$$\begin{cases} J = \overset{k}{\oplus} J_k \\ J_{ab} = \oplus J_{k,ab}; \end{cases}$$

(v) the fibration $\mathcal{E}_2 \to \mathcal{E}_1$ is given by

$$\left\{\begin{array}{ccc} \dfrac{\overset{k}{\oplus}\,\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-2})}{F^0\,\mathrm{Hom}_{\mathbb{C}} + \mathrm{Hom}_{\mathbb{Z}}} & \longrightarrow & \mathcal{E}_2 \\ & & \downarrow \\ & & \dfrac{\overset{k}{\oplus}\,\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-1})}{F^0\,\mathrm{Hom}_{\mathbb{C}} + \mathrm{Hom}_{\mathbb{Z}}}, \end{array}\right.$$

or equivalently by

(II.3)

$$\left\{\begin{array}{ccc} \dfrac{\mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{C}}(V)}{F^0\,\mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{C}}(V) + \mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{Z}}(V)} & \longrightarrow & \mathcal{E}_2 \\ & & \downarrow \\ & & \dfrac{\mathrm{Gr}^W_{-1}\,\mathrm{End}(V)}{F^0\,\mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{C}}(V) + \mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{Z}}(V)} \end{array}\right.$$

where the F^0's in the denominator are induced by the Hodge filtration on the numerator.

(vi) the topological line bundles over J are uniquely specified by their Chern classes. Noting that $\mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{Z}}(V)$ is the lattice that defines J as a quotient of a complex vector space we have

$$\left\{ \begin{array}{c} \text{topological line} \\ \text{bundles over } J \end{array} \right\} \cong \wedge^2 \mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{Z}}(V)^*.$$

Using the mapping

$$\mathrm{Gr}^W_{-1}\,\mathrm{End}(V) \otimes \mathrm{Gr}^W_{-1}\,\mathrm{End}(V) \to \mathrm{Gr}^W_{-2}\,\mathrm{End}(V)$$

given by composition, by dualizing we obtain

$$\text{(II.4)} \qquad \begin{array}{ccc} \mathrm{Gr}^W_{-2}\,\mathrm{End}_{\mathbb{Z}}(V)^* & \longrightarrow & \wedge^2 \mathrm{Gr}^W_{-1}\,\mathrm{End}_{\mathbb{Z}}(V)^* \\ & & \wr\| \\ & & H^2(J,\mathbb{Z}). \end{array}$$

The elements in $\mathrm{Gr}^{W}_{-2}\,\mathrm{End}_{\mathbb{Z}}(V)^*$ that map to (1,1) classes in $H^2(J,\mathbb{Z})$ give rise to holomorphic line bundles on J; they are well defined up to translation.
We next observe that $\mathrm{Gr}^{W}_{-2}\,\mathrm{End}_{\mathbb{Z}}(V)^*$ is naturally isomorphic to H^1(fibres of (II.3), $\mathbb{Z}$).The mapping (II.4) may be identified with the transgression mapping

$$H^0(H^1(\text{fibre})) \xrightarrow{d_2} H^2(\text{base})$$

in the Leray spectral sequence of the fibration (II.3).

Summarizing in words: *For a set of mixed Hodge structures with fixed associated graded Hodge structure, the level 1 extension data is a (direct sum of) compact complex tori. The level 2 extension data is a complex manifold that fibres holomorphically over the level 1 extension data with fibres consisting of semi-abelian-tori.*

We emphasize that although we cannot identify the V with a fixed vector space along the extension data of levels $\leqq 2$, because of the special action of monodromy the above identifications have intrinsic meaning.

Appendix to Section II

We will give a proof that for mixed Hodge structures A, B

(A.1) $$\mathrm{Ext}^2_{\mathrm{MHS}}(B, A) = 0.$$

The idea behind the argument can be used to show that for $k \geqq 2$ all of the $\mathrm{Ext}^k_{\mathrm{MHS}}(B, A) = 0$.

Recall that $\mathrm{Ext}^k_{\mathrm{MHS}}(B, A)$ is generated by exact sequences of mixed Hodge structures

$$0 \longrightarrow A \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_k \longrightarrow B \longrightarrow 0$$

modulo equivalences generated by commutative diagrams

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A & \longrightarrow & E'_1 & \longrightarrow & \cdots & \longrightarrow & E'_k & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_k & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where the maps are morphisms of MHS's. The zero element is the class of

$$0 \longrightarrow A \xrightarrow{=} A \longrightarrow 0 \cdots 0 \longrightarrow B \xrightarrow{=} B \longrightarrow 0.$$

To establish (A.1) we will proceed in two steps.

Step one: Given

$$0 \longrightarrow A \xrightarrow{i} E_1 \xrightarrow{f} E_2 \xrightarrow{\pi} B \longrightarrow 0$$

then for $C = \operatorname{im} f$ we have the solid arrows in

(A.2)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & {\scriptstyle c}\uparrow & & \uparrow & & \\
 & & & & B & = & B & & \\
 & & & & \uparrow & & \uparrow{\scriptstyle \pi} & & \\
0 & \dashrightarrow & A & \overset{a}{\dashrightarrow} & G & \overset{b}{\dashrightarrow} & E_2 & \dashrightarrow & 0 \\
 & & \| & & {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & E_1 & \xrightarrow{h} & C & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We claim that if we can find a mixed Hodge structure G so that the dotted arrows can be filled in to give a commutative diagram, then we will have (A.1).
To verify this, given (A.2) we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & E_1 & \xrightarrow{g\circ h} & E_2 & \xrightarrow{\pi} & B & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \mathrm{id}\oplus\pi} & & \| & & \\
0 & \longrightarrow & A & \xrightarrow[a]{} & G & \xrightarrow[b\oplus c]{} & E_2 \oplus B & \xrightarrow[\pi\oplus\mathrm{id}]{} & B & \longrightarrow & 0.
\end{array}$$

One checks commutativity so that this diagram gives an equivalence of extensions.
Next we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{=} & A & \xrightarrow{0} & B & \xrightarrow{=} & B & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 0\oplus\mathrm{id}} & & \| & & \\
0 & \longrightarrow & A & \xrightarrow[a]{} & G & \xrightarrow[b\oplus c]{} & E_2 \oplus B & \xrightarrow[\pi\oplus\mathrm{id}]{} & B & \longrightarrow & 0
\end{array}$$

which also gives an equivalence of extensions. Combining these gives (A.1).

Step two: We note that the desired G has a filtration

$$A \subset E_1 \subset G$$

with graded pieces A, C, B. To construct it we need

$$e_1 \in \mathrm{Ext}^1_{\mathrm{MHS}}(B, E_1);$$

i.e.,

$$e_1 \in \frac{\mathrm{Hom}_{\mathbb{C}}(B, E_1)}{F^0 \mathrm{Hom}_{\mathbb{C}}(B, E_1) + \mathrm{Hom}_{\mathbb{Z}}(B, E_1)} \xrightarrow{\quad h \quad} \frac{\mathrm{Hom}_{\mathbb{C}}(B, C)}{F^0 \mathrm{Hom}_{\mathbb{C}}(B, C) + \mathrm{Hom}_{\mathbb{Z}}(B, C)}$$

gives the extension class of E_2. But

$$h : \mathrm{Hom}(B, E_1) \to \mathrm{Hom}(B, C)$$

is surjective since both $E \twoheadrightarrow C$ and $B \oplus E_1 \twoheadrightarrow B \oplus C$ are surjective.

If G is the extension defined by e_1, then we have

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & B & = & B & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & E_2 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and we are done. □

The key is given A, B, C where $0 \to A \to E_1 \to C \to 0$ to be able to construct G with these graded pieces. This is possible because there are a lot of mixed Hodge structures.

III. Extension data for limiting mixed Hodge structures

We assume given a limiting mixed Hodge structure $(V, Q, W(N), F^\bullet)$. With the standard notations ([GGLR]) we have

- $N : W_k(N) \to W_{k-2}(N)$ and

$$N^k : \mathrm{Gr}^{W(N)}_{n+k}(V) \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{n-k}(V);$$

- $Q : V \otimes V \to \mathbb{Q}$ and $N \in W_{-2}(N)\,\mathrm{End}(V)$ preserves Q.

The Q will always be present but we shall omit it in the notation; thus, e.g., it is understood that

$$\mathrm{End}(V) = \mathrm{End}(V, Q);$$

- From an equivalent but alternative perspective, there is a non-degenerate pairing

$$\mathrm{Gr}^{W(N)}_{n+k} V \otimes \mathrm{Gr}^{W(N)}_{n-k} \to \mathbb{Q}$$

given by

$$u \otimes v \to Q(u, v).$$

This gives an isomorphism

$$H^i(-(n-i))^* \cong H^i$$

and then

$$N^{n-i} \in H^i(-(n-i))^* \otimes H^i \cong H^i \otimes H^i.$$

In fact, N^{n-i} lies in the symmetric part

$$N^{n-i} \in S^2 H^i \cong S^2 H^i(-(n-i))^*.$$

We shall use these identifications without further comment.

- acting on $\mathrm{Gr}_{\bullet}^{W(N)}(V)$, N can uniquely be completed to an $\mathrm{sl}_2\{N, H, N^+\}$ and preserving Q; here $H = (\ell - n)\,\mathrm{Id}$ on $\mathrm{Gr}_{\ell}^{W(N)}(V)$;
- we decompose $\mathrm{Gr}_{\bullet}^{W(N)}(V)$ into a direct sum of irreducible sl_2-modules; the resulting summands will be called N-strings;
- under this decomposition the $\mathrm{Gr}_{k}^{W(N)}(V)$'s are direct sums of polarizable Hodge structures; by abuse of language we shall simply refer to $\mathrm{Gr}_{k}^{W(N)}(V)$ as a polarized Hodge structure.

Now we come to the main point:

(III.1) *Over the level one extension data there is a canonical line bundle* $L_N \to J$ *such that*

$$L_N \to J_{ab}$$

is positive (ample).

Proof: This is a consequence of the following diagram in which it is to be understood that we are restricting to the lattices $V_{\mathbb{Z}} \subset V$ and that $N : V_{\mathbb{Z}} \to V_{\mathbb{Z}}$ (we may have to replace N by a multiple to achieve this), and where the top horizontal arrow is defined by commutativity of the diagram:

$$
\text{(III.2)}\qquad
\begin{array}{ccc}
\mathrm{Gr}^{W(N)}_{-1}\,\mathrm{End}_{\mathbb{Z}}(V) \otimes \mathrm{Gr}^{W(N)}_{-1}\,\mathrm{End}_{\mathbb{Z}}(V) & \longrightarrow & \mathbb{Z} \\
\downarrow & & \nearrow_{\mathrm{Tr}} \\
\mathrm{Gr}^{W(N)}_{-2}\,\mathrm{End}_{\mathbb{Z}}(V) & & \\
\downarrow N^{+} & & \\
\mathrm{Gr}^{W(N)}_{0}\,\mathrm{End}_{\mathbb{Z}}(V); & &
\end{array}
$$

here the top vertical arrow is composition of endomorphisms and Tr is the trace. Referring to (II.4) one may check that

- the element of $H^2(J,\mathbb{Z})$ defined by (II.4) via (III.2) is of Hodge type (1,1);
- as a consequence of the 2^{nd} Hodge-Riemann bilinear relation for polarized Hodge structures this class is represented by a positive (1,1) form.

- In general the Hodge classes in $\mathrm{Gr}_{-2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)$ are dual to the Hodge classes in $\mathrm{Gr}_{+2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)$, which in particular are $(1,1)$ classes. This is the reason that N^+ appears, here using that

$$Q(N, N^+) \text{ is a positive integer.}$$

In general if we have a several variable limiting mixed Hodge structure that defines a monodromy cone

$$\sigma \subset \mathrm{Gr}_{-2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)$$

consisting of Hodge classes, there is a dual cone

$$\begin{aligned} \check{\sigma} \subset \mathrm{Gr}_{-2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)^* &\cong \mathrm{Gr}_{2}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V) \\ &\cong H^2(J, \mathbb{Z}). \end{aligned}$$

Here we are using that any $N \in \sigma$ defines the same weight filtration. Although $W(\sigma)$ would probably be better notation, we will continue to use $W(N)$, keeping in mind that the sl_2 and resulting polarizations on the $\mathrm{Gr}_k^{W(N)}(V)$ depend on the particular N.
For any $A \in \check{\sigma} \otimes \mathbb{Z}$ the above identification defines a line bundle $L_A \to J$ with the property that

$$L_A \to J_{ab} \text{ is positive if } A \in \check{\sigma}.$$

In words,

> *The integral elements $A \in \check{\sigma} \otimes \mathbb{Z}$ define line bundles $L_A \to J$ over the level 1 extension data. For $A \in \check{\sigma}$ the line bundle $L_A \to J_{ab}$ is ample over the Hodge part J_{ab} of J.*

We note that to define L_A with the above properties we must have the structure of level 2 extension data plus the properties of LMHS's.

The above raises two natural questions. The first is that for a limiting mixed Hodge structure the level 1 extension data is a direct sum of compact tori that using

$$H^i(-(n-i))^* \cong H^i$$

occur in dual pairs, with the understanding that when n is even the middle term is self dual. One may ask if the Poincaré line bundle is a part of the picture, and the answer is that it doesn't seem to be.

More interesting are the following questions: There is a bijection

$$\left\{\begin{matrix}\text{equivalence classes of}\\ \text{limiting mixed Hodge}\\ \text{structures}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{equivalence classes}\\ \text{of nilpotent orbits}\end{matrix}\right\}$$

What is the relation between the extension data associated to a given limiting mixed Hodge structure and the extension data associated to an equivalent limiting mixed Hodge structure? What is the relation between the natural line bundles over each? And referring to the above diagram containing the $\leftrightarrow$, what , if any, is the relation between the natural line bundles over the level 1 extension data on the left and the 1^{st} order "smoothing" variation of LMHS's that arise on the right?[‡] The interesting and somewhat subtle answer to this question will be taken up in the next section.

[‡]In the geometric case where we have a family of smooth varieties X_b parametrized by B and singular varieties lying over Z, then for each point $b_0 \in Z$ there is an equivalence class of limiting mixed Hodge structures $\lim_{b \to b_0} H^n(X_b)$. Thus in general taking the limiting mixed Hodge structure at b_0 and moving it out into B in a normal direction to Z may be thought of as "smoothing" the LMHS at b_0.

IV. Period mappings to extension data (A)

With the setup and notation explained in Section I, we denote by B_p a connected component of a fibre of the extended period mapping Φ_e. We recall the following general facts:

- B_p is complete and is contained in the closure of $\overline{Z}_I$ of a unique minimal stratum $Z_I = \bigcap_{i \in I} Z_i$;
- $B_p^* := B_p \backslash$(the union of B_p intersect the strata $Z_{I \cup \{j\}}$ for $j \notin I$) is a Zariski open $B_p \cap Z_I^*$ in B_p;
- along B_p^* we have a variation of limiting mixed Hodge structures with locally constant associated graded pure Hodge structures; the behavior of the variation of limiting mixed Hodge structures along Z_I^* at the intersection with other strata is developed in [CKS];

- we denote by $\mathcal{E}$ the set of extension data, as described in sections II and III above, for the locally constant associated graded to the LMHS's along B_p^*;
- Γ_I will denote the action of the monodromy on $\mathcal{E}$ induced by the monodromy action of $\pi_1(B_p^*)$ on the family of limiting mixed Hodge structures along B_p^*.

We will see that this monodromy action acting on the complex torus J given by the level 1 extension data is at most a finite group, one that we shall at least initially ignore. Thus in the obvious way we may define an Abel-Jacobi type mapping

$$\Phi_1 : B_p^* \to J \tag{IV.1}$$

by assigning to each point $b \in B_p^*$ the level 1 extension data in the limiting mixed Hodge structure $\Phi_e(b)$. The main result is the

Theorem (IV.2): *The mapping* (IV.1) *extends to a mapping on all of B_p. There it maps to a translate of the abelian subvariety J_{ab} of the complex torus J and we have*

$$\Phi_1^* L_A = -\sum_k \langle A, N_k \rangle \, [Z_k] \big|_{B_p}. \tag{IV.3}$$

Here $L_A \to J$ is the line bundle defined for $A \in \check{\sigma}_I \otimes \mathbb{Z}$ in Section III above. The sum is over all the line bundles $[Z_k]\big|_{B_p}$, noting that this bundle is trivial if $Z_k \cap B_p = \emptyset$ is empty.

Corollary (IV.4): *If $B_p \subset Z_i$ does not meet other strata and $\langle A, N_i \rangle > 0$, then $\Phi_1^* L_A \to B_p$ is a positive line bundle.*

Proof of (IV.3): We begin by explaining the central idea behind the first two assertions with some of the details given below and the rest to be provided elsewhere. For the first statement the mapping (IV.1) induces the top row in

$$\begin{array}{ccc} H_1(B_p^*, \mathbb{Z}) & \xrightarrow{\Phi_{1,*}} & H_1(J, \mathbb{Z}) \\ & \searrow \qquad \nearrow & \\ & H_1(B_p, \mathbb{Z}) & \end{array}$$

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Remark: In (IV.3) we have implicitly thought of A and the N_k as being in $\mathrm{End}(V)$ for a fixed vector space V. The point is the pairing

$$\mathrm{Gr}_{+2}(\mathrm{End}\, V) \otimes \mathrm{Gr}_{-2}(\mathrm{End}\, V) \to \mathbb{Q}$$

is monodromy invariant. The detailed properties of monodromy will be in the Lie theoretic approach that will appear elsewhere.

This top row is a morphism of mixed Hodge structures. By a weight argument the kernel of the slanted solid arrow is of strictly lower weight than the weight of the pure Hodge structure on $H_1(J, \mathbb{Z})$. Therefore the mapping $\Phi_{1,*}$ factors in the way indicated by the dotted arrow in the diagram. Thus there is an induced morphism of Hodge structures

$$H_1(\mathrm{Alb}\, B_p, \mathbb{Z}) \to H_1(J, \mathbb{Z}) \tag{IV.5}$$

where $\mathrm{Alb}\, B_p$ is the Albanese variety of (any desingularization of) the complete variety B_p. From this it follows that Φ_1 in (IV.1) extends to give the composed mapping in

$$B_p \to \mathrm{Alb}\, B_p \to J$$

where the second arrow is induced by the mapping in (IV.5).

For the second part of the theorem, $\mathrm{Gr}_{-1}^{W(N)} \mathrm{End}(V)$ is a pure Hodge structure of weight -1. The Hodge decomposition of this pure Hodge structure looks like

$$(m-1,-m) + \cdots + \underbrace{(0,-1) + (-1,0)} + \cdots + (-m, m-1).$$

The induced morphism of Hodge structures

$$H_1(\mathrm{Alb}\, B_p) \to H_1(J)$$

has image in a Tate twist of the term over the brackets in the above direct sum. It also lands in a subgroup of $H_1(\mathrm{Gr}_{-1}^{W(N)} \mathrm{End}(V), \mathbb{Z})$, which implies that $\Phi_1(B_p)$ lands in a translate of J_{ab}.

In somewhat more detail and from a slightly different perspective, suppose we inductively define maps $\Phi_1, \Phi_2, \ldots$ by

- Φ_1 maps B_p to level 1 extension data;
- Φ_2 maps the fibres of Φ_1 to level 2 extension data

and so forth. Let $C \subset B_p$ be an irreducible curve that is a fibre of $\Phi_1, \ldots, \Phi_{k-1}$. Then there is a Zariski open set $C \backslash D$ in C and a map

$$\Phi_k : C\backslash D \longrightarrow \frac{\mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \, \mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{C}}(V) + \mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)}$$

when the RHS is a semi-abelian-torus. This maps factors through

$$C\backslash D \longrightarrow \frac{H^1(C, D; \mathbb{C})}{F^1 H^1(C, D; \mathbb{C}) + H^1(C, D; \mathbb{Z})}.$$

The numerator is the mixed Hodge structure $H^1(C, D)$ with associated graded described by

$$0 \longrightarrow \frac{H^0(D)}{H^0(C)} \longrightarrow H^1(C, D) \longrightarrow H^1(C) \longrightarrow 0.$$

We then have

$$\begin{array}{ccc} C \backslash D & & \\ \downarrow & \searrow & \\ \frac{H^1(C,D;\mathbb{C})}{F^1 H^1(C,D;\mathbb{C}) + H^1(C,D;\mathbb{Z})} & \longrightarrow & \frac{\mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \, \mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{C}}(V) + \mathrm{Gr}_{-k}^{W(N)} \mathrm{End}_{\mathbb{Z}}(V)}. \end{array}$$

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Then $\Phi_{k,*}$ lands in $F^{-1}\,\mathrm{Gr}_{-k}^{W(N)}\,\mathrm{End}_{\mathbb{C}}(V)$, which has Hodge type

$$(-1,-(k-1))+(0,-k)+\cdots+(-(k-1),-1).$$

It follows that (with the hopefully obvious notation)

$$H^1(C,D;\mathbb{Z})\to F^{-1}\,\mathrm{Gr}_k\cap\overline{F^{-1}\,\mathrm{Gr}_k}.$$

For

$$\begin{array}{ll} k=1 & \text{this is } (-1,0)+(0,-1)\\ k=2 & \text{this is } (-1,-1)\\ k\geqq 2 & \text{this is empty.}\end{array}$$

As a consequence

(IV.6) *For $k\geqq 3$ the maps Φ_k are determined up to constants by Φ_1,Φ_2.*

This is a rather remarkable geometric fact. It says that if in the fibres of Φ_e we consider the successive maps to extension data of increasingly higher levels, then up to constants these maps are already determined by what happens at the first two levels.

In the classical case this is perhaps not so surprising. For $n = 1$ there are only two levels. Any Hermitian symmetric domain may be equivariantly embedded in the period domain for polarized Hodge structures of weight $n = 1$.

In the non-classical case it is consistent with the general philosophy that even when the period domain is not Hermitian symmetric, period mappings behave in much the same way as in the classical case.

We now turn to a discussion of the main formula (IV.3). A formal proof of this will be given elsewhere. Here we will present some special cases that illustrate why the result should be true. We begin with the simplest non-trivial case:

- $n = 1$ and $g = 2$ (here g is $h^{1,0}$ for a weight 1 PHS (V, Q, F));
- $\dim B = 2$ and there are local coordinates (t, w) where Z is given by $t = 0$;
- the normalized period matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \lambda \\ \beta & \alpha \end{pmatrix}$$

where $\alpha(t, w), \lambda(t, w)$ with $\operatorname{Im} \lambda > 0$ are holomorphic and

$$\beta(t, w) = \ell(t) + b(t, w)$$

where $\ell(t) = \log t/2\pi i$ and $b(t, w)$ is holomorphic.

The fibres of Φ_e are given by $\lambda =$ constant. In this case the Q, N and weight filtration are given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \begin{matrix} \\ \\ \\ \} W_0 \end{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} W_1 \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} W_2 = V$$

$$F^1 \cap W_1 = \begin{pmatrix} 0 \\ 1 \\ \lambda \\ \alpha \end{pmatrix}.$$

The level 1 extension data is given by $\alpha(t, w)$ and the level 2 extension data by $\beta(t, w)$. An element A in the dual cone $\check{\sigma}$ is given by a positive integer a, and the pairing that defines the line bundle L_A works out to give that

$$\exp(2^{\text{nd}} \text{ level extension data}) = \exp(2\pi i a\beta(t, w)) = t^a \exp(2\pi i b(t, w))$$

is a nowhere vanishing section of L_{-A}. On the other hand, the LHS gives the transition functions for the line bundle L_{-A}. From this the result follows.

Continuation of example: The connection between the variation of the extension data along a fibre of Φ_e and sections of the canonical bundle to Z along the fibre was first suggested by a calculation that unexpectedly popped out a theta function. Since that calculation is central to these notes we will give it here. This is a continuation of the earlier example with the notation changes $\alpha \to a$ and $\beta \to b$.

The point is to consider the action on the period matrix of monodromy, both from turning around a singular fibre (action of $\exp N$) and by the global action of π_1 (fibre).[§] The latter is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & \lambda \\ b & a \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ c & 1 \\ a+d & \lambda \\ b+c-\frac{c}{2}(d-c\lambda) & a+d-c\lambda \end{pmatrix}.$$

[§] The normalized period matrix is only locally well defined along the fibre. Globally the action of monodromy must be considered; we seek quantities that are single-valued under this action.

By elementary column operations after renormalizing the period matrix we obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a + d - c\lambda & \lambda \\ b + c - \frac{c}{2}(d - c\lambda) + ca & a + d - c\lambda \end{pmatrix}.$$

Here $c \in \mathbb{Z}$ and for purposes of illustration we use $d \in 2\mathbb{Z}$ so that $\frac{cd}{2} \in \mathbb{Z}$. Then

$$a \in \mathbb{C}/\{\mathbb{Z}, \lambda\mathbb{Z}\}$$

which just says that $a \in \mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^2)$. Under the action of monodromy we have

$$e^{2\pi ib} \to e^{2\pi ib} e^{2\pi i(ac - \frac{c}{2}(d - c\lambda))} = e^{2\pi ib} e^{2\pi i(ca - \frac{c}{2}^2\lambda)}.$$

This is the functional equation for the classical θ-function. We may normalize t by $t \to e^{u(t,w)t}$ to have $b = \ell(t)$ and then the above calculation says that

$$t \times \left\{ \begin{matrix} \text{section of the dual} \\ \text{of the } \theta\text{-line bundle} \end{matrix} \right\}$$

is *globally* well defined; i.e.,

> *the pullback of the θ-line bundle under the extension class mapping is the co-normal bundle.*

It was this calculation that suggested (IV.3).

An interpretation of equation (IV.3): It is well known that given a VHS with unipotent monodromy $T = \exp N$ over the punctured disc $\Delta^* = \{0 < |t| < 1\}$, there is a well-defined *equivalence* class of limiting mixed Hodge structures $(V, Q, W(N), F)$ at the origin. The equivalence relation is

$$(V, Q, W(N), F) \sim (V, Q, W(N), F') \iff F' = \exp(zN) \cdot F$$

for some $z \in \mathbb{C}$. To get rid of the equivalence relation one frequently says

> *There is a unique limiting mixed Hodge structure associated to the VHS over Δ^* together with a choice of non-zero $\eta \in T_0^*\Delta$.*

Given the situation of the notes where for simplicity of illustration we assume that Z is irreducible, for a fibre $B_p \subset Z$ at each point $b \in B_b$ there is a well-defined equivalence class of limiting mixed Hodge structures.

These limiting mixed Hodge structures all have the same associated graded polarized Hodge structure. What equation (IV.3) does is to give a line bundle $\Phi_1^* L_{N^+} \to B_p$ such that the points in this line bundle minus the 0-section define a unique limiting mixed Hodge at b.

Returning to the proof of Theorem (IV.2), the points in the argument are as follows:

- For a compact complex torus $T = \mathbb{C}^m / \Lambda$ there is the identification
$$H^2(T, \mathbb{Z}) \cong \wedge^2(\Lambda^*);$$
- For a $(1,1)$ class A in $\wedge^2(\Lambda^*)$ there is a holomorphic line bundle $L_A \to T$, defined up to translation and whose Chern class is A;
- The pullback to $\mathbb{C}^m$ of this bundle is trivial, and it is defined by "transition functions" that are exponentials of linear functions on $\mathbb{C}^m$ whose coefficients are linear in the entries of A;

- Thus the sections of $L_A \to T$ are given by holomorphic functions $\theta(z)$ satisfying $\theta(z+\lambda) = exp\,\langle \ell(A,\lambda), z\rangle$ where $\ell(A,\lambda)$ is linear in A and satisfies a cocycle rule in $\lambda \in \Lambda$. In the above special case the period matrix has been normalized to be of the form $\binom{I}{\Omega}$. In the above special case where $\Omega = \left(\begin{smallmatrix} \alpha & \lambda \\ \beta & \alpha \end{smallmatrix}\right)$ the quotient of $\mathbb{C}^4$ by I-part of the lattice gives transition functions that are trivial; i.e., the line bundle L_A descends to $\mathbb{C}^* \times \mathbb{C}^*$. The transition functions for the second part of the lattice are given by exponentials of the Ω-part of the period matrix. If we cover Z by the open sets $\mathcal{U}_k$ where Z is defined by $t_k = 0$, then in $\mathcal{U}_k \cap \mathcal{U}_\ell$ we have $t_k = f_{k\ell} t_\ell$ where $f_{k\ell}\big|_Z$ give the transition functions for $N^*_{Z/\overline{B}}$. This then identifies the transition functions for $N^*_{Z/\overline{B}}$ with those of $\Phi_1^* L_{-A}$.

- In Section III above we have explained how level 2 extension data gives $(1, 1)$ classes on the compact, complex torus J given by level 1 extension data.

The above calculation carries out this general procedure in the special case described there. The normal bundle to Z appears because the level 2 extension data has an $\ell(t)N$ term in the matrix representing it and exponentiating $\ell(t)N+$ (holomorphic term) produces a "t" that is a local section of $N^*_{Z/\overline{B}}$. For $n \geqq 2$ the period matrix will generally contain $(\ell(t)N)^m$ terms where $m \geqq 2$. The general rule is

- the fibres of the level k extension data over the level $k-1$ extension data are contained in

$$\text{(IV.7)} \qquad \mathrm{Ext}^2_{\mathrm{MHS}}(H^{m+k}, H^m)\text{'s;}$$

- for $k = 2m$, $m \geqq 1$ we get $(\ell(t)N)^m$-terms appearing in the $W_{-k}\,\mathrm{End}(V)$'s;
- thus no matter what the weight n of the original polarized Hodge structure is, only for level 2 extensions corresponding to $W_{-2}\,\mathrm{End}(V)$ (thus $m = 1, k = 2$) do we get $\ell(t)N$'s in the group (IV.7)

A consequence of (IV.6) is

(IV.8) *Let B_p be a connected fibre of Φ_e and*

$$B_p \to \Gamma_I \backslash \mathcal{E}$$

the map to the extension data of the equivalence classes of limiting mixed Hodge structures along B_p. Then this map is constant if, and only if, the maps Φ_1 and Φ_2 to extension data of levels 1 and 2 are constant.

We remark that some further discussion is needed to make this statement precise; this will be provided in the work giving the Lie-theoretic approach.

We conclude this section with a result that although the hypotheses are quite restrictive is a harbinger of what one would like to prove and also indicates that the analysis given above will have interesting consequences.

Suppose that we have a variation of Hodge structure over B with an extension to $\overline{B}$ as described in Section I. The general question/conjecture is to show that *under local Torelli type assumptions there are non-negative integers a_i and an m_0 such that the line bundle*

$$mL - \sum a_i[Z_i] \tag{IV.9}$$

is ample for $m \geqq m_0$.

We shall show that this result holds under the following assumptions:

(i) the differential of Φ_e is 1-1 except along the fibres B_p;

(ii) along the fibres B_p the differential of Φ_1 is 1-1;

(iii) Z has one component; and

(iv) the cone $\mathrm{Eff}^1(\overline{B})$ of effective 1-cycles on $\overline{B}$ is finitely generated.

Then under these assumptions there is an m_0 such that

$$mL - [Z] \text{ is ample for } m \geqq m_0.$$

Proof: Given an irreducible curve $C \subset \overline{B}$ we have to show that

$$(mL - [Z]) \cdot C := \deg \left((mL - [Z])\big|_C \right) > 0 \qquad \text{(IV.10)}$$

for $m \gg 0$.

If C is not a fibre B_p of Φ_e, then using assumption (i) this is a consequence of

$$L \cdot C = \deg(L\big|_C) > 0.$$

If $C \subset B_p$, then by the basic formula (IV.3) using assumption (ii) we have

$$\text{(IV.3)} = \deg\left(N^*_{Z/\overline{B}}\big|_C\right) > 0.$$

Assumption (iii) is used in that C does not meet any other strata of Z, and (iv) is used in order for it to be sufficient to show (IV.10) for any fixed curve C. □

In general, from the construction proposed in [GGLR] for the Satake-Baily-Borel completion $\overline{P}$ of the image $P = \Phi(B) = \Gamma\backslash D$ of a period mapping, it is expected that for fibres $B_p = \Phi_e^{-1}(p)$, $p \in \overline{P}$ we will have positivity of the bundles

$$N^*_{Z/\overline{B}}\big|_{B_p} \to B_p. \tag{IV.11}$$

Modulo issues of the finite generation of the effective cone of curves on a variety, for smooth B_p's the ampleness of the above line bundle is equivalent to

$$\deg\left(N^*_{Z/\overline{B}}\big|_C\right) > 0 \tag{IV.12}$$

for test curves $C \subset B_p$. (IV.9) above is a result in this direction.

Given C there will be a smallest stratum Z_I with

$$C \subset Z_I$$

(thus I is a maximal index set with $C \subset Z_i, i \in I$). We also set $J = \{j : j \in I \text{ but } Z_j \cap C \neq \emptyset\}$.

We will show

(IV.13) *If $\Phi_1\big|_C$ is non-constant and the N_i, $i \in I$, are linearly independent, then* (IV.12) *holds.*

Proof: The cone σ_I is a face of $\sigma_{I\cup J}$, and for $A \in \check{\sigma}_I$ we have

$$\begin{cases} \langle A, N_i \rangle > 0, & i \in I \\ \langle A, N_j \rangle \geq 0, & j \in I. \end{cases}$$

For such A from (IV.3) and setting $d_i = \deg\left(N^*_{Z_i/\overline{B}}\big|_C\right)$ we have

$$0 < \deg\left(\Phi_1^* L_A\big|_C\right) = \sum_{i\in I} \langle A, N_i \rangle \, d_i - \sum_{j\in J} \langle A, N_j \rangle \, (Z_j \cdot C).$$

This gives

$$0 \leq \sum_{j \in J} \langle A, N_j \rangle (Z_j \cdot) < \sum_{i \in I} \langle A, N_i \rangle d_i .$$

Using the assumed linear independence of the N_i, $i \in I$ and letting A vary over $\check{\sigma}_I$ we may conclude (IV.12). □

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Discussion of the proof of (IV.3)

We know of two arguments for (IV.3). The first is an elaboration and extension of the above period matrix one. This is presented here because period matrix collaborations were how many aspects of Hodge theory were first understood. The second, to be given in a separate publication, will be the Lie theoretic, a method that (the very general cohomological/D-module techniques not withstanding) remains an extremely powerful approach to Hodge theory.

> *What do we mean by a period matrix associated to a limiting mixed Hodge structure?*

We will illustrate this when $n = 2$ and $N^2 \neq 0$. In this case the associated graded to the LMHS is a direct sum

$$H^0 \oplus H^1 \oplus \underbrace{H^0(-1) \oplus H^2}\oplus H^1(-1) \oplus H^0(-2)$$

where H^i is a Hodge structure of weight i and the arrows are the action of N (N-strings). We choose a basis for $V_{\mathbb{C}}$ adapted to $W(N)$ and shall use H to stand for a generic holomorphic term in (t, w). The period matrix is then

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(IV.14)

$$
\begin{array}{c}
\\
H^0(-2) \\
H^1(-1) \\
H^0(-1) \oplus H^3 \\
H^1 \\
H^0
\end{array}
\left(
\begin{array}{ccc}
H^0 & F^1 H^1 & F^2 H^2 \\
I & 0 & 0 \\
\boxed{H} & \begin{smallmatrix} I \\ \Lambda \end{smallmatrix} & 0 \\
\ell(t)N + H & \boxed{H} & I \\
H & \ell(t)N + H & \boxed{H} \\
(\ell(t)N)^2 + \ell(t)N + H & \ell(t)N + H & H
\end{array}
\right)
$$

The way to read this is to imagine a family of surfaces X_t degenerating to a surface $X = \cup X_i$ that has normal crossings with smooth components X_i, a double curve $D = \bigcup_{i<j} X_i \cap X_j$ and triple points $T = \bigcup_{i<j<k} X_i \cap X_j \cap X_k$.

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We denote by $\Omega^2_X(\log D)$ the sheaf of meromorphic 2-forms on X having log poles on D and with opposite residues on a component $X_i \cap X_j$ of D, and by $\Omega^1_D(\log T)$ the meromorphic 1-forms on D with log poles at the triple point p with a relation among the three residues from the branches of D through p. Then

$$\lim_{t\to 0} H^0\left(\Omega^2_{X_t}\right) = H^0\left(\Omega^2_X(\log D)\right)$$

where the RHS here is filtered by

(IV.15)

$$\begin{pmatrix}\text{double and single}\\ \text{residues vanish}\end{pmatrix} \subset \begin{pmatrix}\text{double residues}\\ \text{vanish}\end{pmatrix} \subset H^0\left(\Omega^2_X(\log T)\right)$$

$$\begin{matrix} \| & & \| \\ \begin{pmatrix}\text{kernel of the map}\\ \text{of } H^0\left(\Omega^2_X(\log D)\right) \text{ to}\\ H^0(T) \text{ and } H^0\left(\Omega^1_D(\log T)\right)\end{pmatrix} & & \begin{pmatrix}\text{kernel of the}\\ \text{map to } H^0(T)\end{pmatrix}\end{matrix}$$

The entries in the matrix (IV.14) are obtained by taking the rows to correspond to the associated graded to the weight filtration and the columns to the associated graded to the filtration (IV.15). The entries in the matrix (IV.14) represent $\mathrm{End}_{\mathbb{C}}(V)$ with the zero blocks in the upper right corresponding to the modding out by $F^0\,\mathrm{End}_{\mathbb{C}}(V)$.

Now we come to the punch line.

- the terms in the boxes $\square$ represent $\mathrm{Gr}_{-1}\,\mathrm{End}_{\mathbb{C}}(V)$;
- the terms in the circles $\bigcirc$ represent $\mathrm{Gr}_{-2}\,\mathrm{End}_{\mathbb{C}}(V)$;
- the terms in squiggles represent $\mathrm{Gr}_{-4}\,\mathrm{End}_{\mathbb{C}}(V)$.

This is consistent with the observation that in a period matrix only the terms in $\mathrm{Gr}_{-2m}\,\mathrm{End}_{\mathbb{C}}(V)$ have logarithmic entries and there the leading term is $\ell(t)^m$.

Recalling that the compact torus J is a quotient of its tangent space $\widetilde{J} := \mathrm{Gr}^W_{-1} \mathrm{End}_{\mathbb{C}}(V)/F^0 \,\mathrm{Gr}^W_{-1} \mathrm{End}_{\mathbb{C}}(V)$ by a lattice Γ, and that line bundles on $J = \widetilde{J}/\Gamma$ arise from Hodge classes in $\mathrm{Gr}^W_{-2} \mathrm{End}_{\mathbb{C}}(V)$ and they are constructed by taking quotients of the trivial bundle on $\widetilde{J}$ by cocycles of the form $\exp(2\pi i \times \text{linear function on } \widetilde{V}, \gamma)$ where $\gamma \in \Gamma$, we see why only $\exp \ell(t)$'s and not $\exp(\ell(t)^m)$'s for $m \geqq 2$ enter. Now $\exp \ell(t) = t$ is a local section of $N^*_{Z/\overline{B}}$, and this is what is behind the mechanism that relates information along a fibre B_p to information normal to fibres in $\overline{B}$.

Discussion of the proof of (IV.6)

We will illustrate by a period matrix calculation in the case $n = 2$ where the limiting mixed Hodge structure is Hodge-Tate. Then the period matrix is

$$\begin{array}{c} H^0(-2) \\ N \big\downarrow \\ H^0(-1) \\ N \big\downarrow \\ H^0 \end{array} \qquad \begin{pmatrix} I \\ \\ A = \ell(t) I + A_0 \\ \\ B = \frac{\ell(t)^2}{2} I + \ell(t) B_1 + B_0 \end{pmatrix}$$

where A_0, B_1, B_0 are holomorphic. We may take

$$Q = \begin{pmatrix} 0 & 0 & I \\ 0 & -I & 0 \\ I & 0 & 0 \end{pmatrix}.$$

Then the 1^{st} Hodge-Riemann bilinear relation is

$$B + {}^tB = {}^tAA.$$

The symmetric part of B

$$B_s = \frac{1}{2}{}^tAA$$

is determined by A.
The IPR is

$$dB = {}^tA\,dA. \tag{IV.16}$$

The level 2 extension data is given by A and the level 4 extension data by B. From (IV.16) in this case we see explicitly how the extension data of higher level is determined up to constants by the extension data of levels 1,2.

Allowing the fibre B_p to vary

We have defined the extension data mapping Φ_1 on a single fibre B_p of $\Phi_0 = \Phi_e$, and have used the level 2 extension data to define the line bundles $L_A \to B_p$ and derive the basic formula (IV.3). Here we will discuss the case when the fibre B_p varies.
With our usual notation $\Phi_I = \Phi_e\big|_{Z_I^*}$ we have

$$\text{(IV.17)} \qquad \Phi_I : Z_I^* \to P_I \subset \Gamma_I \backslash D_I.$$

This is a period mapping, but it is unlike just having a $\Phi : B \to P \subset \Gamma \backslash D$ because of the normal information to $Z_I^* \subset \overline{B}$. As reflected in the basic formula (IV.3) this gives additional structure to Φ_I. We shall now extend this picture, first under the simplifying assumption that (IV.16) is a fibration. Following that we shall comment on the steps needed to extend the discussion to the general case.

Thus we assume that (IV.17) is a fibration with fibres B_p^*. The completion B_p will meet other strata Z_j, $j \notin I$, in lower dimensional subvarieties. We have proved (Theorem IV.2) that the level 1 extension data mapping $B_p^* \to J_{p,ab}$ extends to

$$\text{(IV.18)} \qquad \Phi_1 : B_p \to J_{p,ab}.$$

The relative version is first that the level 1 extension data gives a family $\mathcal{J}_I \xrightarrow{\pi} P_I$ of compact, complex tori. Here the monodromy action arising from the representation of $\pi_1(P_I)$ given by the local system $R^1_\pi \mathbb{Z}$ is in general not finite.

By a Mumford-Tate group argument, at a very general point of P_I there is a maximal family, preserved by monodromy,

$$\pi : \mathcal{J}_{I,ab} \to P_I$$

of abelian subvarieties $J_{p,ab} \subset J_p$ whose fibres are Tate twists of weight 1 Hodge structures.

If we let

$$\widetilde{Z}_I = \begin{Bmatrix} \text{partial completion of } Z_I^* \text{ given} \\ \text{by completing the } B_p^*\text{'s to } B_p\text{'s} \end{Bmatrix},$$

then we have

(IV.19) $$\mathcal{J}_{I,ab} \to P_I$$

and we have the basic observation

(IV.20) *the level 1 extension data mappings* (IV.19) *give*

$$\Phi_1 : \widetilde{Z}_I \to \mathcal{J}_{I,ab};$$

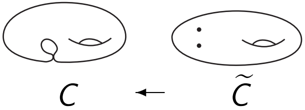
here we have pulled the family (IV.19) back to $\widetilde{Z}_I$. Moreover, for $A \in \check{\sigma}_I$ there is a line bundle $\mathcal{L}_A \to \mathcal{J}$ that restricts to a line bundle

$$\mathcal{L}_A \to \mathcal{J}_{ab}$$

that is positive on each fibre. Although we haven't checked it seems plausible that there is a metric in $\mathcal{L}_A$ whose Chern form is positive.

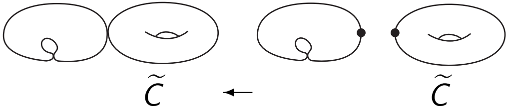
Example: Referring to the first example discussed in the appendix to Sections IV and V,

- A point of B_p^* is the limiting mixed Hodge structure associated to a curve C as in the picture



  Being in a fibre B_p^* means that the normalizations $\widetilde{C}$ are fixed elliptic curves and $J_p = J(\widetilde{C})$.

- The points that are added to B_p^* to complete it correspond to curves



  where the vanishing cycle is trivial. The level 1 extension data doesn't change.

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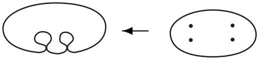
- When we seek to complete $\widetilde{Z}_I$ we will need to add curves such as



When this is done the family of abelian varieties

$$\mathcal{J}_{ab} \to P_I$$

will be completed by adding semi-abelian varieties.¶ We have not worked things out but see no reason why the existing theory of degenerating abelian varieties should not be adaptable to the situation at hand.

¶In this case a $\mathbb{C}^*$.

V. Period mappings to extension data (B)

Reviewing briefly, for the complete varieties

$$B_p = \Phi_e^{-1}(p), \quad p \in \overline{P}$$

we have defined a complex torus J_{ab}, holomorphic line bundles $L_A \to J_{ab}$ and an Abel-Jacobi mapping $\Phi_1 : B_p^* \to J_{ab}$ where B_p^* is the Zariski open obtained by removing from B_p the lower dimensional intersections with other strata. We then showed that the above mapping extends to

$$\Phi_1 : B_p \to J_{ab},$$

and from (IV.3) we have

$$\Phi_1^* L_A = -\sum_i \langle A, N_i \rangle \, N^*_{Z_i/\overline{B}} \Big|_{B_p}.$$

On the fibres of Φ_1 there is a mapping

(V.1)

$$\Phi_2 : \{\text{Zariski open in } \Phi_1^{-1} \text{ points}\} \to \{\text{suitable quotient of the level 2 extension data}\}.^{\|}$$

The purpose of this section is to begin to analyze this mapping. We denote by S a typical fibre of (V.1) and assume for simplicity that S is irreducible. Then there will be a maximal index set I such that

$$S \subset Z_I .$$

‖ This will be made precise in the Lie theoretic treatment of the subject.

There will be another index set $J = \{j \notin I : S \cap Z_j \neq \emptyset\}$. We let

$$S^* = S \setminus \left(\bigcup_{j \in J} S \cap Z_j \right)$$

be the Zariski open obtained by removing from S the lower dimensional intersections with the other strata of Z.

In general the level 2 extension data is a direct sum of

$$\mathrm{Ext}^1_{\mathrm{MHS}}\left(H^{k+2}, H^k\right)\text{'s.} \tag{V.2}$$

These are quotients of Hodge structures of weight -2. The integral classes of type $(-1,-1)$ then project to a subgroup, denoted by $\mathrm{Hg}\otimes\mathbb{C}^*$, of the direct sum of the terms (V.2). We thus have

$$0 \to \mathrm{Hg}\otimes\mathbb{C}^* \to \begin{pmatrix}\text{extension data}\\ \text{of level 2}\end{pmatrix} \to T \to 0 \tag{V.3}$$

where T is a quotient of a $\mathbb{C}^m$ by a discrete group, which in general is not a full lattice. The notation is chosen because $\mathrm{Hg}\otimes\mathbb{C}^*$ is constructed from a product of $\mathbb{C}^*$'s.

In (V.2) the H^p's are graded quotients of a filtration on the fixed vector space V. When we vary along S^* we have to take into account the monodromy action. We will denote by M, to be described explicitly later, the resulting quotient.

Proposition V.4: *The set-theoretic mapping* (V.1) *induces a holomorphic mapping*

$$\Phi_2 : S^* \to M.$$

Proof: The point here is that the differential of Φ_2 maps to zero in the quotient T in (V.3). This is essentially the argument just above (IV.6) in the case $k = 2$ there. We will give the details here since contrary to the case of Φ_1 monodromy along S^* enters the picture.

The map (V.1) is induced by passing to the quotient by $\sigma_{\mathbb{C}}$ of the locally defined mapping

$$\text{(V.5)} \qquad S^* \to \frac{\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \, \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V) + \mathrm{End}_{\mathbb{Z}}(V)}.$$

Now

$$\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V) = \bigoplus_{p+q=-2} \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V)^{p,q}$$

$$\frac{\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \, \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V)} \cong (-1,-1) \oplus (-2,0) \oplus (-3,1) \oplus \cdots .$$

We let

$$\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)\right) := \left((-1,-1) \text{ summand } \cap \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)\right).$$

Then

$$\frac{\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)\right) \otimes \mathbb{C}}{\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)\right)} \hookrightarrow \frac{\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V) + \mathrm{End}_{\mathbb{Z}}(V)}.$$

This gives

$$\mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)\right) \otimes \mathbb{C}^* \hookrightarrow \frac{\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V)}{F^0 \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{C}}(V) + \mathrm{End}_{\mathbb{Z}}(V)}.$$

As noted above, the quotient is generally a non-compact $\mathbb{C}^m/\Lambda$.**

**A model to keep in mind is for a smooth algebraic surface X we have

$$H^2(X,\mathbb{C}) \cong \left(\mathrm{Hg}^1(X) \otimes \mathbb{C}\right) \oplus H^2(X,\mathbb{C})_{\mathrm{tr}}.$$

Then

$$\frac{H^2(X,\mathbb{C})}{F^2 H^2(X,\mathbb{C}) + H^2(X,\mathbb{Z})} \cong \left(\mathrm{Hg}^1(X) \otimes \mathbb{C}^*\right) \oplus H^2(X,\mathbb{C})_{\mathrm{tr}}/(\text{integral stuff}).$$

By the argument just above (V.6) the derivative of the map (V.5) lands in the $(-1,-1)$ part of the Hodge decomposition of the tangent space. On the other hand, the derivative is the complex linear map induced by

$$H_1(S^*, \mathbb{Z}) \to \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)/\sigma_{\mathbb{Z}}$$

where the $\sigma_{\mathbb{Z}} = \mathrm{span}_{\mathbb{Z}}\{N_i : i \in I\}$ reflects the action of monodromy. It follows that (V.5) is a map

$$S^* \to \mathrm{Hg}\left(\mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)/\sigma_{\mathbb{Z}}\right) \otimes \mathbb{C}^* \tag{V.6}$$

where the RHS defines the M in the statement of Proposition V.4.

Remark: We have an exact sequence of mixed Hodge structures

$$0 \to H_0(S \backslash S^*, \mathbb{Z})(-1) \to H_1(S^*, \mathbb{Z}) \to H_1(S, \mathbb{Z})$$

and by a weight argument any map of mixed Hodge structures

$$H_1(S, \mathbb{Z}) \to \mathrm{Gr}_{-2}^{W(\sigma)} \mathrm{End}_{\mathbb{Z}}(V)$$

is zero. This suggests that the mapping (V.6) should in some sense be determined by the $H_0(S \backslash S^*, \mathbb{Z})$ part of $H_1(S^*, \mathbb{Z})$. As will now be explained, this is indeed the case.
For simplicity we assume that S is an irreducible curve.

Proposition V.7: *The mapping Φ_2 in (V.6) is determined up to a constant by the discrete data*

$$N_k \text{ for } k \in I \cup J; \ \deg N^*_{Z_I/\overline{B}}\Big|_S \text{ for } i \in I,$$

$$\text{and the intersection numbers } Z_j \cdot S \text{ for } j \in J.$$

Proof: A formal proof will be given elsewhere. Here we will give a period matrix argument as in the proof of (IV.3). The period matrix is

$$\begin{pmatrix} I & 0 \\ 0 & I \\ \alpha & \lambda \\ \beta & {}^t\alpha \end{pmatrix}$$

where the entries are now block matrices. Using coordinates $t_i, i \in I$ and $t_j, j \in J$ we have

$$\beta = \sum_{i \in I} \ell(t_i) N_i + \sum_{j \in J} \ell(t_j) N_j + H$$

where H is holomorphic

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Moreover β is a coordinate representation of the level 2 extension information. For a symmetric matrix A we have

$$e^{2\pi i\langle A,\beta\rangle} = \prod_{i\in I} t_i^{\langle A,N_i\rangle} \prod_{j\in j} t_j^{\left\langle A,N_j\right\rangle} e^{2\pi iH}.$$

Now we use that S lies in a fibre of Φ_2. The line bundle $\Phi_1^* L_{-A}$ is then a trivial bundle on S. As in the argument for (IV.3), $e^{2\pi i\langle A,\beta\rangle}$ is the local coordinate representation of a *global* holomorphic section of this trivial bundle. Thus

$$e^{2\pi i\langle A,\beta\rangle} = \text{constant}$$

(this is the constant in the statement of V.7). Continuing to follow as in the argument for (IV.3), $e^{-2\pi iH}$ is a section over S of the line bundle $\prod_{i\in I} N_{Z_i/\overline{B}}^{*\otimes\langle A,N_i\rangle}$ that vanishes to order $\langle A, N_j\rangle\,(Z_j\cdot S)$ at the points of $S\cap Z_j$, $j\in J$. □

We have seen that for a fibre $Y \subset B_p$ of $\Phi_e = \Phi_0$ there is an abelian variety J_{ab} and a 1^{st} order extension data mapping

$$\Phi_1 : Y \to J_{ab}.$$

The next question is

What does a fibre S of Φ_1 map to?

This question was partially addressed in Proposition V.4, but the object "M" in the statement of that proposition was not identified. That is what we will do now.

For the discussion it will be convenient to continue to think of S as a curve; the extension to the case where $\dim S$ is arbitrary will hopefully be reasonably clear. We assume S is contained in a minimal stratum Z_I, and that S meets the Z_j for $j \in J$ in points.

Referring to the proof of Proposition V.7 we then have

$$\text{(V.8)} \qquad \prod_{i \in I} t_i^{\langle A, N_i \rangle} \prod_{j \in J} \langle A, N_j \rangle (Z_j \cdot S) \cdot e^{2\pi H} = \text{constant.}$$

Now t_i is a local section of $N^*_{Z_i/\overline{B}}\big|_S$. We let $A_1, \ldots, A_m$ be a basis for $\mathrm{span}_{\mathbb{Z}}\{N_j : j \in J\}^*$. Then we may interpret (V.8) as defining a mapping

$$\boxed{\text{(V.9)} \qquad \widetilde{\Phi}_2 : \mathbb{P}N_{Z_I/\overline{B}}\big|_S \to \mathbb{P}\left(\bigoplus_{j=1}^{m} L^*_{A_j}\right)}$$

Reprise on level 2 extension data

Summarizing and rephrasing the above, along the fiber S^* of the level 1 extension data map Φ_1 there are two types of level 2 extension data:*

Type I: Then S^* maps to

$$\frac{\mathrm{Gr}_{-2}^{W(N_I)}\mathrm{End}(V)}{F^0\,\mathrm{Gr}_{-2}^{W(N_I)}\mathrm{End}(V)+\mathrm{Gr}_{-2,\mathbb{Z}}^{W(N_I)}\mathrm{End}(V)+\mathrm{span}(\sigma_{I,\mathbb{C}})}.$$

The first two terms in the denominator are the usual ones that appear in an $\mathrm{Ext}^1_{\mathrm{MHS}}(H^{k-2,k})$ at a point of S^*, and the third term reflects the monodromy action around S.

*Here we retain our convention that S is contained in a minimal Z_I, and we denote by $J = j \notin I : Z_j \cap S \neq \emptyset\}$.

The only non-constant part of the level 2 extension data map lands in

$$\frac{\mathrm{span}(N_k : k \in I \cup J)}{\mathrm{span}(N_i, i \in I)} \otimes \mathbb{C}^*.$$

Type II: This consists of sections of line bundles that involve positive powers of the $N^*_{Z_i/\overline{B}}\big|_{S^*}$, $i \in I$, and whose vanishing is prescribed on the intersections $Z_j \cap S$, $j \in J$.

Example: Using the above period matrix

$$\begin{pmatrix} I & 0 \\ 0 & I \\ \alpha & \lambda \\ \beta & {}^t\alpha \end{pmatrix}$$

- to determine α in a fibre Y of Φ_0 we need to map

$$H_1(Y, \mathbb{Z}) \to H_1(E_\lambda, \mathbb{Z})$$

where $E_\lambda = \mathbb{C}/\mathbb{Z} + \mathbb{Z} \cdot \lambda$ plus a constant in E_λ;

- to determine β on a fibre S^* of Φ_1 we need
  - the monodromy logarithm N_k, $k \in I \cup J$;
  - the bundles $N^*_{Z_i/\overline{B}}$ for $i \in I$,
  - the bundles $[Z_j \cap S]$, $j \in J$; and
  - a constant matrix that may be interpreted as sections of specified bundles with specified values along the $Z_j \cap S$.

As an illustration of this we have the

Proposition (V.10): *Let C be an irreducible curve in Z_I such that $C^* = C \cap Z_I^*$ is the fibre of Φ_1. Then*

$$\left\{\begin{matrix} \textit{the types I and II information} \\ \textit{is constant on } C \end{matrix}\right\} \iff C \cap Z_j = \emptyset, j \in J.$$

Proof: By (IV.2)

$$\sum_{i\in I} N_i[Z_i]\big|_C + \sum_{j\in J} N_j[Z_j]\big|_C = 0.$$

The type I and II is constant if, and only if,

- the bundles $\sum_{i\in I} N_i[Z_i]\big|_C$ are all trivial, and
- $\sum_{j\in J} N_j[Z_j]\big|_C = 0$.

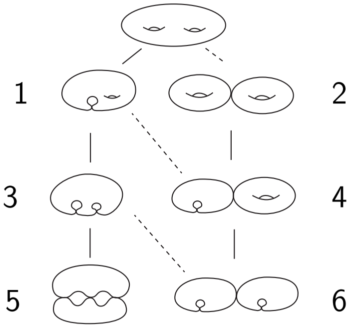
If either of these holds, then they both do, and vice versa.
A similar argument leads to the

Proposition (V.11): *If S^* is an irreducible fibre of Φ_1, then the level II extension data on S is constant if, and only if, $J = \emptyset$.*

Appendix to Sections IV and V: Examples

We shall present some examples that illustrate the constructions in these notes.

Example 1: This is $\overline{\mathcal{M}}_2$, the Deligne-Mumford compactification of genus 2 curves. Although it is the simplest example it illustrates many of the basic constructions. The stratification of $\overline{\mathcal{M}}_2$ may be pictured as



The solid lines represent degenerations with infinite oder monodromy ($N \neq 0$). The dotted ones are degenerations with trivial monodromy ($N = 0$). Although $\overline{\mathcal{M}}_2$ is not smooth the singularities are rational quotient ones and we shall ignore them. For $(\overline{B}, Z)$ we shall use the successive blow ups of $\overline{\mathcal{M}}_2$ along the strata pictured above, beginning with the most singular one. This is just a convenience to fit the following discussion into the general framework of these notes.

The stratum Z_2 whose general member is a curve of type 2

We begin with this one as it is the simplest and also illustrates a general principal we shall use repeatedly without further comment. Since the monodromy is trivial we can extend the original period mapping (here $D = \mathcal{H}_2$ and $\Gamma = \mathrm{Sp}(4, \mathbb{Z})$)

$$\mathcal{M}_2 \xrightarrow{\Phi} \Gamma \backslash D$$

across it. Thus in the $(\overline{B}, Z)$ setting we need not include it in Z.

The stratum Z_1 whose general member is a curve of type 1

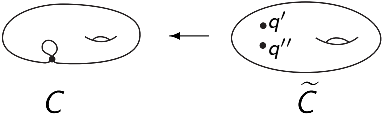
The image $\overline{P}$ of the extended period mapping Φ_e is smooth on $P = \Phi(B)$ and everywhere singular along the boundary; i.e.,

$$\overline{P}_{|sing} = \overline{P} \backslash P.$$

The image $\Phi_e(Z_1) := \overline{P}_1$ is a curve that in an open set in $\overline{P}$ near a curve of type 1 we have a 1-parameter family of simple elliptic surface singularities. A general fibre B_p of the map

$$\Phi_e : Z_1 \to \overline{P}_1$$

consists of curves C whose normalization $\widetilde{C}$ is a fixed elliptic curve

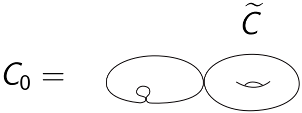


The abelian variety J_{ab} is just the Jacobian $J(\widetilde{C})$ and the mapping

$$\Phi_1 : B_p \to J(\widetilde{C})$$

is $\mathrm{AJ}_{\widetilde{C}}(q' - q'')$ (we are being sloppy about double coverings here). The line bundle L_{N^+} turns out to be 2Θ, twice the theta line bundle on an elliptic curve.

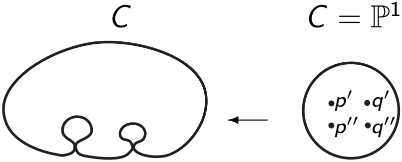
In general we shall let Z_k denote the component of Z obtained by blowing up the stratum in $\overline{\mathcal{M}}_2$ consisting of curves corresponding to k in the diagram. The general fibre of $\Phi_e\big|_{Z_1}$ meets the stratum Z_4 in points corresponding to a curve



The vanishing cycle as $C \to C_0$ is homologically trivial and Φ_1 obviously extends. The intersection $Z_1 \cap Z_4$ occurs when $q' = q''$ in the above picture.
In this case Φ_1 is locally 1-1 and there is no need to consider Φ_2.

The stratum Z_3 whose general member is a curve of type 3

A general point on this stratum is on the blow up of curves of type 3 in the above picture. The picture of the corresponding curve is



Since the limiting mixed Hodge structure is of Hodge-Tate type,

$$\Phi_e(Z_3) = \text{point and } \Phi_1 \text{ is trivial.}$$

Thus the interesting map is

$$\Phi_2 : Z_3^* \to \mathbb{C}^*$$

given by the cross-ratio of $(p', p''; q', q'')$.

Here we should quotient $\mathbb{C}^*$ by the action of monodromy but we shall ignore this. There are two degenerations of the above curve C. One is to a curve of type 6. For the same reason as when a curve of type 1 degenerates to one of type 4, the mapping Φ_2 extends across this locus and continues to map to $\mathbb{C}^*$.

However when C degenerate to a curve of type 5, i.e., when there is a non-trivial vanishing cycle and the monodromy cone goes from dimension 2 to dimension 3, we must add to point to $\mathbb{C}^*$ to receive the image of Φ_2. This is a partial toroidal completion of the type to be discussed elsewhere.

Example: This example illustrates how lifting the mapping

$$\Phi_0 : \overline{B} \to \overline{P}_{\mathrm{SBB}}{}^{\dagger}$$

to the (set-theoretic) mapping

$$\Phi_2 : \overline{B} \to \overline{P}_T$$

may be used to suggest how one may at least partially desingularize completed moduli spaces for general type surfaces. This is in contrast to the case of curves where $\overline{\mathcal{M}}_g$ is essentially smooth and maps to a suitable toroidal completion $(\overline{\Gamma\backslash D})^{\mathrm{tor}}$ where $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ and $D = \mathcal{H}_g$. References to the background and details of the following discussion may be found in [G1], [G2] and [G3].

† $\overline{P}_{\mathrm{SBB}}$ denotes $\overline{P}$ as used elsewhere in these notes.

In summary form the basic points needed for this example are the following (here stated informally):

- For a given type of smooth general type surface X, Kollár-Shepherd-Barron-Alexeev have defined a moduli space $\mathcal{M}$ with a canonical projective completion $\overline{\mathcal{M}}$;
- the surfaces X_0 that are added to compactify the moduli space have $\mathbb{Q}$-Gorenstein canonical divisor class K_{X_0} and semi-log-canonical (slc) singularities;
- for surfaces there is a classification of slc singularities; in the case when X_0 is normal with a singular point p
  - (i) if X_0 is Gorenstein, then p is either simple elliptic or a cusp;
  - (ii) if X_0 is non-Gorenstein, then p is a rational singularity;

- still in the case of surfaces the period mapping $\Phi : \mathcal{M} \to \Gamma\backslash D$ extends to

$$\Phi_0 : \overline{\mathcal{M}} \to \overline{P}_{\mathrm{SBB}} \tag{A.V.1}$$

Remark: The period mapping Φ extends across the locus in $\overline{\mathcal{M}}$ of normal surfaces X_0 of type (ii) (the monodromy around X_0 is finite); for type (i) surfaces in general there is a non-trivial limiting mixed Hodge structure associated to a degeneration $X \to X_0$; it is this case that we shall be concerned with here.

- In contrast to the case of curves, $\overline{\mathcal{M}}$ is singular along the boundary $\partial M = \overline{\mathcal{M}}\backslash\mathcal{M}$; moreover, the mapping (A.V.1) does *not* lift to give the dotted arrow in

$$\begin{array}{ccc} & & \overline{P}_T \\ & \nearrow & \downarrow \\ \Phi_e : \overline{\mathcal{M}} & \longrightarrow & \overline{P}_{\mathrm{SBB}} \end{array}$$

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This suggests that, at least in cases where one has a local Torelli property, one might blow up $\overline{\mathcal{M}}$ to make the dotted arrow well defined and use this to partially desingularize $\overline{\mathcal{M}}$. This turns out to be the case for a very interesting class of surfaces. Referring to [FPR], [G1], [G2], [G3] for more details the example is the following:

- An *I-surface* is a smooth,[‡] minimal general type surface X with
  - $K_X^2 = 1$;
  - $p_g(X) = 2$ and $q(X) = 0$.

  It is known that the moduli space $\mathcal{M}_I$ of I-surfaces is essentially smooth of dimension 28:

$$h^1(T_X) = 28 \text{ and } h^0(T_X) = h^2(T_X) = 0.$$

[‡] Everything that follows works if we assume X has canonical (ADE or DuVal) singularities.

- The period domain D is of dimension 57, the IPR is a contact structure and the Torelli (or period) mapping

$$\Phi : \mathcal{M}_I \to \Gamma \backslash D$$

is locally 1-1;[§] the image $\Phi(\mathcal{M}_I) = P \subset \Gamma \backslash D$ is a contact sub-manifold;

- There is a 20-dimensional boundary component $\mathcal{N}_2 \subset \overline{\mathcal{M}}_I$ whose general point corresponds to a singular I-surface X_0 with the property
X_0 is normal and Gorenstein with a simple elliptic singularity of degree 2.

[§] It is known that the monodromy group Γ is of finite index in the full arithmetic group $G_{\mathbb{Z}}$. Since $K_X^2 = 1$, the intersection form is unimodular on the primitive cohomology $H^2(X, \mathbb{Z})_{\mathrm{prim}} = c_1(K_X)^{\perp}$. The ideal situation would be that $\Gamma = G_{\mathbb{Z}}$ and that global Torelli holds (i.e., Φ is 1-1), but this is not known.

The resolution of this singularity is $(\widetilde{X}, \widetilde{C}) \to (X_0, p)$ where $\widetilde{X}$ is a smooth surface whose minimal model X is a K3 surface and $\widetilde{C} \subset \widetilde{X}$ is an elliptic curve with $\widetilde{C}^2 = -2$. The map $\widetilde{X} \to X$ is given by contracting a -1 curve E with $E \cdot \widetilde{C} = 2$; it follows that the image $C \subset X$ of $\widetilde{C}$ is a curve C with $C^2 = 2$ and one node. From this it follows that X is a 2:1 cover of $\mathbb{P}^2$ branched over a sextic curve B and that C is a double cover of a tangent line ℓ to B;

- the limiting mixed Hodge structure corresponding to X_0 has associated graded

$$\Phi_e(X_0) = H^2(X)_{\mathrm{prim}} \oplus H^1(\widetilde{C});$$

it depends on 20 parameters and up to finite data determines the pair (X, C). In other words we have local Torelli for the boundary component.

What about the extension data in the limiting mixed Hodge structure?

To desingularize $\overline{\mathcal{M}}_I$ along the locus $\mathcal{N}_2$ of these surfaces we need to blow up at a general point corresponding to a surface X_0 as described above. This means that we consider a 1-parameter degeneration $X_t \to X_0$ and do a semi-stable reduction to have a smooth total space with a normal crossing divisor $\widetilde{X}_0$ over the origin. From the Clemens-Schmid exact sequence one may guess that $\widetilde{X}_0$ has a double curve isomorphic to $\widetilde{C}$; i.e.,

$$\widetilde{X}_0 = \widetilde{X} \bigcup_{\widetilde{C}} Y$$

where Y is a smooth surface containing the curve $\widetilde{C}$. If $\widetilde{X}_0$ is the central fibre in a smooth family, then

$$N^*_{\widetilde{C}/\widetilde{X}} \cong N_{\widetilde{C}/Y}.$$

The line bundle on the left has degree 2, and if we think of Y as obtained from a smooth cubic in $\mathbb{P}^2$ by blowing up points q_i, then there must be 7 q_i's in order to have $\deg N_{\widetilde{C}/Y} = 2$.

The 1^{st} order extension data for the limiting mixed Hodge structure is $J = J(\widetilde{C})$ and the 7-parameters in the extension data corresponds to the points in $J(\widetilde{C})$ given by the to the q_i's. Of course there are important details required to make this precise, but this at least illustrates the point that the extension data in the limiting mixed Hodge structure may serve as a guide on how to desingularize some moduli spaces of surfaces.

VI. Local Torelli conditions

Given a variation of Hodge structure over a smooth quasi-projective variety B there is a corresponding period mapping

$$\Phi : B \to \Gamma\backslash D. \qquad \text{(VI.1)}$$

There are two equivalent conditions that the *local Torelli* (LT) property should hold. One is that the differential of the mapping Φ should be injective.

Recall that by definition Φ should be locally liftable and this means that the differential of one, and hence any, local lifting should be injective.[¶]

The other condition is given by considering the Higgs data (E, θ) given by the VHS. Here

- $E = \oplus E^p$ where $E^p = F^p / F^{p+1}$;
- $\theta = \oplus \theta^p$ where $\theta^p : E^p \to E^{p-1} \otimes \Omega^1_B$ is induced by the Gauss-Manin connection.

[¶] There is a subtlety here. In moduli problems where the period mapping of a moduli space $\mathcal{M}$ maps a subvariety of $\mathcal{M}$ to singular points of $\Gamma \backslash D$ corresponding to fixed points of Γ acting on D, the differential in the above sense may not be injective whereas in framework of stacks it should be considered as being injective. The classical example here is the hyperelliptic locus in the moduli space $\mathcal{M}_g$ of smooth curves of genus $g \geqq 3$.

This gives a map

$$\theta : TB \to \mathrm{End}(E) \otimes \Omega^1_B \tag{VI.2}$$

and in the Higgs setting local Torelli the condition is that this mapping should be an injective mapping of vector bundles. The observation then is that

These two local Torelli conditions are equivalent.

This is not entirely trivial. The usual expression for the differential of Φ is given by

$$\Phi_* : TB \to \oplus \mathrm{Hom}(F^p, V_C/F^p).$$

By the IPR this mapping is induced by

$$\begin{array}{rcl} \Phi_* : TB & \longrightarrow & \oplus \mathrm{Hom}(F^p, F^{p-1}/F^p) \\ & & \| \\ & & \oplus \mathrm{Hom}(F^p, E^{p-1}). \end{array}$$

Again by the IPR we have a factorization

$$\begin{array}{ccc} TB & \xrightarrow{\quad\Phi_*\quad} & \mathrm{Hom}(F^p, E^{p-1}) \\ & {\scriptstyle\theta}\searrow \qquad \nearrow{\scriptstyle j} & \\ & \mathrm{Hom}(E^p, E^{p-1}) & \end{array}$$

where j is an inclusion of vector bundles. Thus

$$\ker \Phi_* = \ker \theta,$$

which implies the equivalence of the two LT conditions. Assuming unipotent local monodromies across the irreducible components Z_i of Z, the canonical extension $F_e^p \to \overline{B}$ of the Hodge bundles and the extension of the Gauss-Manin connection induces

$$\text{(VI.3)} \qquad \theta_e : T_{\overline{B}}(-\log Z) \to \mathrm{End}(E_e).$$

Definition (VI.4): *The local Torelli (LT) condition is satisfied on $\overline{B}$ if the map* (VI.3) *of vector bundles is injective.*

One may ask what the geometric meaning of this condition is? Let ω_e be the Chern form of the canonically extended augmented Hodge line bundle L_e. On B from the relation (I.4)

$$\omega(\xi) = \|\Phi_*(\xi)\|^2, \qquad \xi \in T_b B$$

we see that local Torelli is equivalent to the positivity $\omega > 0$ of the Chern form. On $\overline{B}$ the Chern form extends to a closed (1,1) current ω_e whose coefficients are in L^1_{loc} and which defines the Chern class of $L_e \to \overline{B}$. Moreover, along the divisor $Z = \overline{B} \backslash B$ even though ω_e is not smooth the condition

$$\text{(VI.5)} \qquad \omega_e(\xi) = 0, \qquad \xi \in T_b \overline{B}$$

is well defined.

Moreover, the fibres of Φ_e are defined by the exterior differential system $\omega_e = 0$. Since in general the limiting mixed Hodge structure is varying non-trivially along the B_p, $\omega_e(\xi) \neq 0$ does not give the right local Torelli condition to capture the full VLMHS.

This raises the question *What is the geometric meaning of (VI.4)?*

Theorem (IV.6) : *The local Torelli condition (VI.4) is equivalent to*

- $\omega_e(\xi) \neq 0$ *for* ξ *not tangent to a fibre* B_p;*
- *if* $\omega_e(\xi) = 0$, *then* $\Phi_{1,*}(\xi) \neq 0$, *and if both* $\omega_e(\xi) = 0$ *and* $\Phi_{1,*}(\xi) = 0$ *then* $\Phi_{2,*}(\xi) \neq 0$ *for* $\xi \in TB_p$.

*We allow the possibility that $\omega_e(\xi) = \infty$.

Geometrically, we have the set-theoretic maps

$$\text{(VI.7)} \qquad \begin{cases} \Phi_0 : \overline{B} \to \overline{P} \\ \Phi_1 : (\text{fibres } B_p \text{ of } \Phi_0) \to \mathcal{E}_1 \\ \Phi_2 : (\text{fibres of } \Phi_0 \text{ and } \Phi_1) \to \mathcal{E}_2. \end{cases}$$

For each of these maps the kernel of the differential can be defined analytically, and the condition (VI.4) is equivalent to the intersection of the kernels of all these maps should be zero. In words

> *Local Torelli means geometrically that the limiting mixed Hodge structures together with their extension data should to* 1^{st} *order be faithfully captured by the family of Hodge structures and limiting mixed Hodge structures parametrized by* $\overline{B}$*.*

One can imagine that there should be complete algebraic varieties $\overline{P} = P_0, P_I, P_2$ and maps

$$\text{(VI.8)} \qquad \begin{cases} \Phi_0 : \ \overline{B} \longrightarrow P_0 & \text{(equal to } \Phi_e : \overline{B} \to \overline{P}) \\ \Phi_1 : \ \overline{B} \dashrightarrow P_1 & \\ \Phi_2 : \ \overline{B} \dashrightarrow P_2. & \end{cases}$$

where outside of the fibres B_p of Φ_0 the map Φ_1 is equal to Φ_0 and along the B_p it gives the 1^{st} order extension data, and Φ_2 is similarly defined using Φ_1 in place of Φ_0 and it maps to level 2 extension data. One may also imagine that there is a blowing up $\overline{P}_T$ of $\overline{P}$ along $\partial P = \overline{P} \backslash P$ that has the full set of extension data for the LMHS's over ∂P. The maps Φ_k are then quotients of the map to $\overline{P}_T$.

An interesting point here is that no matter what the weight n of the original family of Hodge structures over B is, we need only go to level 2 to capture the full limiting mixed Hodge structure along the boundary Z. Put differently, if we go ahead and inductively define $\Phi_3, \dots, \Phi_n$ as above, then up to finite data arising from the constants of integration

$$\text{fibres of } \Phi_n = \text{fibres of } \Phi_2.$$

This suggests the

Possible definition: $\overline{P} = P_0$ is the *Satake-Baily-Borel completion* of the image $P \subset \Gamma \backslash D$ of the original period mapping, and $\overline{P}_T = P_\infty$ is the *minimal toroidal completion of P.*

Among the properties that a line bundle $L \to X$ over a complete algebraic variety can have are

$$\text{(VI.9)} \qquad \begin{cases} \text{(i)} & L \text{ is nef} \\ \text{(ii)} & L \text{ is big} \\ \text{(iii)} & L \text{ is free} \\ \text{(iv)} & L \text{ is ample.} \end{cases}$$

The first is a numerical property of the line bundle. The second is that powers $L^k \to X$ have lots of sections for $k \gg 0$ in the sense that

$$h^0(X, L^k) = Ck^d + (\text{lower order terms})$$

where $\dim X = d$ and $C > 0$.

One might view this as more of an algebraic rather than a geometric property since the ring $R(X, L) := \overset{k}{\oplus} H^0(X, L^k)$ may not be finitely generated and not a lot seems to be known in general about the stable base locus of the linear systems $|mL|$.
As previously noted, there is considerable literature about (i) and (ii); in these notes we are primarily concerned with (iii) and (iv).
Both (iii) and (iv) are geometric properties in the sense that one may take $\mathrm{Proj}\, R(X, L)$ to have a projective variety; i.e., if either property holds we may actually construct something.
Therefore results giving conditions under which $L \to X$ has either of these properties are of particular geometric interest.

In the situation of these notes we have noted that in [BBT] it is proved that, under the assumption that the monodromy group Γ is arithmetic, the image $P = \Phi(B) \subset \Gamma\backslash D$ of a period mapping is an algebraic variety over which the augmented Hodge line bundle $L_e \to P$ is ample. This means that if we denote by $L_e^{\otimes m}(*Z)$ the sheaf of sections of $L_e^{\otimes m}$ that have zeroes of arbitrary finite order on Z, then for $m \gg 0$ the global sections in $L_e^{\otimes m}(*Z)$ will induce a projective embedding of P.*

*In the classical case these correspond to cusp forms. We note that here we are using sections of $L_e \to \overline{B}$ that vanish, rather than having poles, on Z.

In attempting to refine this result the central issue is to understand the behavior of Φ_e on the boundary. There are a number of results about this; e.g., in Section VII we shall give a fairly complete understanding of it when $\dim B = 2$. Here we shall give an ampleness result that holds when $\dim B$ is arbitrary and strong local Torelli assumptions are made. In addition we shall assume

(VI.10) *the effective cone* $\mathrm{Eff}^1(\overline{B})$ *is finitely generated.*

This assumption is probably unnecessary; we make it in order to be able to focus on the geometric properties of the extended period mapping on the boundary.

Theorem VI.11: *In addition to* (VI.10) *assume that Z has one irreducible component, the local Torelli condition* (VI.4) *is satisfied and that the mapping Φ_1 has injective differential. Then the line bundle*

$$L_m := mL_e - [Z]$$

is ample for $m \geqq m_0$.

As noted above the assumptions in this result are very strong. In particular, the assumption that Z has one component will be removed in Section VIII (cf. VIII.29). The point is to illustrate how the basic formula (IV.4) can be used to draw a conclusion.

Proof: Using the assumption (VI.10) there will be finitely many irreducible test curves C_α such that the result will follow from the existence of an m_0 such that

$$C_\alpha \cdot L_m > 0 \tag{VI.12}$$

for $m \geqq m_0$ and all α. We will do this for C corresponding to any one of the C_α's.

If $\Phi_e\big|_C$ is non-constant, then $L_e \cdot C > 0$ and we will have $L_m \cdot C > 0$ for $m \gg 0$. So the interesting case is when $L_e \cdot C = 0$; i.e., when C is in a fibre B_p of Φ_e.

Let C be an irreducible test curve. Since we have only one Z the basic formula (IV.4) reduces to

$$\Phi^*_{1,C} L_{N^+} = \left\langle N^+, N \right\rangle \left(N^*_{Z/\overline{B}}\big|_C \right)$$

which has positive degree.

The hierarchical nature of $K_{\overline{B}}(\log Z)$: For later use we will give some properties of the line bundle

$$K_{\overline{B}} + [Z] = K_{\overline{B}}(Z) = K_{\overline{B}}(\log Z);$$

all three notations will be used depending on the context. We also set

$$Z_{I^c} = \bigcup_{j \in I^c} Z_j = \sum_{j \in I^c} Z_j,$$

where again both the second union sign and third summation notation will be used.

Each of the assertions below may be verified using local coordinates:

$$\text{(VI.13)} \qquad K_{\overline{B}}(\log Z)\big|_{Z_I} \cong K_{Z_I}(\log Z_{I^c}),$$

$$\text{(VI.14)} \qquad T_{Z_I}(-\log Z_{I^c}) \longrightarrow F^{-1}\,\mathrm{End}\left(\mathrm{Gr}^{W(\sigma_I)} E_e\big|_{Z_I}\right).$$

The notation $\mathrm{Gr}^{W(\sigma_I)} E_e\big|_{Z_T}$ means the restriction of $\mathrm{Gr}^{W(\sigma_I)} E_e$ to Z_I, where $\mathrm{Gr}^{W(N_I)} E_e$ is the associated graded to the filtration on E_e induced by any $N \in \sigma_I$.

(VI.15) $T_{\overline{B}}(-\log Z) \longrightarrow F^{-1}\,\mathrm{End}(E_e)$ *is injective on* Z_I if, and only if, the following conditions are satisfied:

(i) the $\{N_i : i \in I\}$ *are linearly independent*

(ii) $T_{Z_I}(-\log Z_{I^c}) \to \bigoplus_k F^{-1}\,\mathrm{End}\left(\mathrm{Gr}_k^{W(N_I)} E_e\big|_{Z_I}\right)$ *is injective.*

VII. The case when $\dim B = 2$

We consider the case where $\dim B = 2$ and make the following assumptions:

(VII.1) *The differential of Φ is everywhere injective;*

(VII.2) *The group $Eff^1(\overline{B})$ of effective 1-cycles on $\overline{B}$ is finitely generated;*

Theorem (VII.3): *Under these assumptions there are $a_i \geq 0$ such that for $c \gg 0$ the line bundle*

$$M = cL_e - \sum_i a_i[Z_i]$$

is ample.

The assumption (VII.1) is a local Torelli one on B; it is a reasonable one to have the result. If one weakens it to the assumption that Φ_* is just generically 1-1, i.e., the image $\Phi(B) = P \subset \Gamma\backslash D$ is a surface, and there is still a result that can be proved.

The assumption (VII.2) is not necessary and the proof that this is so will be also given later. The objective here is to give a simple clear statement whose proof illustrates the essential ideas behind a more general result.

Proof: Using (VII.2) there will be a finite set of irreducible "test curves" C that generate $Eff^1(\overline{B})$. We have to show that there exists c and the a_i such that for the line bundle M in the statement of the theorem

$$M \cdot C = \deg M\big|_C > 0$$

for all such test curves. For this we separate the Z_i into

(1) the Z_i where $\Phi_e(Z_i)$ is a curve; call these Z_α's;

(2) the Z_i where $\Phi_e(Z_i)$ is a point; we continue to call these Z_i's.

Part of the reason for the notation is that for a Z_α in the first group we will have $a_\alpha = 0$.

Let C be a test curve. There are three possibilities:

(a) $C \cap B$ is a Zariski open set in C (i.e., $C \cap Z$ is a finite set of points on C);

(b) C is a curve Z_α from group (1) above;

(c) C is a curve Z_i from group (2) above.

For C in either group (a) or group (b) we have

$$L_e \cdot C > 0.$$

We will see below that the a_i are determined by the intersection matrix $\|Z_i \cdot Z_j\|$. Then for large enough c we will have

$$\left(cL_e - \sum_i a_i Z_i \right) \cdot C > 0.$$

For C in group (c) we have $L_e \cdot C = 0$. By the Hodge index theorem the intersection matrix $A =: \|Z_i \cdot Z_j\|$ is negative definite. There is then the following linear algebra result:

Lemma: *Let A be an integral negative definite symmetric matrix where all the off-diagonal entries are $\geqq 0$. Then A has an eigenvector $a = {}^t(a_1, \ldots, a_m)$ with all $a_i > 0$.*

If μ is the eigenvalue, then

$$Aa = \mu a$$

where $\mu < 0$ since A is negative definite.

Proof: Let $a =^t (a_1, \ldots, a_m)$ be an eigenvector with maximal eigenvalue. Let

$$|a| = \begin{pmatrix} |a_1| \\ \vdots \\ |a_m| \end{pmatrix}.$$

Then

$$
\begin{aligned}
{}^t|a|A|a| &= \sum_i A_{ii}|a_i|^2 + \sum_{i\neq j} A_{ij}|a_i|\,|a_j| \\
&\geqq \sum_i A_{ii}a_i^2 + \sum_{i\neq j} A_{ij}a_i a_j \text{ (because } A_{ij} \geqq 0).
\end{aligned}
$$

Thus $|a|$ is an eigenvector with a maximal eigenvalue. Next we claim that all $a_i > 0$. Let $I = \{i : a_i > 0\}$. By connectivity of Z there is $j \in I^c$ with $Z_i \cdot Z_j \neq 0$. Let $e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, where the 1 is the j^{th} spot. Set $a_\epsilon = a + \epsilon e_j$. Then ${}^t a_\epsilon A a_\epsilon = {}^t aAa + 2\epsilon \sum_i A_{ij} a_i + \epsilon^2 a_{ii}$. Now $\|a_\epsilon\|^2 = \|a\|^2 + \epsilon^2$ so

$$
\frac{{}^t a_\epsilon A a_\epsilon}{\|a_\epsilon\|^2} = \frac{{}^t aAa}{\|a\|^2} + 2\epsilon \frac{\sum A_{ik} a_k}{\|a\|^2} \mod \epsilon^2
$$

which contradicts minimality. □

Applying this to the case at hand gives

$$\sum_j (a_j Z_j) \cdot Z_i < 0$$

for each Z_i. Thus

$$\left(cL_e - \sum_j a_j [Z_j] \right) \cdot Z_i > 0$$

and we are done. □

The point here is that a simpler result such as

$$cL_e - \sum_i [Z_i] \text{ is ample}$$

is not true. The coefficients a_i are necessary; they reflect a property of the singularity that Z is contracted to.

We next recall Grauert's contactability theorem in the surface case (cf. [BS]):

Theorem: *Let X be a normal complex analytic variety, $Z \subset X$, a compact local complete extension subvariety and assume that the normal bundle*

$$N^*_{Z/X} \to Z$$

is ample. Then there exist a complex variety Y, a proper holomorphic mapping $f : X \to Y$ and $p \in Y$ such that

$$\begin{cases} f(Z) = p \ \text{ and} \\ f : X \backslash Z \xrightarrow{\sim} Y \backslash \{p\} \text{ is biholomorphic.} \end{cases}$$

Suppose now that $\dim X = 2$ and $Z = Z_1 \cup \cdots \cup Z_m$ are smooth curves forming a connected normal crossing divisor. Then

$$[Z] = \sum_{i=1}^{m} [Z_i],$$

and

$$N_{Z/X} = [Z]\big|_Z.$$

The condition that $N^*_{Z/X} \to Z$ be ample is

$$\left(\sum_{i=1}^{m} Z_i \right) \cdot Z_j < 0 \text{ for all } j.$$

We will show this implies that the intersection matrix

$$A := \|Z_i \cdot Z_j\| < 0$$

is negative definite.[†]

For the proof using the above lemma we assume that A has an eigenvector a with maximal eigenvalue μ. Then all $a_i > 0$. We want to show that $\mu < 0$. Now

$$\left(\sum_i a_i Z_i\right) \cdot Z_j = \mu a_j \text{ for all } j.$$

Suppose that $\mu \geq 0$ and renumber so that we have $a_1 \geqq a_i$ for all i.

[†]The converse is not true. The matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ satisfies the first condition $\iff a + b < 0$ and $b + c < 0$, while the second condition is $a < 0, c < 0$ and $ac - b^2 < 0$. These are not equivalent conditions (e.g., take $\left(\begin{smallmatrix} -5 & 2 \\ 2 & -1 \end{smallmatrix}\right)$).

Then

$$a_1 Z_1 + \cdots + a_m Z_m = a_1(Z_1 + \cdots + Z_m) - \sum_{i>1}(a_1 - a_i)Z_i$$

$$\left(\sum_i a_i Z_i\right) \cdot Z_1 = a_1 \left(\sum_i Z_i\right) \cdot Z_1 - \left(\sum_{i>1}(a_1 - a_i)Z_i\right) \cdot Z_1.$$

By the ampleness assumption the first term is negative, and since $Z_i \cdot Z_1 \geqq 0$ for $i \neq 1$

$$-\sum(a_1 - a_i)Z_i \cdot Z_i \leqq 0. \qquad \square$$

The point here is that the argument applies to our Hodge-theoretic situation if we take X to be a tubular neighborhood of Z in $\overline{B}$. If we know that $N^*_{Z/X}$ is ample, then we don't have to use the global Hodge index theorem to conclude that $\|Z_i \cdot Z_j\| < 0$.

This is a purely Hodge theoretic argument using only the behavior of Φ_e in a neighborhood of Z. We finally consider the question

> *Without assuming $N^*_{Z/\overline{B}}$ is ample, can we have $\Phi_1 =$ constant on Z?*

To give a partial answer to this we will show

(VII.5) Assuming that $\Phi(B)$ is a surface, Φ_1 *is non-constant on some component Z_i of Z.*

The proof is in two steps. We recall the notation $\omega_e \in H^2(\overline{B})$ for the Chern class of the canonically extended augmented Hodge line bundle $L_e \to \overline{B}$.

Step one: In this argument we denote by $[Z_i] \in H^2(\overline{B})$ the Chern class of the line bundle $[Z_i]$. If $\omega_e^2 > 0$ and $\omega_e \cdot Z_i = 0$ for all i, then we claim that $[Z_1], \ldots, [Z_m]$ are linearly independent.

Proof: If $\sum_i a_i[Z_i]$ is a primitive relation,[‡] let

$$K_+ = \{i : a_i > 0\}, \quad K_- = \{i : a_i < 0\}.$$

Assume $K_+ \neq \emptyset$ and $K_- \neq \emptyset$. We have

$$\sum_{i \in K_+} a_i[Z_i] = \sum_{j \in K_-} (-a_j)[Z_j].$$

By primitivity of the relation $\sum_{i \in K_+} a_i[Z_i] = 0$ so

$$\left(\sum_{i \in K_+} a_i[Z_i] \right)^2 < 0.$$

[‡]This means that it cannot be non-trivially broken into a sum of two relations.

But since $Z_i \cdot Z_j \geqq 0$ for $i \in K_+$, $j \in K_-$,

$$\left(\sum_{i \in K_+} a_i [Z_i]\right)^2 = \left(\sum_{i \in K_+} a_i [Z_i]\right) \left(\sum_{j \in K_-} (-a_j)[Z_j]\right) \geqq 0,$$

which is a contradiction. Thus either K_+ or K_- is empty; say $K_- = \emptyset$. Then we have a relation among the $[Z_i]$ with positive coefficients that cannot happen on a Kähler manifold.

Step two: Using the basic formula

$$\Phi_1^* L_A = -\sum_i \langle A, N_i \rangle [Z_i],$$

if Φ_1 is constant on each Z_i, we have

$$\sum_i \langle A, N_i \rangle Z_i \cdot Z_j = 0$$

for all j. Then

$$\sum \langle A, N_i \rangle [Z_i] \in \operatorname{span}([Z_1], \dots, [Z_m]) \cap \operatorname{span}([Z_1], \dots [Z_m])^{\perp},$$

which is zero by the Hodge index theorem. Varying A over $\check{\sigma}$ we may conclude that all $N_i = 0$. □

VIII. Period mappings and the canonical bundle

As was noted above for line bundles $L \to X$ over a smooth variety among the measures of positivity are the properties of being

$$\left.\begin{array}{rl}\text{(i)} & \text{nef}\\ \text{(ii)} & \text{big}\end{array}\right\}\text{numerical}$$

$$\left.\begin{array}{rl}\text{(iii)} & \text{free}\\ \text{(iv)} & \text{ample}\end{array}\right\}\text{geometric}$$

The geometric ones are generally more difficult to establish, and generally speaking the criteria for their existence in one way or another seems to involve the canonical bundle K_X. In this section we shall give some results concerning (iii) and (iv). In these either directly or indirectly $K_{\overline{B}}$ will play a central role.

The assumptions for these results will be considerably stronger than what should be the case for optimal ones. The intent here is to isolate the geometric and Hodge theoretic aspects of the situation.

The two main technical assumptions we shall make are

$$\text{(VIII.1)} \qquad \begin{cases} \text{(a)} & \text{the cone } \mathit{Eff}^1(\overline{B}) \text{ of effective 1-cycles} \\ & \text{on } \overline{B} \text{ is finitely generated; and} \\ \text{(b)} & \text{the maps } \Phi_I : Z_I^* \to P_I \subset \Gamma_I \backslash D_I \text{ are} \\ & \text{fibrations.} \end{cases}$$

For a line bundle $M \to \overline{B}$ and curve $C \subset \overline{B}$ we set

$$C \cdot M = \deg \left(M\big|_C \right).$$

Then (a) imples that for any curve

$$\text{(VIII.2)} \qquad M \text{ is ample } \iff C \cdot M > 0.$$

As for (b), on a fibre $B_p^* \subset Z_I^*$ of Φ_I the associated graded to the limiting mixed Hodge structures along B_p^* are locally constant. Hence Φ_I extends to the closure $B_p \subset Z_I$ of B_p^*. We recall our notation

$$\widetilde{Z}_I = \bigcup_{p \in P_I} B_p \subset Z_I$$

and (b) means that the proper holomorphic mapping

$$\Phi_I : \widetilde{Z}_I \to P_I \tag{VIII.3}$$

should be a complex analytic fibration in the usual sense. As mentioned (a) and (b) are technical assumptions that almost certainly are not required for optimal results.

Remark: For (b) the map (VIII.3) is a proper holomorphic mapping. We may stratify P_I and $\widetilde{Z}_I$ so that the map Φ_I restricts to a fibration on each stratum and then try to adapt the argument below to each straturm.

Our first result is the

Theorem (VIII.4): *Assume the local Torelli condition (VI.4) and that the Gauss mappings associated to Φ_1 have injective differentials. Then the line bundle $K_{\overline{B}} + Z$ is ample.*

There is an interesting subtlety here, as illustrated by the following

Example*(VIII.5): Let $\overline{\mathcal{A}}_g$ be a toroidal compactification of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. Then $K_{\overline{B}} + Z$ is not ample.

Proof: From [M] we will assume that $\overline{\mathcal{A}}_g$ is smooth and that $W = \overline{\mathcal{A}}_g \backslash \mathcal{A}_g$ is a local normal crossing divisor. The actual singularities of $\overline{\mathcal{A}}_g$ are mild quotient singularities that do not effect the argument, and similarly the fact that W is only a local rather than global normal crossing divisor makes no essential difference. We shall ignore both of these issues.

*We are indebted to Kang Zuo for calling our attention to this.

If $\mathcal{A}_g^*$ is the Satake-Baily-Borel completion of $\mathcal{A}_g$, then the natural mapping

(VIII.6) $$\overline{\mathcal{A}}_g \xrightarrow{\pi} \mathcal{A}_g^*$$

is a resolution of the singularities of $\mathcal{A}_g^*$. There is an ample "automorphic form" line bundle $\mathcal{O}_{\mathcal{A}_g^*}(1) \to \mathcal{A}_g^*$, and from (3.4) in [M] we have

(VIII.7) $$K_{\overline{\mathcal{A}}_g} + W = \pi^* \mathcal{O}_{\mathcal{A}_g^*}(1).$$

In particular $K_{\overline{\mathcal{A}}_g} + W$ is not ample over $\overline{\mathcal{A}}_g$.

Remark: The fibres of (VIII.6) are abelian varieties. We will see below that the fibres B_p of (VIII.3) have the property

(VIII.8) *The Gauss mapping on the fibres of π is constant.*

Proof of Theorem (VIII.4): With the assumptions LT (local Torelli) and (VIII.1), what has to proved is

(VIII.9) $$C \cdot (K_{\overline{B}} + Z) > 0$$

for any irreducible curve $C \subset \overline{B}$. There are three cases to consider:

(a) $C \cap B := C^*$ is a Zariski open in C;
(b) $C \subset Z$ is not in a fibre of Φ_0;
(c) $C \subset Z$ is in a fibre B_p of Φ_o.

Before treating these cases one by one we will give some differential geometric preliminaries. Setting $\dim B = d$, the vector bundles

$$F^{-1}\,\mathrm{End}(E_e) \text{ and } H_d := \wedge^d F^{-1}\,\mathrm{End}(E_e)$$

has a singular metric induced from those in the Hodge bundles $F_e^p \to \overline{B}$.

The singularity properties of these metrics, their curvatures and resulting Chern forms have been extensively treated in the literature (cf. [CKS] and also [GGLR], [GG] for a summary of the relevant properties). We denote by Θ_{H_d} the curvature matrix of H_d and recall the notation ω_e for the Chern form of the augmented Hodge line bundle L_e.

Lemma (VIII.10): *The curvature form for H_d is non-positive, and there are positive constants c_1, c_2 such that*

$$c_1\omega_e \leqq -\text{Trace } H_d \leqq c_2\omega_e.$$

We will give the proof of this lemma below after the argument for the theorem.

For the holomorphic sub-bundle

$$T_{\overline{B}}(-\log Z) \subset F^{-1}\,\mathrm{End}(E_e)$$

there is an induced singular metric that in turn induces one on the line sub-bundle

$$(K_{\overline{B}} + Z)^* = \wedge^d T_{\overline{B}}(-\log Z) \subset H_d.$$

We denote by

$$\kappa = \text{Chern form of } K_{\overline{B}} + Z.$$

Then we have

$$\text{(VIII.11)} \qquad \kappa = -\left(\text{Trace } \Theta_{H_d}\Big|_{\wedge^d T_{\overline{B}}(-\log Z)}\right) + \sigma$$

where σ is a $(1,1)$ form computed from the second fundamental form of $T_{\overline{B}}(-\log Z)$ in $F^{-1}\,\mathrm{End}(E_e)$.

Lemma (VIII.12): (i) *The* $(1,1)$ *form* σ *is non-negative and bounded;* (ii) *the* $(1,1)$ *form* κ *is a closed* $(1,1)$ *current that represents* $c_1(K_{\overline{B}} + Z)$ *and satisfies*

$$c\omega_e + \sigma \leqq \kappa$$

for a positive constant c*;* (iii) *the restriction of* κ *to* B_p *is well defined and is given by*

$$\kappa\Big|_{B_p} = G(\Phi_1)^*\psi \tag{VIII.13}$$

where $T(\Phi_1) : TB_p \to T\,\mathrm{Grass}(k, T_e J_{ab})$ *is the Gauss map and* ψ *is the Chern form of the Plücker line bundle over the Grassmanian.*

We now turn to the proof of Theorem (VIII.4).

Case (a): From (VIII.12) we have

$$C \cdot (K_{\overline{B}} + Z) = \int_{C^*} \kappa \geqq c \int_C \omega > 0.$$

Case (b): We have

$$\Phi_I := \Phi_e : \widetilde{Z}_I \to P_I$$

and $\Phi_I(C) := \widetilde{C} \subset P_I$ is a curve. Moreover, the augmented Hodge line bundle $Z_I \to P_I$ pulls back under Φ_I to $L_e \to \widetilde{Z}_I$. Then if m is the degree of the mapping $\Phi_I : C \to \widetilde{C}$

$$0 < m \deg \left(L_I \big|_{\widetilde{C}} \right) = \deg \left(L_e \big|_C \right).$$

Case (c): This is the most interesting case. We have

$$\omega_e \big|_{B_p} = 0.$$

Here we note that even thoughω_e is a singular $(1,1)$ form, the restriction of ω_e to B_p is well defined (cf. [GG]). Thus

$$C \cdot (K_{\overline{B}} + Z) = \int_e \sigma = \int_C G(\Phi_1)^* \psi > 0. \qquad \square$$

Proof of Lemma (VIII.12): We will give both a computational proof and a conceptual argument.

The computational proof is based on the curvature calculation in [CM-SP], especially Section 13.6. At a point of B a tangent vector $\xi = T_b B$ is given by $\xi = \oplus \xi^p$ where

$$\xi^p : E_b^p \to E_b^{p-1}$$

the fibres E_b^p have Hermitian metrics and we denote by ξ^* the adjoint of ξ.

Then the curvature

$$\Theta_E(\xi, \bar{\xi}) = -[\xi, \xi^*] \in \mathrm{End}(E_b). \tag{VIII.14}$$

When we come to the tangent bundle TB with the metric induced by the inclusion

$$\mathrm{TB} \hookrightarrow F^{-1}\,\mathrm{End}(E) \tag{VIII.15}$$

by standard results from Hermitian differential geometry the curvature of TB is the sum of the restriction to TB of the curvature of $F^{-1}\,\mathrm{End}(E)$ plus a term coming from the second fundamental form of the sub-bundle (VIII.15). We will later deal with the second fundamental form term and here will discuss the term arising from $F^{-1}\,\mathrm{End}(E)$.

In this case (VIII.14) gives

$$\Theta_{F^1 \operatorname{End}(E)}(\xi, \bar{\xi}) = -\operatorname{ad}_{[\xi,\xi^*]}$$

where "ad" refers to the adjoint action in the Lie algebra $\mathrm{gl}(E_b)$, which in this case is just the bracket. Then $(\operatorname{ad}_\xi)^* = \operatorname{ad}_{\xi^*}$, and the Jacobi identity gives

$$\operatorname{ad}_{[\xi,\xi^*]} = [\operatorname{ad}_\xi, \operatorname{ad}_{\xi^*}].$$

The crucial observation is that $T_b B$ is an integrable subspace of $E^{-1} \operatorname{End}(E_b)$ (cf. [CM-SP]and [Z]) and this gives $\operatorname{ad}_\xi \eta = 0$ for any two tangent vectors. Then

$$\text{(VIII.16)} \qquad \left(\Theta_{F^{-1} \operatorname{End}(E_b)} \Big|_{T_b B} \right) (\xi, \xi) = -\| \operatorname{ad}_\xi \xi^* \|^2.$$

Now we want to apply this not to TB but to $\wedge^d \mathrm{TB} \subset \wedge^d F^{-1} \operatorname{End}(E)$.

For $\eta_1, \dots, \eta_d$ a unitary basis for the subspace $T_b B \subset F^{-1}\,\mathrm{End}(E_b)$ we obtain
(VIII.17)

$$\Theta_{\wedge^d F^{-1}\,\mathrm{End}(E_b)}\Big|_{\wedge^d T_b B}(\eta_1 \wedge \cdots \wedge \eta_d)(\xi, \bar{\xi}) = -\sum_i \|\,\mathrm{ad}_\xi(\eta_i)\|^2.$$

It is a general fact that the right-hand side of (VIII.17) is positive. We shall give a conceptual argument for this below.

The final step in that is that we need to show the existence of $c_1 > 0$ so that

$$\sum_i |\,\mathrm{ad}_\xi(\beta_i)\|^2 \geqq c_1 \|\xi\|^2 = \omega(\xi). \qquad \text{(VIII.18)}$$

The period domain D is a homogeneous space and all the metrics are invariant. Therefore it suffices to show that (VIII.18) holds at one point.

The variety of d-dimensional horizontal subspaces of the tangent space is a subvariety of the Grassmannian of d-planes in the tangent space, and inside that Grassmannian the abelian subspaces form a closed analytic subvariety. By compactness we conclude the existence of a $c_1 > 0$ such that (VIII.18) holds. □

Proof of Lemma (VIII.12): We begin with some general remarks in Hermitian differential geometry.

Given a holomorphic bundle $R \to X$ over a complex manifold, associated to a Hermitian metric in R there is a canonical Chern connection

$$D_X : A^0(X, R) \to A^1(X, R)$$

characterized by

$$\text{(VIII.19)} \qquad \begin{cases} D''_X = \overline{\partial} \\ d(r, r') = (D_x r, r') + (r, D_X r') \end{cases}$$

for C^∞ sections r, r' of $R \to X$.

If $Y \subset X$ is a complex submanifold, then we have the functoriality property

$$D_X\big|_Y = D_Y. \tag{VIII.20}$$

The left-hand side means: take any local section of $R_Y := R\big|_Y$. Extend it to a local section of R, take D_X and restrict to Y. That is equal to applying D_Y to the original section.

For the next property, suppose that $S \subset R$ is a holomorphic sub-bundle with quotient bundle Q. Then there is a second fundamental form

$$P_X : A^0(X, S) \to A^{1,0}(X, Q)$$

that measures the extent to which the Chern connection for $R \to X$ fails to map S to itself. The curvature operators for R and S are related by

$$\Theta_S = \Theta_R\big|_S - (P_X, P_X). \tag{VIII.21}$$

The second functoriality property of the Chern connection is

$$P_Y = P_X\big|_Y; \tag{VIII.22}$$

it follows from the first functoriality property (VIII.20).
Suppose now that

$$R\big|_Y \text{ is a flat; i.e., } D_Y^2 = 0. \tag{VIII.23}$$

In fact, suppose we can identify the fibres of $R\big|_Y$ with a fixed $\mathbb{C}^m$ having the standard metric. The sub-bundle $S\big|_Y \subset R\big|_Y$ is then given by pulling back the universal sub-bundle under a holomorphic mapping

$$\varphi : Y \to \mathrm{Grass}(k, m)$$

to a Grassmannian.

Moreover, the second fundamental form is

$$P_Y = \varphi_*. \tag{VIII.24}$$

It follows from (VIII.21) and the functoriality properties that

$$\Theta_S\big|_Y(\eta) = \|\varphi_*(\eta)\|^2. \tag{VIII.25}$$

We now apply the general considerations to the situation in Lemma (VIII.12). We have

$$\begin{aligned} X &= \overline{B}, \\ Y &= B_p = \text{fibre of } \Phi_e, \\ R &= F^{-1}\,\mathrm{End}(E_e), \\ S &= T_{\overline{B}}(-\log Z) \subset R\big|_{B_p}. \end{aligned}$$

Although the Hodge metrics are singular, the analysis in [CKS] and [GG] enables us to apply the above as if the metrics were smooth. In particular, since

$$\omega_e\big|_{B_p} = 0$$

the induced metric is $F^{-1}\operatorname{End}(E_e)\big|_{B_p}$ is flat. Moreover the mapping φ above is just the Gauss mapping associated to

$$\Phi_1 : B_p \to J_{ab}.$$

This completes the proof of Theorem (VIII.4).

Differential geometric discussion: There were some loose ends in the above (e.g., having *constants* c_1, c_2 in (VIII.10)). We will now deal with these by bringing in some differential geometric aspects that are of interest in their own right.

The Chern form ω of the augmented Hodge line bundle defines a Kähler metric on B.* From [Z], [GG] and the references given there this metric has the following properties:

- the holomorphic bisectional curvatures $R(\xi, \eta) \leqq 0$;
- the holomorphic sectional curvatures $R(\xi) \leqq -c$ where $c > 0$ is a constant independent of (B, Φ);
- the Ricci ternsor $R_{i\bar{j}}$ satisfies

$$\left(\frac{i}{j}\right) \sum_{j,k} R_{j\bar{k}} dz^j \wedge dz^{-k} < -c\omega$$

  for a constant $c > 0$.

*This metric is complete if all $N_i \neq 0$.

Referring to the proof of Lemma (VIII.10), the above computation shows that at a point b of B the curvature form of H_d evaluated on decomposable elements in $\wedge^d T_{\overline{B}}(-\log Z)_b$ is a $(1,1)$ form bounded below by $c_1\omega$. Moreover, *this curvature form is the same as the one induced on* $\wedge^d T_B$ *by the curvature of the Kähler metric given by* $\omega\big|_B$. It then follows from the above discussion that we have the uniform constant c_1 in (VIII.10).

Remark: The geometry at infinity, i.e., in the boundary of tubular neighborhoods of Z, is of interest. One interpretation of the arguments given above is that there is "flatness" of this boundary along the fibres B_p of Φ_e. This is perhaps not unreasonable since $\omega_e\big|_{B_p} = 0$.

Remark: Let ω_e be the Chern form of $L_e \to \overline{B}$ and

$$\Omega_e = \underbrace{\omega_e \wedge \cdots \wedge \omega_e}_{d}$$

the corresponding volume form. Then it is known (cf. [GGLR] and [GG]) that the coefficients of Ω_e are in L^1_{loc} on $\overline{B}$, and therefore Ω_e defines a singular volume form. Moreover it defines a pseudo-metric in $K_{\overline{B}} + Z := K_{\overline{B}}(Z)$, meaning that it is a metric on B and along Z_I it vanishes like $\mathcal{O}\left(\prod_{i\in I} \frac{1}{(\log|t_i|)^2}\right)$. This follows from the properties of ω_e given in [CKS] and [GG].

The toy example of this is the area form on $\mathbb{P}^1$ induced from the universal covering $\mathcal{H} \to \mathbb{P}^1 \backslash \{0, 1, \infty\}$. Near, say, the origin $z = 0$ the form Ω_e looks like the Poincaré form $\pi := \frac{dz\wedge d\bar{z}}{|z|^2(-\log|z|)^2}$ plus lower order terms. Then $\mathrm{Ric}\,\pi = \pi + \{0\}$.

Although we won't use it we want to mention without proof the following.

(VIII.26) *The volume* $\int_{\overline{B}} \Omega_e < \infty$ *is finite, and the Ricci form*

$$\operatorname{Ric}\Omega_e = c_1(K_{\overline{B}}) + Z$$

where $c_1(K_{\overline{B}})$ *is a closed* $(1,1)$ *current with coefficients in* L^1_{loc}*, with Lelong numbers equal to zeros and which represent* $c_1(K_{\overline{B}})$ *in* $H^2(\overline{B})$*.*

On B we have (up to a positive constant)

$$\operatorname{Ric}\Omega_e = -\left(\frac{1}{2\pi i}\right)\sum_{j,k} R_{j\bar{k}} dz^j \wedge d\bar{z}^k$$

where $R_{j\bar{k}}$ is the Ricci tensor of the Kähler metric ω.

We now turn to other properties of the line bundle $K_{\overline{B}} + Z$. These are summarized in the

Theorem (VIII.27):

(i) *If generic local Torelli holds, then $K_{\overline{B}} + Z$ is big;*

(ii) *if generic local Torelli holds and Φ_1 has no positive dimensional fibres, then $K_{\overline{B}} + Z$ is nef and $K_{\overline{B}} + (1+\epsilon)Z$ is free for small ϵ;*

(iii) *if local Torelli holds on the Gauss map $\mathbb{G}(\Phi_1)$ has no positive dimensional fibres, then $K_{\overline{B}} + Z$ is ample.*

The result (i) says that $(\overline{B}, Z)$ is log canonical. In various forms it has appeared several times in the literature (cf. [LSZ]). A proof also follows from the arguments given above for Theorem (VIII.4). The result (iii) is a restatement of that theorem.

The interesting result is (ii). There $K_{\overline{B}} + (1+\epsilon)Z$ is a $\mathbb{Q}$-line bundle and the statement means that for $m \gg 0$

$$mK_{\overline{B}} + (m+1)Z \text{ is free.}$$

For the proof we shall use the base-point-free theorem (BPFT) from [KM]. We need that

- $mK_{\overline{B}} + (m+1)Z$ is big for $m \gg 0$;
- $mK_{\overline{B}} + (m+1)Z - K_{\overline{B}}$ is nef.

The first follows from the analysis given in the proof of Theorem (VIII.4) and the results in [D]. The second also follows from that analysis and the differential geometric discussion following the proof of (VIII.4)

Theorem (VIII.28): *Assume that generic local Torelli holds and that Φ_1 has no positive dimensional fibres. Then there exists an $\ell_0 > 0$ and a $k_0 = k_0(\ell_0)$ such that for $k \geqq k_0$*

$$kL - \ell_0 Z \text{ is free.}$$

The assumption that generic local Torelli holds is equivalent to

$$\dim \Phi(B) = \dim B,$$

or equivalently that Φ_* is somewhere injective.

A corollary of this theorem is a special case of a more general version of the result in [BBT]:

(VIII.29): $\mathrm{Proj}(kL - l_0 Z)$ *exists and defines a projective algebraic variety containing $\Phi(B)$ as a Zariski open set to which the augmented Hodge line bundle descends and is ample.*

This result is more general than [BBT] in that the monodromy group is not assumed to be arithmetic; additionally the order of vanishing along Z of the sections of $L^{\otimes k} \to \overline{B}$ used to projectively embed $\Phi(B)$ is given.
Remark that we suspect that the various assumptions made in the above results can be considerably relaxed once one has a better understanding of the level 2 extension data mapping Φ_2.
For the proof of (VIII.29) using the BPFT basically we have to show that there exists an ℓ_0 such that for $k \gg 0$

$$kL - \ell_0 Z - K_{\overline{B}} \text{ is nef.} \tag{VIII.30}$$

By arguments used above the issue is to show that for curves C such that $L \cdot C = 0$ there is an ℓ_0 such that

$$C \cdot (\ell_0 Z + K_{\overline{B}}) \leqq 0. \tag{VIII.31}$$

Such a curve C is in a fibre $B_p \subset Z_I$. Then as above we have

$$\begin{cases} K_{\overline{B}}\big|_{B_p} \cong K_{B_p} \otimes N^*_{Z_I/\overline{B}}\big|_{B_p} \cong K_{B_p} - \sum_{i\in I}[Z_i]\big|_{B_p} \\ \ell Z\big|_{B_p} = \sum_{i\in I} \ell[Z_i]\big|_{B_p}. \end{cases}$$

From the basic formula (IV.3), for $A \in \check{\sigma}_I$ and using $Z_j \cdot C \geqq 0$ we have for $d_i = \deg[Z_i]\big|_C$

$$\sum_{i\in I} \langle A, N_i \rangle \, d_i < 0.$$

Since the N_i are linearly independent this implies that

$$d_i < 0, \quad i \in I.$$

Then

$$\deg\left([Z]\big|_C\right) = \sum_{i\in I} d_i < 0.$$

We then may choose ℓ_0 so that (VIII.30) holds, from which it follows that the conditions of the BPFT are satisfied.

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