

# COMPLETION OF PERIOD MAPPINGS AND AMPLESNESS OF THE HODGE BUNDLE

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## I. INTRODUCTION

This paper is in part motivated by the following:

Let  $\mathcal{M}$  be the KBSA moduli space for smooth varieties  $X$  of general type and with given numerical characters. Then  $\mathcal{M}$  is quasi-projective and has a canonical projective completion  $\overline{\mathcal{M}}$ .<sup>1</sup> The singular varieties  $X_0$  corresponding to points of the boundary  $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$  are locally described by the requirement to have only semi-log-canonical (slc) singularities, but very little is known about the global structure of the  $X_0$ 's, or about the global structure (stratification) of  $\partial\mathcal{M}$ .<sup>2</sup> A natural invariant of  $\mathcal{M}$  is the period mapping  $\Phi : \mathcal{M} \rightarrow \Gamma \backslash D$  whose image  $\Phi(\mathcal{M})$  is a quasi-projective algebraic subvariety of the complex analytic variety  $\Gamma \backslash D$ . One seeks a canonical completion  $\Phi(\mathcal{M})_e$  of  $\Phi(\mathcal{M})$  and an extension  $\Phi_e : \overline{\mathcal{M}} \rightarrow \Phi(\mathcal{M})_e$  of the period mapping. We note that

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<sup>1</sup>Our general references for the theory of moduli are of the expository papers by Kollár [Kol1] and [Kov1].

<sup>2</sup>One exception is the recent work [F-P-R1], [F-P-R2], [F-P-R3].

if  $\dim X \geq 2$ ,  $\Gamma \backslash D$  is never an algebraic variety ([G-R-T]) so that analytic methods seem to be required for the study of  $\Phi$  and  $\Phi_e$ . From the theory of degeneration of polarized Hodge structures ([C-K-S1] and extensive subsequent works; cf. [Ca]<sup>3</sup> for a recent overview), one may hope that this theory will lead to analysis of the behavior of  $\Phi_e$  on  $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ , and then this information can in turn lead to an understanding of  $\partial\mathcal{M}$  and of the global structure of the  $X_0$ 's.

Various aspects of this and related topics are work in progress by a number of people including the four of us, and are the subject of other papers currently in preparation. Here we shall be concerned with the study of a *period mapping*

$$(I.1) \quad \Phi : B \rightarrow \Gamma \backslash D$$

where  $B$  is a smooth, quasi-projective variety having a smooth projective completion  $\overline{B}$  where  $Z = \overline{B} \setminus B$  is a reduced normal crossing divisor. We assume that the local monodromies  $T_i$  around the irreducible branches  $Z_i$  of  $Z$  are unipotent. For example, this situation might arise starting with a  $\Phi : \mathcal{M} \rightarrow \Gamma \backslash D$  as above and then passing to a finite covering  $B$  of a desingularization of  $\mathcal{M}$  and completing  $B$  to a  $\overline{B}$  as described. Given (I.1) the image  $M = \Phi(B)$  is a quasi-projective algebraic subvariety

$$M \subset \Gamma \backslash D$$

of the moduli space of  $\Gamma$ -equivalence classes of polarized Hodge structures parametrized by the period domain  $D$ .<sup>4</sup> The primary objectives of this paper are

- (i) to construct a canonical completion  $\overline{M}$  of  $M$  as a compact, complex analytic variety, one that is minimal in a sense to be explained; and

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<sup>3</sup>The volume in which this reference appears contains a number of papers that provide general references to the material in this paper.

<sup>4</sup>If we assume, as we may, that  $\Phi$  has been extended over the  $Z_i$  where  $T_i$  is trivial, then the resulting period mapping  $\Phi : B \rightarrow \Gamma \backslash D$  is *proper*, and hence the image is a closed analytic subvariety. The quasi-projectivity result has a long history that will be discussed below. One of the earliest results is given in [Som].

(ii) to show that the Hodge line bundle  $\Lambda \rightarrow M$  extends to an ample line bundle  $\Lambda_e \rightarrow \overline{M}$ .

This provides a canonical, projective Hodge-theoretic object that may be used to study moduli of varieties of general type. Some early examples suggest that it can give a very effective tool for this study.<sup>5</sup> A consequence of this result and other work in progress is that one may now define the *Satake-Baily-Borel (SBB) completion* of the image of the period mapping  $\Phi : \mathcal{M} \rightarrow \Gamma \backslash D$  to be  $\text{Proj}(\Lambda_e)$ .<sup>6</sup> Examples show that this SBB completion can be very effective to organize the boundary structure of some surfaces of general type.<sup>7</sup>

We will now explain points (i) and (ii) in further detail.<sup>8</sup> To begin, we recall that given the data  $(V, Q)$ , where  $V$  is a  $\mathbb{Q}$ -vector space containing a lattice  $V_{\mathbb{Z}}$  and

$$Q : V \otimes V \rightarrow \mathbb{Q}$$

is a non-degenerate bilinear form satisfying  $Q(u, v) = (-1)^n Q(v, u)$ , the period domain  $D$  is the set of polarized Hodge structures (PHS)  $F^\bullet = \{F^p\}$  of weight  $n$ . The conditions that the decreasing Hodge filtration  $F^\bullet$  give a Hodge structure is that for each  $p$  with  $1 \leq p \leq n$  we have the isomorphism

$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}.$$

Equivalently, setting  $V^{p,q} = F^p \cap \overline{F}^q$  there is the Hodge decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \quad \overline{V}^{p,q} = V^{q,p}.$$

<sup>5</sup>An informal overview of some aspects of this is given in [G3].

<sup>6</sup>A Lie-theoretic Satake-Baily-Borel completion of  $\Gamma \backslash D_n$  of  $\Gamma \backslash D$  is also under construction. It is hoped to be able to relate  $\overline{M}$  and  $\Gamma \backslash D_h$ .

<sup>7</sup>Some examples are discussed in [G3], and there is currently further work in progress in this area. Basically the boundary of the Gorenstein part  $\mathcal{M}^{\text{Gor}}$  is stratified according to the associated graded of the mixed Hodge structure, modulo the  $\text{Hg}^1$ -part of that mixed Hodge structure.

<sup>8</sup>A general reference for period domains and period mappings is [C-M-S-P].

The relation between the two is  $F^p = \bigoplus_{p' \geq p} V^{p', n-p'}$ . Defining as usual the Weil operator by

$$C(v) = i^{p-q}v, \quad v \in V^{p,q},$$

the polarization condition is given by the two Hodge-Riemann bilinear relations

$$\begin{cases} Q(F^p, F^{n-p+1}) = 0, \\ Q(Cv, \bar{v}) > 0, \quad 0 \neq v \in V_{\mathbb{C}}. \end{cases}$$

Setting  $G = \text{Aut}(V, Q)$ , the period domain is a homogeneous complex manifold

$$D = G_{\mathbb{R}}/H$$

where  $G_{\mathbb{R}}$  is the real Lie group associated to the  $\mathbb{Q}$ -algebraic group  $G$  and  $H \subset G_{\mathbb{R}}$  is the compact subgroup leaving invariant a reference polarized Hodge structure. A period mapping (I.1) is given by a monodromy group  $\Gamma \subset G_{\mathbb{Z}}$  and a locally liftable holomorphic mapping (I.1) that satisfies the infinitesimal period relation

$$\Phi_* : TB \rightarrow I$$

where  $I \subset TD$  is the invariant distribution defined by

$$\dot{F}^p \subset F^{p-1}.$$

The differential of  $\Phi$  is a map  $\Phi_* : TB \rightarrow \bigoplus^{\lfloor \frac{n+1}{2} \rfloor} \text{Hom}(F^p, F^{p-1}/F^p)$  and we shall denote by

$$\Phi_{*,n} : TB \rightarrow \text{Hom}(F^n, F^{n-1}/F^n)$$

the end piece of  $\Phi_*$ .<sup>9</sup>

**Definition:** The *Hodge vector bundle*  $F \rightarrow D$  is the homogeneous vector bundle whose fibre at  $F^\bullet \in D$  is  $F^n$ . The *Hodge line bundle* is the line bundle

$$\Lambda = \det F.$$

We shall frequently refer to  $\Lambda$  as simply the *Hodge bundle*, but we will always use “vector” when discussing the Hodge vector bundle. We

<sup>9</sup> $\Phi_{*,n} = \Phi_*$  when the weight  $n = 1, 2$ .

have chosen  $\Lambda$  as the notation for the Hodge bundle by analogy to the standard notation  $\lambda$  for the Hodge bundle over  $\mathcal{M}_g$ .

There are other vector bundles such as those with fibres  $F^{p-1}/F^p$  associated to period mappings, but for application to moduli of varieties of general type the Hodge vector and line bundles are especially important due to the identification  $F^n = H^{n,0}(X) = H^0(K_X)$  where  $X$  is a smooth projective variety of dimension  $n$  and  $H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X)$  the Hodge decomposition of its  $n^{\text{th}}$  cohomology group.

The Hodge-Riemann bilinear relations induce invariant Hermitian metrics in  $F$  and  $\Lambda$ , and denoting by  $\Omega$  the curvature form in the pullback  $\Lambda \rightarrow B$  to  $B$  of the Hodge bundle over  $D$ ,<sup>10</sup> the positivity of the Hodge bundle over  $B$  is expressed by

$$(I.2) \quad \Omega(\xi) = \|\Phi_{*,n}(\xi)\|^2, \quad \xi \in TB.$$

Here  $\Phi_{*,n}$  is the component of  $\Phi_*$  in  $\text{Hom}(F^n, F^{n-1}/F^n)$ . Thus  $\Omega$  is a positive semi-definite (1,1) form, and its null space is the kernel of the end component of the differential of the period mapping. In particular,  $\Omega$  is positive if, and only if,  $\Phi_{*,n}$  is injective. This covers the particularly interesting case of moduli of surfaces of general type for which local Torelli holds.

It is standard that the data of a period mapping (I.1) is equivalent to that of a (polarized) *variation of Hodge structures* (VHS)  $(\mathbb{V}, \mathcal{F}^\bullet, Q, \nabla)$  over  $B$ . Here  $\mathbb{V}$  is a local system with Gauss-Manin connection  $\nabla : \mathcal{O}_B(\mathbb{V}) \rightarrow \Omega_B^1(\mathbb{V})$  where  $\mathcal{O}_B(\mathbb{V}) = \mathbb{V} \otimes \mathcal{O}_B$ , the  $\mathcal{F}^p \subset \mathcal{O}_B(\mathbb{V})$  are holomorphic sub-bundles,  $Q : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{Q}$  is a horizontal bilinear form and where this data induces at each point of  $B$  a polarized Hodge structure. The infinitesimal period relation (IPR) is

$$(I.3) \quad \nabla \mathcal{F}^p \subseteq \Omega_B^1 \otimes \mathcal{F}^{p-1}.$$

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<sup>10</sup>A more precise notation would be  $\Phi^* \Lambda \rightarrow B$ . But since we will mainly be working with the Hodge bundle and the Hodge vector bundle over  $B$  we shall omit the  $\Phi^*$ 's.

The Hodge vector bundle  $F$  is  $\mathcal{F}^n$ . We shall use interchangeably the data of period mappings and of variations of Hodge structure.

Given a period mapping (I.1) and completion  $B \subset \overline{B}$  as described above, we set

$$Z_I = \bigcap_{i \in I} Z_i$$

where  $Z_i$  are the irreducible components of the divisor at infinity  $Z = \overline{B} \setminus B$ . We denote by  $Z_I^* \subset Z_I$  the open strata obtained by removing from  $Z_I$  the lower dimensional sub-strata. Under the assumption that the local monodromies around the branches  $Z_i$  of  $Z$  are unipotent with logarithms  $N_i$ , it is well known ([C-K-S1] and [P-S]) that there are canonical extensions of the Hodge filtration bundles  $\mathcal{F}^p$  to vector bundles  $\mathcal{F}_e^p \rightarrow \overline{B}$  where the infinitesimal period relation (I.3) becomes

$$\nabla \mathcal{F}_e^p \subseteq \Omega_{\overline{B}}^1(\log Z) \otimes \mathcal{F}_e^{p-1},$$

and where up to a factor of  $2\pi\sqrt{-1}$

$$\text{Res}_{Z_i} \nabla = N_i.$$

We denote by  $\mathcal{F}_e$  or  $F_e$  the canonical extension of the Hodge vector bundle, and by  $\Lambda_e = \det F_e$  the canonically extended Hodge (line) bundle.

Setting  $N_I = \sum_{i \in I} N_i$  we obtain a nilpotent operator to which there is canonically associated a weight filtration  $W_\bullet(N_I)$  uniquely defined by  $N_I : W_m(N_I) \rightarrow W_{m-2}(N_I)$  and  $N_I^k : \text{Gr}_{n+k}^{W(N_I)} \xrightarrow{\sim} \text{Gr}_{n-k}^{W(N_I)}$ .<sup>11</sup> It is a consequence of [C-K-S1] that there is over each open stratum  $Z_I^*$  a polarizable variation of mixed Hodge structure (VMHS) with weight filtration  $W(N_I)$ .<sup>12</sup> Passing to the primitive parts of this variation of mixed Hodge structure gives period mappings

$$(I.4) \quad \Phi_I : Z_I^* \rightarrow \Gamma_I \backslash D_I.$$

<sup>11</sup>We are centering the weight filtration at  $n$ , which is the weight of the polarized Hodge structures under consideration, rather than at zero which is perhaps more customary.

<sup>12</sup>In the literature the VMHS is frequently referred to as the *limiting mixed Hodge structures* associated to the VHS over  $B$ . For the general theory of variations of mixed Hodge structures we refer to [S-Z].

We note the important relation

$$(I.5) \quad \Lambda_e|_{Z_I^*} = \Lambda_I$$

where the left-hand side is the restriction to  $Z_I^*$  of the canonically extended Hodge line bundle and the right-hand side is the Hodge line bundle associated to the period mapping  $\Phi_I$ . It is this relation that relates the canonical extension  $\Lambda_e \rightarrow \overline{B}$  of the Hodge bundle to the Hodge bundles on the open strata of the boundary  $Z = \overline{B} \setminus B$  of  $B$ .

We set

$$M_I = \Phi_I(Z_I^*)^\circ.$$

Here the exponent  $^\circ$  refers to a covering space of the image  $\Phi_I(Z_I^*)$  obtained by a Stein factorization of the map (I.4). As noted above,  $M_I$  is a quasi-projective algebraic variety, which may be described as a finite-covering space of the set of equivalence classes of PHS's given by the primitive parts of the VMHS arising from the canonical extension of  $(\mathbb{V}, \mathcal{F}^\bullet, Q, \nabla)$  to  $\overline{B}$ . Basically,  $M_I$  is obtained by taking the limiting mixed Hodge structures along the open strata  $Z_I^*$ , throwing out the extension data in the mixed Hodge structures, and passing to a finite covering of what is left, the purpose for this is to have connected components of the fibres of  $\Phi_I : Z_I^* \rightarrow \Gamma_I \setminus D_I$ .

**THEOREM A:** *There exists a canonical extension  $\overline{M}$  of  $M$ , which is a compact complex analytic variety and where there is an extension*

$$\Phi_e : \overline{B} \rightarrow \overline{M}$$

*of the period mapping (I.1). As a set*

$$\overline{M} = M \amalg \left( \coprod_I M_I \right).$$

In first approximation,  $\overline{M}$  is obtained from  $M$  by attaching the associated graded PHS's to the limiting mixed Hodge structures arising from the period mapping (I.1) and describing how these fit together. The precise statement differs in that we pass to the  $\Gamma$ -equivalence classes of the set-theoretic mapping described above, and then we pass

to a finite covering arising from a Stein factorization in the construction. The mechanism for fitting the pieces  $M_I$  together in a consistent way to obtain the structure of a complex analytic variety is the most subtle part of the construction.

A natural way to try to prove the above theorem would be to show that there is an extension

$$\Gamma \backslash D \subset \Gamma \backslash D_h$$

of the moduli space of  $\Gamma$ -equivalence classes of polarized Hodge structures as a complex analytic variety, one to which the period mapping (I.1) extends and whose image provides the  $\overline{M}$  in the theorem. In the classical case when  $D$  is a Hermitian symmetric domain and  $\Gamma$  is an arithmetic group such a completion is provided by the Satake-Baily-Borel compactification ([Sa] and [B-B]). However, in the non-classical case the construction of such an extension encounters new and significantly different phenomena that are both Lie-theoretic and Hodge-theoretic in character;<sup>13</sup> it is currently a work in progress that has only been partially carried out.

Our approach will be to construct  $\overline{M}$  by mapping neighborhoods  $\mathcal{U} \cong \Delta^{*k} \times \Delta^\ell$  at infinity in  $B$  to  $\mathbb{C}^N$  and showing that

- (i) the maps  $\mu : \mathcal{U} \rightarrow \mathbb{C}^N$  extend to  $\overline{\mathcal{U}} = \Delta^k \times \Delta^\ell$  and have image an analytic subvariety;
- (ii) these extended maps have natural compatibility properties (explained below) when restricted to sub-strata of  $\overline{\mathcal{U}}$ ; and
- (iii) the extended maps glue together in the intersections  $\overline{\mathcal{U}} \cap \overline{\mathcal{U}'}$  of the open sets in  $\overline{B}$ .

We may think of the  $\mu : \mathcal{U} \rightarrow \mathbb{C}^N$  as local *charts* for the image of the period mapping around the points at infinity.<sup>14</sup> They are not local coordinates in the traditional sense, but operationally serve a similar

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<sup>13</sup>These phenomena are related to the fact noted above that  $\Gamma \backslash D$  is not an algebraic variety, so that considerations of a different sort than the classical case must enter into the story.

<sup>14</sup>To be precise they serve as local charts up to the finite coverings that arise from a Stein factorization of a holomorphic mapping.



function. Even in the classical case they seem to provide a different way of approaching the construction of compactifications.

These maps will first be constructed in case the restriction

$$\Phi_{\mathcal{U}} : \mathcal{U} \rightarrow \Gamma_T \backslash D$$

of the period mapping is a nilpotent orbit. Here,  $\mathcal{U} = \Delta^{*k}$  and  $\Gamma_T = \{T_1^{\mathbb{Z}}, \dots, T_k^{\mathbb{Z}}\}$  is the local monodromy group generated by the monodromies  $T_i$  around the branches of  $Z_i \cap \bar{\mathcal{U}}$ . Denoting by  $(t_1, \dots, t_k)$  coordinates in  $\Delta^{*k}$ , the to be constructed  $\mu$  will be a *monomial mapping*

$$\mu(t) = (t^{A_1}, \dots, t^{A_N})$$

where

$$\begin{aligned} A_j &= (a_{j_1}, \dots, a_{j_k}) \in \mathbb{Z}^{\geq 0}, \\ t^{A_j} &= t_1^{a_{j_1}} \dots t_k^{a_{j_k}}. \end{aligned}$$

Of the properties listed above, the definition and combinatorial properties of the  $A_j$ 's needed to satisfy (ii) are the most subtle and require both non-trivial results from combinatorics and the *relative weight filtration* property of limiting mixed Hodge structures.<sup>15</sup> We note the similarity to aspects of toroidal geometry; in fact, one may reasonably expect that around points of  $Z$  the completion has locally the structure of a normal toroidal variety.

A general period mapping  $\Phi_{\mathcal{U}}$  is well approximated by a nilpotent orbit and a further argument is used to establish the properties (i)–(iii) for such a  $\Phi_{\mathcal{U}}$ . The approximation that will be used has evolved over time from the one in [C-K-S1]; here we will use the version given in [C-K2].

An interesting point concerns passing from the image of the localized at infinity period mapping to the quotient by the full monodromy

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<sup>15</sup>Basically what has to be proved is that for index sets  $I \subset J$  the closure in  $\Delta_J^* \subset \overline{\Delta_I^*}$  of the level sets of the to be constructed  $\mu_I : \Delta_I^* \rightarrow \mathbb{C}^{N_I}$  will be contained in the level sets of  $\mu_J : \Delta_J^* \rightarrow \mathbb{C}^{N_J}$ , and that the level sets distinguish the associated graded polarized Hodge structures to the limiting mixed Hodge structures.

group; i.e., to

$$(I.6) \quad \mathcal{U} \xrightarrow{\Phi_{\mathcal{U}}} \Gamma_T \backslash D \rightarrow \Gamma \backslash D.$$

Denoting by  $\mathcal{H}$  the upper-half-plane in  $\mathbb{C}$  and by  $\tilde{\mathcal{U}} \cong \mathcal{H}^k \times \Delta^\ell$  the universal covering of  $\mathcal{U}$  with coordinates  $(z_1, \dots, z_k, w_1, \dots, w_\ell)$ , we let  $\mathcal{S} = \{(z, w) : |\operatorname{Re} z_j| \leq 1/2 \text{ and } \operatorname{Im} z_j \geq c > 0\}$  be a fundamental domain for the action of  $\Gamma_T$  on  $\tilde{\mathcal{U}}$ . Then if there were a Siegel set  $\Sigma \subset D$  with the property that the lifted period mapping

$$\tilde{\Phi}_{\mathcal{U}} : \tilde{\mathcal{U}} \rightarrow D$$

maps  $\mathcal{S}$  to the Siegel set; i.e., that

$$(I.7) \quad \tilde{\Phi}_{\mathcal{U}}(\mathcal{S}) \subset \Sigma,$$

then there would be only finitely many  $\gamma \in \Gamma/\Gamma_T$  such that the image  $\tilde{\Phi}_{\mathcal{U}}(\mathcal{S})$  meets the translate  $\gamma\tilde{\Phi}_{\mathcal{U}}(\mathcal{S})$  of that image; i.e., finitely many  $\gamma$  such that

$$(I.8) \quad \gamma(\tilde{\Phi}_{\mathcal{U}}(\mathcal{S})) \cap \tilde{\Phi}_{\mathcal{U}}(\mathcal{S}) \neq \emptyset. \quad {}^{16}$$

In this case passing to the quotient by the full monodromy group  $\Gamma$  in (I.6) would be a finite operation. However, the Siegel set property (I.7) is *not* true in general. But we will see that as a consequence of the results given in [C-D-K] the set of  $\gamma \in \Gamma/\Gamma_T$  satisfying (I.8) is finite, so that the discrepancy between working with the local monodromy group  $\Gamma_T$  or the global one  $\Gamma$  is a finite set. One may phrase this informally as saying

*Variations of Hodge structure have the Siegel set property.*

The second main result in this paper is the

**THEOREM B:** *The Hodge bundle extends to a holomorphic Hodge line bundle on  $\overline{M}$ , and there  $\Lambda_e \rightarrow \overline{M}$  is ample.*

In the classical case when  $D$  is a Hermitian symmetric domain and  $\Gamma$  is an arithmetic group this result is a consequence of the properties of

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<sup>16</sup>For the definition and properties of Siegel sets we refer to [B-B]. Informally  $\Sigma$  may be thought of as an approximate fundamental domain for the action of  $\Gamma$  on  $D$ .

the Satake-Baily-Borel construction; it is a global one in that sections of  $\Lambda_e^{\otimes m}$  that give a projective embedding of  $\Gamma \backslash D_h$  are constructed using modular forms. For the reasons explained above, such an approach is not possible in the non-classical case. Our proof is in spirit analogous to the one used by Kodaira to show that over a compact, complex manifold a line bundle with positive Chern class in the differential-geometric sense is ample. The proof of the result here depends on some rather subtle properties of the Chern form  $\Omega$  of the Hodge bundle. It is well known that, as suggested by (I.2), the Hodge bundle has positivity properties. It is due to [C-K-S1] with an important amplification in [Kol2] that  $\Omega$  defines a closed  $(1, 1)$  current  $\Omega_e$  on  $\overline{B}$ . For the proof of Theorem B we need to significantly refine this in several ways.

First, since currents are differential forms with distribution coefficients, the singular support  $\text{sing } \Omega_e$  of  $\Omega_e$  is defined, and assuming as we may that all monodromy logarithms  $N_i \neq 0$  we have

(a) *the singular support*  $\text{sing } \Omega_e = Z$ .

Next, it is well known that distributions and currents cannot in general be restricted to submanifolds. To get around this one needs a more subtle notion than just the singular support. Associated to a current  $\Psi$  on a general manifold  $Y$  is its *wave front set*

$$\text{WF}(\Psi) \subset T^*Y.$$

If  $W \subset Y$  is a submanifold whose tangent spaces are transverse to the wave front set in the sense that

$$TW \subset \text{WF}(\Psi)^\perp,$$

then the restriction  $\Psi|_W$  is a well-defined current on  $W$ . Applying this to the situation at hand where  $Y = \overline{B}$  and  $W = Z_I^*$ , in first approximation we will show that

$$(b) \text{WF}(\Omega_e) \subseteq \bigcup_I N_{Z_I^*/\overline{B}}^*;$$

that is, the wave front set of the Chern form  $\Omega_e$  is contained in the co-normal bundles of the open strata  $Z_I^*$ . The proof will also give the

conditions under which equality holds in (b). It will follow that the restriction  $\Omega_e|_{Z_I^*}$  is well defined, and we then have

(c)  $\Omega_e|_{Z_I^*} = \Omega_I$  is the Chern form of the Hodge bundle  $\Lambda_I \rightarrow Z_I^*$ .

We have used the qualifier “in first approximation” because what will actually be proved is not (b) with the usual definition of the wave front set, but rather we will have (b) with an extended definition of what is meant in this paper by the wave front set, a definition that has for our purposes the all important property that the restrictions  $\Omega_e|_{Z_I^*}$  are defined. We might say that “we have (b) adapted to the Hodge-theoretic situation at hand.”

With this understanding we may summarize the above by the following

**THEOREM C:** *The Chern form  $\Omega$  of the Hodge bundle  $\Lambda \rightarrow B$  extends to a current  $\Omega_e$  on the completion  $\overline{B}$  of  $B$ . There it has singularities as described in (a), (b), and (c) above. In particular it represents the Chern class of the Hodge bundle  $\Lambda_e \rightarrow \overline{B}$ .*

Regarding the positivity of the extended Chern form  $\Omega_e$ , an interpretation of the analysis behind the properties (a) and (b) above may be informally expressed as saying that *the more singular the extended period mapping is, the more positive  $\Omega_e$  is*. This heuristic will be further explained and used in Section VI where we discuss curvature properties of the extended Hodge vector bundle.

We will give two arguments for Theorem C. The first, in Section III, will be geometric and essentially deals with the case of 1-parameter degenerations. The result (Proposition III.9) is a statement about the curvature properties of the Hodge vector bundle and will be used later in the paper. An additional aspect of the argument is that it displays the estimates on the Hodge norms and the resulting connection and curvature forms giving descriptions that are more precise than the existing ones in the literature.

There is a short appendix to Section III in which we show that in the geometric case the Hodge theoretically defined polarizations on

the limiting mixed Hodge structure coincide, up to constants, with standard ones derived from geometry.

The second proof given in Section IV establishes the result in full generality and exhibits in detail how the very special and subtle general properties mentioned above of several variable degenerations of polarized Hodge structures come into play.

It is worth noting that in [C-K-S1] the asymptotic properties of several parameter degenerations of polarized Hodge structure, properties which lead to the analysis of the degenerating Hodge metrics, involve choosing an ordering of the parameter coordinates and dividing the universal covering  $\mathcal{H}^k$  of the parameter space  $\Delta^{*k}$  into sectors. Similar issues arise — in perhaps a more transparent way — here. The underlying question is one that is interesting in its own right. Let  $P(x_1, \dots, x_k)$  and  $Q(x_1, \dots, x_k)$  be homogeneous polynomials that are positive in the sense that they are positive if all  $x_i > 0$ . Suppose that  $\deg P = \deg Q + 1$ . Then the conditions that  $\lim_{x_i \rightarrow 0} Q(x)/P(x) = 0$  be defined, and therefore be zero, requires special properties of the monomials in  $Q$  relative to those in  $P$ .<sup>17</sup> In the situation at hand,  $P(x)$  arises from the determinants of  $\sum_i x_i N_i$  acting on the associated graded to the limiting mixed Hodge structures along the  $\Delta_I^*$ . The resolution of the issue raised above requires a classical result from combinatorics and linear programming and a suitable interpretation of the relative weight filtration property of several parameter limiting mixed Hodge structures.

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<sup>17</sup>For example,  $\lim_{x_1, x_2 \rightarrow 0} \frac{x_1 + x_2}{x_1 x_2}$  is not defined.

Turning to the proof of Theorem B, using the properties of the Chern form we will prove successively that

$$\Lambda_e \rightarrow \overline{M} \text{ is } \begin{cases} \text{strictly nef} \\ \text{big} \\ \text{free.} \end{cases}$$

Here “strictly nef” means that for any curve  $C \subset \overline{B}$

$$\Lambda_e \cdot C > 0$$

unless  $C$  lies in a fibre of the map  $\overline{B} \rightarrow \overline{M}$  in Theorem A. These three properties will combine to give that  $\Lambda_e \rightarrow \overline{M}$  is ample. Of the three the most difficult is the third. In fact, we shall show that  $\Lambda_e \rightarrow \overline{B}$  is free, and since  $\Lambda_e$  is trivial on the connected fibres of  $\overline{B} \rightarrow \overline{M}$  this will give that  $\Lambda_e$  is free on  $\overline{M}$  as well.

In addition to the three principal results stated above, in Section VI we will discuss curvature properties of the Hodge vector bundle  $F \rightarrow B$ . Here although there is only the one result stated below, the main point is to raise some questions concerning positivity properties of the Chern classes of the Hodge vector bundle. At the end of that section we will discuss how these positivity properties enter in important results of Viehweg and others ([V1], [V2], [Kol2]), together with a few historical comments concerning the evolution of these properties.

Given a holomorphic vector bundle  $E \rightarrow Y$  with a Hermitian metric over a complex manifold  $Y$ , we recall that there is a unique Chern connection

$$D : A^0(E) \rightarrow A^1(E)$$

with the properties

$$\begin{cases} D'' = \bar{\partial} \\ d(e, e') = (De, e') + (e, De'), \quad e, e' \in A^0(E). \end{cases}$$

The curvature  $\Theta_E \in A^{1,1}(\text{End } E)$  is as usual defined by

$$\Theta_E(e) = D^2e$$

where  $e \in A^0(E)$ . In terms of a local unitary frame  $e_\alpha$  for  $E$  with  $e_\beta^*$  the unitary dual to  $e_\beta$  and local holomorphic coordinates  $z^i$  on  $Y$ ,

$$\Theta_E = \sum \Theta_{\beta i \bar{j}}^\alpha e_\alpha \otimes e_\beta^* \otimes dz^i \wedge d\bar{z}^j$$

where  $\Theta_{\beta i \bar{j}}^\alpha + \bar{\Theta}_{\alpha i \bar{j}}^\beta = 0$ . The *curvature form* is defined by

$$\Theta_E(e, \xi) = \langle (\Theta_E(e), e), \xi \wedge \bar{\xi} \rangle \quad e \in E_y, \xi \in T_y Y.$$

For  $e = \sum v_\alpha e_\alpha, \xi = \sum \xi^i \partial / \partial z^i$  this is the bi-quadratic form

$$\sum \Theta_{\beta i \bar{j}}^\alpha v_\alpha \bar{v}_\beta \xi^i \bar{\xi}^j.$$

The bundle  $E \rightarrow Y$  is defined to be *positive* if there exists a Hermitian metric such that for non-zero  $e, \xi$

$$\Theta_E(e, \xi) > 0.$$

If we only have

$$\Theta_E(e, \xi) \geq 0$$

then  $E$  is said to be *semi-positive*. The condition of positivity is too strong for many purposes, while semi-positivity is too weak. For example, the trivial bundle is semi-positive. A more substantial example is that the universal quotient bundle  $Q \rightarrow G(k, n)$  over the Grassmannian of  $k$ -planes in  $\mathbb{C}^n$  is semi-positive, but is not positive except in the case  $k = n - 1$  when  $Q = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ .<sup>18</sup>

From the elementary symmetric functions of the curvature matrix one constructs the Chern forms  $c_k(\Theta_E)$  by

$$\sum_{k=0}^r c_k(\Theta_E) \lambda^{r-k} := \det \left\| \left( \frac{1}{2\pi i} \right) \Theta_E + \lambda I \right\|, \quad r = \text{rank } E.$$

<sup>18</sup>An important algebro-geometric analogue of semi-positive called *weak positivity* was introduced by Viehweg ([V1], [V2]) and has been extensively used by him and others. A relation between the two is discussed in [P], Theorem 2.21.

The Chern forms represent the Chern classes of  $E \rightarrow Y$  in de Rham cohomology. For an index set  $I = (i_1, \dots, i_r)$  with  $i_\alpha \geq 0$  we define the Chern monomials by

$$(I.9) \quad c_I(\Theta_E) = c_1(\Theta_E)^{i_1} \cdots c_r(\Theta_E)^{i_r}.$$

In the following we will use the notations

- $T = T_b B$  for  $b \in B$ ;
- $H^{n,0} = F_b^n$ ;
- $H^{n-1,1} = F_b^{n-1}/F_b^n$ .

We then have

$$\Phi_{*,b} : T \rightarrow \text{Hom}(H^{n,0}, H^{n-1,1}),$$

and we will denote this mapping by

$$T \otimes H^{n,0} \rightarrow H^{n-1,1}.$$

**THEOREM D:** *For a variation of Hodge structure (I.1) the Hodge vector bundle  $F \rightarrow B$  is semi-positive. Moreover, the Chern monomials satisfy*

$$c_I(\Theta_F) \geq 0.$$

We have

- $c_k(\Theta_F) \neq 0$  if, and only if, there exist  $k$ -dimensional subspaces  $A \subset T$  and  $B \subset H^{n,0}$  such that

$$A \otimes B \rightarrow H^{n-1,1}$$

*has no left or right kernel;*

- $c_1(\Theta_F)^k \neq 0$  if, and only if, there exists a  $k$ -dimensional subspace  $A \subset T$  such that

$$A \rightarrow \text{Hom}(H^{n,0}, H^{n-1,1})$$

*is injective.*

Since  $c_1(F) = c_1(\det F) = c_1(\Lambda)$  where  $\Lambda \rightarrow B$  is the Hodge line bundle, the last statement follows from (I.2).

The above raises the general



**Question:** What properties of the variation of Hodge structure (I.1) will imply the positivity of the Chern monomials? This question will be amplified and refined at the end of Section VI.

In this regard we shall give a heuristic argument that suggests the result

*If  $\Phi : B \rightarrow \Gamma \backslash D$  has no trivial factors and if  $h^{n,0} \leq \dim B$ , then  $c_{h^{n,0}}(\Theta_F) > 0$ .*

At the end of Section VI we give a brief historical discussion on the positivity of the curvature of the Hodge bundles up through the period (roughly the late 1980s) when the two basic properties — the sign and the nature of the singularities — were developed in the form most relevant to this paper.<sup>19</sup> Since that period the positivity and singularity structure of the Hodge and related bundles has been and continues to be a very active and interesting area and there are excellent survey articles (cf. [P] and the references cited there) on this topic.

## II. CONSTRUCTION OF A COMPLETION OF THE IMAGE OF A PERIOD MAPPING

In this section we will give a proof of Theorem A as stated in the introduction. We recall that we are seeking to construct a compact, complex analytic variety that completes the image of a period mapping as described in the diagram

$$(II.1) \quad \begin{array}{ccc} B & \xrightarrow{\Phi} & M \subset \Gamma \backslash D \\ \cap & & \cap \\ \overline{B} & \xrightarrow{\Phi_e} & \overline{M}. \end{array}$$

The argument will be in number of steps. For the first we localize the period mapping to a neighborhood  $\mathcal{U} \cong \Delta^{*k} \times \Delta^\ell$  at infinity to have

$$\Phi_{\mathcal{U}} : \mathcal{U} \rightarrow M_{\mathcal{U}} \subset \Gamma_T \backslash D$$

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<sup>19</sup>For us most relevant means that the curvature of the Hodge line bundle *exactly* detects the variation of the associated graded to the limiting mixed Hodge structures along and in the normal directions to the strata of the completed period mapping. It is this geometric property that underlies Theorem B above.

where  $\Gamma_T = \{T_1^{\mathbb{Z}}, \dots, T_k^{\mathbb{Z}}\}$  is the local monodromy group. We want to construct a local extension of  $M_{\mathcal{U}}$  to obtain for  $\overline{\mathcal{U}} = \Delta^k \times \Delta^\ell$  a diagram

$$(II.2) \quad \begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi} & M_{\mathcal{U}} \subset \Gamma_T \backslash D \\ \cap & & \cap \\ \overline{\mathcal{U}} & \xrightarrow{\Phi_e} & \overline{M}_{\mathcal{U}}. \end{array}$$

The steps are

**Step 1 (main step):** We will construct (II.1) when  $\Phi$  is a nilpotent orbit. This will be done in several stages — Step 1a, Step 1b, Step 1c.

**Step 2:** We will extend the construction to the case where  $\Phi$  is a general variation of Hodge structure.

**Step 3:** Finally we will give the construction of (II.1) when the full monodromy group  $\Gamma$  is taken into account.

**Step 1a:** We are given a nilpotent orbit

$$\Phi : \Delta^{*k} \rightarrow \Gamma_T \backslash D$$

where, for notational convenience setting  $\ell(t_j) = (\log t_j)/2\pi i$ ,

$$\Phi(t) = \exp \left( \sum_{j=1}^k \ell(t_j) N_j \right) \cdot F_0$$

with  $F_0 \in \check{D}$  being a reference point. We set  $N = \sum_j N_j$  and denote by

$$\lim_{t \rightarrow 0} \Phi(t) = (V, W(N), F_0)$$

the polarized limiting mixed Hodge structure (LMHS). Our basic references for the definition and properties of limiting mixed Hodge structures is [C-K-S1]. Without loss of generality we may assume that the limiting mixed Hodge structure is  $\mathbb{R}$ -split, and we denote its Deligne decomposition by

$$V_{\mathbb{C}} = \oplus I^{p,q}, \quad \overline{I^{p,q}} = I^{q,p}.$$

Then

$$W(N)_m = \bigoplus_{p+q \leq m} I^{p,q},$$

and for the Hodge filtration  $F_e^\bullet$  of the LMHS we have

$$F_e^p = \bigoplus_{p' \geq p} I^{p',q}.^{20}$$

We ask the question

(II.3) *Which monomials  $t_1^{c_1} \cdots t_k^{c_k}$  are constant on the connected fibres of  $\Phi$ ?*

For the answer we first recall that  $\Phi(t) \in D$  for  $0 < |t_i| < \epsilon$ . Since  $D \cong G_{\mathbb{R}}/H$  where  $H$  is a compact subgroup of  $G_{\mathbb{R}}$ , the vector field on  $\check{D}$  induced by the action of a non-zero nilpotent  $N \in \mathfrak{g}$  is non-vanishing on  $D$ . Thinking of the nilpotent orbit as given by a homomorphism  $\Phi : \mathbb{C}^{*k} \rightarrow G_{\mathbb{C}}$  where the image acts on  $F_0 \in \check{D}$ , passing to the induced mapping on the Lie algebras this implies that for  $0 < |t_i| < \epsilon$

$$\Phi_* \left( \sum_i a_i t_i \partial / \partial t_i \right) \text{ is tangent to a fibre of } \Phi \iff \sum_i a_i N_i = 0.$$

We shall write this condition as

$$\left( \sum_j a_j t_j \partial / \partial t_j \right) \Phi(t) = 0$$

at the point  $t$  in question.

From

$$\left( \sum_i a_i t_i \partial / \partial t_i \right) t_1^{c_1} \cdots t_k^{c_k} = \left( \sum_i a_i c_i \right) t_1^{c_1} \cdots t_k^{c_k}$$

we have the conclusion

$$\left\{ \begin{array}{l} t_1^{c_1} \cdots t_k^{c_k} \text{ is constant on} \\ \text{the connected fibres of } \Phi \end{array} \right\} \iff (c_1, \dots, c_k) \in \text{Rel}(N_1, \dots, N_k)^\perp$$

where  $\text{Rel}(N_1, \dots, N_k)$  is the  $\mathbb{Q}$ -vector space of linear relations among  $N_1, \dots, N_k$ . We may choose a generating set of the  $c_1, \dots, c_k$  where  $c_i \in \mathbb{Z}$ .

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<sup>20</sup>The  $F_0 \in \check{D}$  is only defined to the action of  $\exp(\mathbb{C} \cdot \sum_{i=1}^k N_i)$  on  $\check{D}$ . The fibres  $F_e$  of the extended Hodge vector bundle are however well defined, as is the associated graded  $\text{Gr}(\text{LMHS})$  to the limiting mixed Hodge structures. Since we shall only be concerned with the associated graded to the limiting mixed Hodge structures we hope the ambiguity in the notation will not create a difficulty.

**Step 1b:** For each index set  $I \subset \{1, \dots, k\}$  we let

$$\Delta_I^* = \{t = (t_1, \dots, t_k) : t_i = 0 \text{ for } i \in I \text{ and } j \neq 0 \text{ for } j \in I^c\}$$

denote the open strata in the standard stratification of  $\Delta^k$ . Setting  $N_I = \sum_{i \in I} N_i$  and denoting by  $t_I$  a point of  $\Delta_I^*$ , there is a polarized limiting mixed Hodge structure

$$\lim_{t \rightarrow t_I} \Phi(t) = (V, W(N_I), F_I)$$

where  $F_I = \exp(iN_{I^c}) \cdot F_0$  and with weight filtration  $W(N_I)$ .<sup>21</sup> Taking the primitive parts of the associated graded to these limiting mixed Hodge structures leads to a period mapping

$$\Phi_I : \Delta_I^* \rightarrow \Gamma_{T_I} \setminus D_I$$

where  $\Gamma_{T_I}$  is generated by the  $T_j$  for  $j \in I^c$ . The question arises as to the relation between  $W(N_I)$  and  $W(N_{I \cup J})$  where  $I, J$  are disjoint index sets. The answer is given by the *relative weight filtration property* that we now explain.

(i) Since  $[N_I, N_J] = 0$ , the map  $N_J$  induces a map

$$N_J^o : \mathrm{Gr}_m^{W(N_I)} V \rightarrow \mathrm{Gr}_m^{W(N_I)} V$$

on the graded pieces of the  $W(N_I)$  filtration;

(ii) this nilpotent endomorphism induces a weight filtration  $W(N_J^o)$  on  $\mathrm{Gr}_m^{W(N_I)} V$ ;

(iii)  $W(N_{I \cup J})$  also induces a filtration on  $\mathrm{Gr}_m^{W(N_I)} V$ , and the relative weight filtration property is

$$(II.4) \quad W_{n+m}(N_{I \cup J}) \cap \mathrm{Gr}_m^{W(N_I)} V = W_n(N_J^o) \cap \mathrm{Gr}_m^{W(N_I)} V.$$

In words:

- $N_J$  induces a filtration  $W(N_J^o)$  on  $\mathrm{Gr}_m^{W(N_I)} V$ ;
- $W(N_{I \cup J})$  also induces a filtration on  $\mathrm{Gr}_m^{W(N_I)} V$ ;
- these filtrations agree.

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<sup>21</sup>There is a subtlety here in that the choice of a reference  $F_0$  depends on the index set  $I$  and an ordering of the indices in  $\{1, \dots, k\}$  (cf. [C-K-S1]). However, the associated graded to the limiting mixed Hodge structures are well defined.

This is a *very* special property of a pair of commuting nilpotent endomorphisms of a vector space. That it holds for the  $N_i$  arising from a VHS is a Hodge-theoretic, not a linear algebra, property of the nilpotent Lie algebra generated by  $N_1, \dots, N_k$  (cf. [C-K-S1]).<sup>22</sup>

**Step 1c:** The action of  $N_I$  on  $V$  induces one on  $\mathfrak{g} \subset \text{Hom}(V, V)$  and we denote by  $W(N_I)\mathfrak{g}$ , or just  $W(N_I)$  when no confusion is possible, the weight filtration induced by this action. The analogue of the condition  $\sum_i a_i N_i = 0$  in Step 1a is now that the condition

$$\left( \sum_{j \in I^c} a_j t_j \partial / \partial t_j \right) \Phi_I = 0$$

on  $\Delta_I^*$  is equivalent to

$$\sum_{j \in I^c} a_j N_j \in W_{-1}(N_I).$$

It is here that one sees operationally how passing to the associated graded of the LMHS's enters the picture; the condition just above means that the vector field  $\sum_j a_j t_j \partial / \partial t_j$  is tangent to the fibres of the period mapping  $\Phi_I$ . We note that in both cases above the sum is over all  $j$ ; in the first sum we have  $t_i = 0$  on  $\Delta_I^*$  for  $i \in I$ , and in the second sum  $N_i \in W_{-1}(N_I)$ . What we want to prove is

$$(II.5) \quad \begin{array}{l} \text{For } I \subset I', \text{ if } t_1^{c_1} \cdots t_k^{c_k} \text{ is constant on the fibres for } \Delta_I^*, \\ \text{then } t_1^{c_1} \cdots t_k^{c_k} \text{ is constant on the fibres of } \Delta_{I'}^* \subset \overline{\Delta_I^*}. \end{array}$$

The proof will be given by two propositions, the first of which is

**PROPOSITION II.6:** *Set  $S_I = \{\sum a_i N_i : a_i \in \mathbb{R} \text{ and } \sum_i a_i N_i \in W_{-1}(N_I)\}$ . Then for  $I \subset I'$*

$$S_I \subseteq S_{I'}.$$

*Proof.* Let  $J = I' \setminus I$ . Setting  $N_I = \sum_{i \in I} N_i$ , as noted above the nilpotent operator  $N_I$  induces a weight filtration  $W(N_I)$  on  $\mathfrak{g}$ . The relative weight filtration property says that you add the weight of  $\text{Gr}_0^{W(N_I)} N_J$

<sup>22</sup>Cf. also [S-Z] for an extensive discussion of the linear algebra involved here.

acting on  $\bigoplus_m \text{End}(\text{Gr}_m^{W(N_I)} \mathfrak{g})$  to  $m$ . Now

$$\sum_i a_i N_i \in \mathcal{Z}(N_J) = \text{centralizer of } N_J \text{ in } \mathfrak{g}.$$

Thus for  $N_{i,0} = \text{Gr}_0^{W(N_I)} N_i$  we have

$$\sum_i a_i N_{i,0} \in \mathcal{Z}\left(\text{Gr}_0^{W(N_I)} N_J\right)$$

which implies that

$$\sum_i a_i N_{i,0} \in W_0\left(\text{Gr}_0^{W(N_I)} N_J\right).$$

If  $\sum a_i N_i \in W_{-1}(N_I)\mathfrak{g}$ , then by the relative weight filtration property we have

$$\sum_i a_i N_i \in W_{-1+0}(N_{I+J})(\mathfrak{g}) = W_{-1}(N_{I'}).$$

This gives the statement in the proposition.  $\square$

**Remark:** The relative weight filtration property implies the following. If  $N, N', N''$  are in the closure of the monodromy cone, then

$$N'' \in W_{-1}(N) \implies N'' \in W_{-1}(N + N').$$

Thus

$$\sum_j a_j N_j \in W_{-1}(N_I) \implies \sum_j a_j N_j \in W_{-1}(N_I + N_k) \text{ for any } k \notin I.$$

The other key to the construction of the monomial charts is a second proposition given below following two preliminary lemmas whose statements are necessary for the formulation of the proposition.

Let  $Q^+ = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_i \geq 0\}$  be the first quadrant, and let  $S \subset \mathbb{R}^k$  be a linear subspace.

**LEMMA II.7:** *If  $S \cap Q^+ = 0$ , then  $S^\perp \cap Q^+ \neq 0$ .*

The proof of this lemma is a consequence of the following result from classical linear programming.

FARKAS ALTERNATIVE LEMMA:<sup>23</sup> For  $A$  a real  $m \times k$  matrix and  $b \in \mathbb{R}^k$ , of the following two problems exactly one has a solution:

- (i)  $Ax = b$  where  $x \in \mathbb{R}^k$  and  $x_i \geq 0$  (i.e.,  $x \in Q^+$ );
- (ii)  $y^t A \geq 0$  and where  $y \in \mathbb{R}^m$  and  $y^t b \leq 0$ .

The second preliminary lemma is

LEMMA II.8: Given  $Q^+, S$  as above, there is a unique partition  $K$  of  $\{1, \dots, k\}$  for which there exist

$$\begin{aligned} v_1 &\in Q^+ \cap S, x_i > 0 \text{ if } i \in K \text{ where } v_1 = (x_1, \dots, x_k) \\ v_2 &\in Q^+ \cap S, x'_i > 0 \text{ if } i \in K^c \text{ where } v_2 = (x'_1, \dots, x'_k). \end{aligned}$$

Before giving the proof of the lemma we note that it is possible that  $K = \emptyset, v_1 = 0$  or  $K^c = \emptyset, v_2 = 0$ . From the lemma we have the

COROLLARY: If  $Q^+ \cap S = 0$ , then there exists  $v = (v_1, \dots, v_k) \in S^\perp$  with  $v_i > 0$  for all  $i$ . If  $Q^+ \cap S^\perp = 0$ , then there exists  $v \in S$  with  $v_i > 0$  for all  $i$ .

For the proof of Lemma II.8 we have from Lemma II.7 that either

$$\text{there exists } v_1 \neq 0 \text{ in } Q^+ \cap S \text{ or } v_2 \neq 0 \text{ in } Q^+ \cap S^\perp.$$

By symmetry it suffices to consider the case where we have  $v_1 = (v_{1,1}, \dots, v_{1,k}) \neq 0$  in  $Q^+ \cap S$ . If  $v_{1,i} > 0$  for all  $i$  we may take  $K = \{1, \dots, k\}$ . If not we have  $J \neq \emptyset$  with  $v_{1,i} = 0 \iff i \in J$ . For  $\mathbb{R}^{|J|} =$  subspace of  $\mathbb{R}^k$  corresponding to vectors with coordinate entries corresponding to the index set  $J$ , we let  $\pi : \mathbb{R}^k \rightarrow \mathbb{R}^{|J|}$  be the orthogonal projection. For  $e_1, \dots, e_k$  the standard basis for  $\mathbb{R}^k$ , if there exists  $w \in S$  with  $\langle w, e_i \rangle \geq 0$  for  $i \notin J$  and  $\langle w, e_{i_0} \rangle > 0$  for some  $i_0 \in J$ , then  $v_1 - \epsilon w \in Q^+ \cap S$  and has

$$(v_1 - \epsilon w)_i = 0 \iff i \in J' \text{ with } J' \subset J \text{ (} i_0 \notin J').$$

We may assume  $J$  is minimal, so  $v_{1,i} > 0$  if  $i \in J^c$ . Then there does not exist  $w \in S$  such that  $\langle w, e_i \rangle \geq 0$  for  $i \in J$  and  $\langle w, e_{i_0} \rangle > 0$  for

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<sup>23</sup>An informative discussion is in Wikipedia:  
[https://en.m.wikipedia.org/wiki/Farkas' lemma](https://en.m.wikipedia.org/wiki/Farkas'_lemma)

some  $i_0 \in J$ . Equivalently

$$\pi(S) \cap (Q^+ \cap \mathbb{R}^{|J|}) = 0$$

by induction on  $k$ , for some  $v_1 \neq 0$  so that  $\dim \mathbb{R}^{|J|} < k$ . By the above corollary there exists  $u \in \mathbb{R}^{|J|} \cap (\pi(S))^\perp$  with coordinates  $u_i > 0$  for all  $i \in J$ . Lift  $u$  to  $v_2 \in \mathbb{R}^k$  by taking  $v_{2,i} = 0$  if  $i \in J^c$ . Then  $v_2 \in S^\perp$ , and  $v_1, v_2$  satisfy the conditions in the lemma for  $I = J$ .

To prove uniqueness, if  $K$  and  $K'$  both satisfy the conditions in the lemma, then we have

$$v_1, v_2, K, K^c \text{ and } v'_1, v'_2, K', K'^c.$$

If  $K \cap K'^c \neq \emptyset$  we would have  $\langle v_1, v'_2 \rangle > 0$  contradicting  $v'_1 \in S$ ,  $v_2 \in S^\perp$ . Then  $K \cap K' \neq \emptyset$  and  $K^c \cap K' = \emptyset$  gives uniqueness.  $\square$

For  $I \subset \{1, \dots, k\}$  we set

$$S_I = \{(v_1, \dots, v_k) : v_i \in \mathbb{R} \text{ and } \sum_i v_i N_i \in W_{-1}(N_I)\}.$$

The crucial second proposition is

**PROPOSITION II.9:** *Let  $K_I$  be the  $K$  associated to  $S_I$  by Lemma II.8. Then*

$$S_I = S_{I'} \text{ for all } I \subseteq I' \subseteq I + K_I.$$

*Proof.* Let  $v_1 = \sum_{i \in K_I} m_i N_i$  where all  $m_i > 0$ . Then by the defining property of  $K_I$

$$\sum_{i \in K_I} m_i N_i \in W_{-1}(N_I).$$

From the property of relative weight filtrations we have

$$W_\bullet(N_I) = W_\bullet(N_{I+K_I}).$$

Then  $S_I = S_{I+K_I}$  which using  $S_I \subseteq S_{I'} \subseteq S_{I+K_I}$  gives the proposition.  $\square$

We are now ready to construct the monomials that give the map  $\mu : \Delta^k \rightarrow \mathbb{C}^m$  with the desired properties. We note that

$$c \in S_I^+ \cap Q^+ \cap \mathbb{Z}^k \iff \left\{ \begin{array}{l} t_1^{c_1} \cdots t_k^{c_k} \text{ is constant on the connected} \\ \text{fibres of the nilpotent orbit correspond-} \\ \text{ing to } \Phi_I \end{array} \right\}.$$



If  $I \subseteq I'$  there are the two cases

- (a)  $I' \cap (I + K_I)^c \neq \emptyset$ ;
- (b)  $I' \subseteq I + K_I$ .

In case (a) there is  $t_1^{c_1} \cdots t_k^{c_k}$  with all  $c_i > 0$  for  $i \in (I + K_I)^c$  and all other  $c_i = 0$ . Such a monomial restricts to 0 on  $\Delta_{I'}^*$ , and therefore it can be used as a component of  $\mu$ . The set of monomials corresponding to  $S_I^\perp \cap Q^+ \cap \mathbb{Z}^k$  distinguish the points in  $\Delta_I$ .

In case (b),

$$S_{I'}^\perp \cap Q^+ \cap \mathbb{Z}^k = S_I^\perp \cap Q^+ \cap \mathbb{Z}^k = S_{I+K_I}^\perp \cap Q^+ \cap \mathbb{Z}^k.$$

Then since  $I' + K_{I'} = I + K_I$  every monomial corresponding to  $S_{I'}^\perp \cap Q' \cap \mathbb{Z}^k$  restricts to a monomial on  $\Delta_{I'}$  corresponding to a vector in  $S_{I'}^\perp \cap Q^+ \cap \mathbb{Z}^k$ .

In summary we have obtained the following

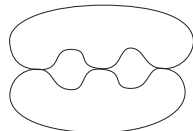
(II.10) **Prescription for constructing the monomial map:** *For each  $I$  use all monomials corresponding to  $S_I^\perp \cap Q^+ \cap \mathbb{Z}^k$  with  $c_i > 0$  if, and only if,  $i \in (K_I + I)^c$ .*

When we restrict to  $\Delta_{I'}$  where  $I \subseteq I'$ , then the restrictions of the monomials in the prescription vanish (case (a) above), or they restrict to monomials corresponding to  $S_{I'}^\perp \cap Q^+ \cap \mathbb{Z}^k$  with  $c_i > 0$  if, and only if,  $i \in (K_{I'} + I')^c = (K_I + I)^c$  (case (b) above). We note that the crucial step in the construction is the choice of the exponents in  $t_1^{c_1} \cdots t_k^{c_k}$  exactly corresponds to  $\sum_i c_i m_i = 0$  whenever  $\sum_i m_i N_i \in W_{-1}(N_I)$ . It is this last relation that means we are picking up the associated graded to  $\lim_{t \rightarrow \Delta_I^*} \Phi(t)$ , and it is the relative weight filtration property that ensures that the maps fit together across the strata.

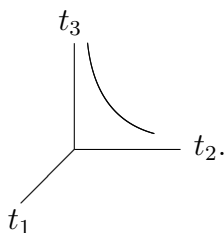
This completes the construction of the monomial map charts for the case of nilpotent orbits.

The final step is for a general VHS to suitably perturb the given monomial map chart for the approximating nilpotent orbit to obtain a chart for it. We will turn to this after we discuss the following

**Example:** The maximal degeneration in  $\overline{\mathcal{M}}_2$  of a genus 2 curve is the nodal curve<sup>24</sup>



In  $\overline{\mathcal{M}}_2$  a neighborhood of this curve is a  $\Delta^3$  and we will (i) describe the corresponding nilpotent orbit, and (ii) illustrate how the above proof works in this case. The picture is



The maximally degenerate curve corresponding to the origin is the one above, the curves on the axes outside the origin will be



those on the 2-planes outside the axes will be



and those in the interior will be smooth. Our conclusion will be that the monomial mapping is

$$\begin{array}{ccc} \Delta^3 & \xrightarrow{\mu} & \mathbb{C}^4 \\ \cap & & \cap \\ (t_1, t_2, t_3) & \longrightarrow & (t_1 t_2, t_1 t_3, t_2 t_3, t_1 t_2 t_3) \end{array}$$

We note that

<sup>24</sup>As usual we denote by  $\mathcal{M}_g$  the moduli space for smooth curves of genus  $g \geq 2$ , and by  $\overline{\mathcal{M}}_g$  its canonical compactification (Deligne-Mumford) obtained by adding stable curves  $C$  with arithmetic genus  $p_a(C) = g$  and ample dualizing sheaf  $\omega_C$ .

- *the axes all map to a point.*

The curves on the axes have moduli; their normalizations are  $(\mathbb{P}^1, 4 \text{ points})$  and the cross ratio of the 4 points gives the modulus. But this parameter is in the extension data of the limiting mixed Hodge structure and it disappears when we pass to the associated graded.

- *the level sets on the faces are parabolas  $t_1 t_2 = c$ , etc.*

The curves on the faces have two moduli; their normalizations are  $(E, 2 \text{ points})$  where  $E$  is an elliptic curve. The 2 points are in the extension data of the limiting mixed Hodge structure; the associated graded is the Hodge structure of  $E$ . Note that as  $c \rightarrow 0$  the level sets tend to the two axes; this is the compatibility result in this case.

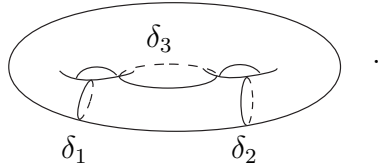
For the computation, we may choose a symplectic basis for  $V = \mathbb{Z}^4$  with the standard alternating form so that

$$N_i = \begin{pmatrix} 0 & S_i \\ 0 & 0 \end{pmatrix}$$

where

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For example we may take for the vanishing cycles  $\delta_1, \delta_2, \delta_3$  giving rise to the Picard-Lefschetz transformations corresponding to the  $N_i$  to be those in the picture



Here the standard basis for  $\mathbb{Z}^4$  is  $e_1, e_2, f_1, f_2$ , where  $e_i$  corresponds to  $\delta_i$  for  $i = 1, 2$  and  $f_i$  is the dual 1-cycle to  $\delta_i$ . The period matrix for the nilpotent orbit is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \ell(t_1 t_3) & \ell(t_3) \\ \ell(t_3) & \ell(t_2 t_3) \end{pmatrix}.$$

The  $N_i$  are linearly independent so that the period mapping has no positive dimensional fibres in  $\Delta^{*3}$ . On the face corresponding to  $t_1 = 0$  which corresponds to the monodromy logarithm  $N_1$ , we have

$$e_1 \in W_1(N_1)V, f_1 \in W_{-1}(N_1)V \text{ and } e_2, f_2 \in W_0(N_1)V.$$

Thus  $W_{-1}(N_1)\mathfrak{g}$  takes

$$\begin{cases} e_1 \rightarrow \mathbb{Z}f_2 \\ e_2 \rightarrow \mathbb{Z}f_1 \end{cases}$$

and so it is the set of endomorphisms  $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \in \text{End}(V)$  where

$$S = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

This gives

$$N_3 - N_2 - N_1 \in W_{-1}(N_1)\mathfrak{g}$$

and the monomial constant on the fibres is  $t_2t_3$ .

As a final comment we note that

- *the image of the monomial mapping  $\Delta^3 \rightarrow \mathbb{C}^4$  lies in the toroidal algebraic variety*

$$z_4^2 = z_1z_2z_3.$$

As one sees from (II.10), this is as a general feature of monomial mappings associated to nilpotent orbits.

**Step 2:** We want to extend the prescription (II.10) to a general VHS

$$\Phi : \Delta^{*k} \times \Delta^\ell \rightarrow \Gamma_T \backslash D.$$

We shall give an argument in the weight 1 case and then take up the general case.

In the weight 1 case it is classical (cf. [C-M-S-P] or [Ca]) that a symplectic  $\mathbb{Q}$ -basis for  $V$  may be chosen so that the period matrix  $\Omega(t) \in \mathcal{H}_g$  has the form

$$\Omega(t) = \begin{pmatrix} \sum_j \ell(t_j)S_j + A(t) & B(t) \\ {}^tB(t) & C(t) \end{pmatrix} \begin{matrix} \} g' \\ \} g'' \end{matrix}$$

where

- $S_j = {}^tS_j \geq 0$  are integral symmetric matrices and  $\sum_j S_j > 0$ ;

- $A(t) = {}^t A(t), B(t), C(t)$  are holomorphic and  $C(t) \in \mathcal{H}_{g''}$ .

Here we are using the customary notation  $\mathcal{H}_m$  for the Siegel generalized upper-half-plane of  $m \times m$  symmetric matrices  $Z$  with  $\text{Im } Z > 0$ .

Let  $P =$  space of  $g' \times g'$  symmetric matrices,  $R = \text{span}_{\mathbb{C}}\{S_1, \dots, S_k\}$  and  $R^\perp$  the orthogonal space to  $R$  so that

$$P = R \oplus R^\perp.$$

Under a reparametrization  $\tilde{t}_j = t_j e^{f_j(t)}$  we have

$$A(t) \rightarrow A(t) + \sum_j f_j(t) \cdot S_j.$$

Thus we may choose coordinates so that  $\Omega(t)$  has the above form with  $A(t) \in R^\perp$  in the above decomposition. Consequently the level sets for  $\sum_j \ell(t_j) S_j + A(t)$  are intersections of the level sets for the nilpotent orbit  $\sum_j \ell(t_j) S_j$  and for the holomorphic matrix  $A(t)$ . Thus the desired result in the weight 1 case is reduced to

- the case (II.10) of nilpotent orbits;
- a standard foundational result in complex analysis.

The foundational result is the following

(II.11) *Let  $F : U \rightarrow \mathbb{C}^N$  be a holomorphic mapping of a neighborhood  $U$  of the origin in  $\mathbb{C}^n$ . Setting  $F(0) = p \in \mathbb{C}^N$  by shrinking  $U$ ,  $F(U)$  will be an analytic subvariety defined in a neighborhood of  $p$ .*

Without normalizing as above, for the component

$$Z(t) = \sum_j \ell(t_j) S_j + A(t)$$

of the period matrix we may proceed as follows (ignoring the  $2\pi i$  factors):

$$dZ(t) = \sum_j S_j \frac{dt_j}{t_j} + dA(t).$$

If  $\mu(t) = t_1^{c_1} \cdots t_k^{c_k}$  is a monomial for the nilpotent orbit, then we have from

$$d \log \mu = \sum_j c_j \frac{dt_j}{t_j}$$

that

$$d \log \mu \equiv 0 \text{ modulo the entries in } d \left( \sum_j \ell(t_j) S_j \right)$$

$$\Updownarrow$$

$$\sum_m c_j m_j = 0 \text{ whenever } \sum_j m_j N_j = 0.$$

If we consider

$$f(t) = \mu(t) e^{u(t)}$$

then we have

$$d \log f(t) \equiv 0 \text{ modulo the entries in } dZ(t)$$

if

$$(i) \quad \sum_j c_j m_j = 0 \text{ whenever } \sum_j m_j N_j = 0$$

and

$$(ii) \quad du(t) \equiv 0 \text{ entries in } dA(t).$$

If we choose separately  $\mu(t)$  such that (i) is satisfied and such that (ii) is satisfied, then  $f(t)$  is constant on the connected components of the  $Z(t) = \text{constant}$  part of the VHS. The remaining parts fall under (II.11).

We note the implication of the above argument:

(II.12) If we perturb the nilpotent orbit to a nearby VHS, then the level sets will drop in dimension. This is what is expected by upper-semi-continuity. However the equations of the level set *separate* into the ones arising from the nilpotent orbit (the logarithmic ones) and the rest (the holomorphic ones).

We will give an argument following (II.14) that suitably interpreted this implication remains valid for a general weight.

For this we shall use [C-K1] to normalize the expression for the VHS.

For

$$\Phi(t) = \exp \left( \sum_j \ell(t_j) N_j \right) \cdot u(t) \cdot F_o$$

where  $F_o \in \check{D}$  and  $u(t) \in G_{\mathbb{C}}$ , we may choose  $u(t)$  such that the normalization condition

$$\text{Ad } u(t) \left( \frac{1}{2\pi i} \right) \frac{N_j}{t_j} + u(t)^{-1} \partial u(t) / \partial t_j \in \mathfrak{b}$$

is satisfied, where

$$\mathfrak{b} = \bigoplus_{\substack{p \leq -1 \\ q}} \mathfrak{g}^{p,q}$$

is a nilpotent sub-algebra of  $\mathfrak{g}_{\mathbb{C}}$  and  $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p,q} \mathfrak{g}^{p,q}$  is the Deligne decomposition of  $\mathfrak{g}_{\mathbb{C}}$  relative to the  $\mathbb{R}$ -split limiting mixed Hodge structure associated to  $\Phi(t)$ . If the vector field  $\sum_j a_j \partial \setminus \partial t_j$  with  $a_j = a_j(t_1, \dots, t_k)$  is tangent to the fibres of the variation of Hodge structure  $\Phi(t)$ , then

$$F^0 \cap \mathfrak{g}_{\mathbb{C}} = \bigoplus_{\substack{p' \geq 0 \\ q}} \mathfrak{g}^{p',q}$$

for the limiting mixed Hodge structure gives

$$F^0 \mathfrak{g}_{\mathbb{C}} \cap \mathfrak{b} = 0.$$

The condition that  $F_o$  not move then becomes

$$\sum_j a_j \left( \left( \frac{1}{2\pi i} \right) \frac{N_j}{t_j} + \frac{\partial u}{\partial t_j} u^{-1} \right) = 0.$$

Setting  $b_j = a_j/t_j$  this gives

$$(II.13) \quad \sum_j b_j N_j + (2\pi i) b_j t_j \frac{\partial u}{\partial t_j} u^{-1} = 0.$$

As above denoting by

$$S = \{(c_1, \dots, c_k) : \sum_j c_j N_j = 0\}$$

the relations among the  $N_j$ , we decompose

$$b = b_S + b_{S^\perp}$$

into its  $S$  and  $S^\perp$  components. Then (II.13) gives

$$\sum_j b_{S^\perp, j} N_j + 2\pi i \sum_j b_j t_j \frac{\partial u}{\partial t_j} u^{-1} = 0.$$

This gives a bound

$$\|b_{S^\perp}\| \leq C \|b\| \|t\|.$$

In other words, by normalizing the  $u(t)$  that gives the perturbation of the nilpotent orbit approximating  $\Phi(t)$  the component of  $b$  orthogonal to the relations on the  $N_j$  is small.

The above estimate may be rewritten

$$\frac{\|b_{S^\perp}\|}{\|b\|} \leq C\|t\|;$$

i.e., most of  $b$  is in the space  $S$  of relations among the  $N_j$ . Now let

$$X = \sum_j b_j t_j \partial / \partial t_j, \quad Y = \sum_j b'_j t_j \partial / \partial t_j$$

be tangent to the fibres of  $\Phi(t)$  and chosen so that the bracket  $[X, Y] = 0$ . We will show that

$$(II.14) \quad Yb_{S^\perp} - Xb'_{S^\perp} \in S.$$

*Proof.* From (II.14) we have

$$\begin{cases} Xu = -\left(\frac{1}{2\pi i}\right) \sum_j b_{j,S^\perp} N_j u \\ Yu = -\left(\frac{1}{2\pi i}\right) \sum_j b'_{j,S^\perp} N_j u \end{cases}$$

which gives

$$\begin{aligned} YXu &= \left(\frac{1}{2\pi i}\right) \left( \sum_j Yb_{j,S^\perp} N_j u - \left(\frac{1}{2\pi i}\right) \left( \sum_j b_{j,S^\perp} N_j \right) \left( \sum_m b'_{m,S^\perp} N_m \right) u \right), \\ XYu &= -\left(\frac{1}{2\pi i}\right) \left( \sum_j X'b'_{j,S^\perp} N_j u - \left(\frac{1}{2\pi i}\right) \left( \sum_j b_{j,S^\perp} N_j \right) \left( \sum_m b'_{m,S^\perp} N_m \right) u \right). \end{aligned}$$

Using  $[N_j, N_m] = 0$  and  $[X, Y] = 0$  we have

$$0 = YXu - XYu = \left(-\frac{1}{2\pi i}\right) \left( \sum_j Yb_{j,S^\perp} - Xb'_{j,S^\perp} \right) N_j u$$

which gives

$$\sum_j (Yb_{j,S^\perp} - Xb'_{j,S^\perp}) N_j = 0$$

as desired.  $\square$



We want to find functions  $g$  that are constant on the fibres of the integral varieties of the commuting vector fields  $X = \sum a_j \partial / \partial t_j$  and  $Y = a'_j \partial / \partial t_j$ . The commutation condition is

$$Xa'_j = Ya_j$$

for all  $j$ . If we set

$$g = t_1^{c_1} \cdots t_k^{c_k} e^f,$$

then by the above estimates

$$X = X_0 + \epsilon X_1, \quad Y = Y_0 + \epsilon Y_1$$

where  $X_0, Y_0$  are tangent to the fibres of the nilpotent orbits and  $[X_0, Y_0] = 0$ . Then

$$\begin{aligned} Xg &= (X_0 + \epsilon X_1)g = \overbrace{(X_0(t_1^{c_1} \cdots t_k^{c_k}))}^0 e^f + t_1^{c_1} \cdots t_k^{c_k} (X_0 b) e^f \\ &\quad + \epsilon (X_1 b) t_1^{c_1} t_k^{c_k} e^f + \epsilon (X_1(t_1^{c_1} - t_k^{c_k})) e^f \\ &= (X_0) t_1^{c_1} \cdots t_k^{c_k} e^f + \epsilon e^f X_1(t_1^{c_1} \cdots t_k^{c_k}) \end{aligned}$$

so we want

$$(II.15) \quad Xf = -\epsilon X_1(\log t_1^{c_1} \cdots t_k^{c_k}).$$

Now

$$\begin{aligned} YXf &= -\epsilon YX_1 \log(t_1^{c_1} \cdots t_k^{c_k}), \\ XYf &= -\epsilon XY_1 \log(t_1^{c_1} \cdots t_k^{c_k}) \end{aligned}$$

while

$$\begin{aligned} 0 &= [X, Y] = \underbrace{(X_a Y_0)}_0 + \epsilon (X_0 Y_1 + X_1 Y_0 - Y_0 X_0 - Y_1 X_0) + (X_1 Y_1 - Y_1 X_1) \\ &= \epsilon ((X_0 + \epsilon X_1) Y_1 - (Y_0 + \epsilon Y_1) X_1) + \epsilon (X_1 Y_0 - Y_1 X_0). \end{aligned}$$

Then

$$X_0 \log(t_1^{c_1} \cdots t_k^{c_k}) = 0, \quad Y_0 \log(t_1^{c_1} \cdots t_k^{c_k}) = 0$$

so

$$(XY_1 - YX_1)(\log(t_1^{c_1} \cdots t_k^{c_k})) = 0.$$

Consequently  $YXf = XYf$  if  $f$  satisfies (II.14) for  $X$  and  $Y$ .

We note that if  $X_0g_0 = 0$ ,  $Y_0g_0 = 0$  and if  $g = g_0e^f$ ,

$$Xf = -\epsilon X_1 \log g_0$$

is what we need, since this gives

$$XYf - YXf = -\epsilon(XY_1 - YX_1) \log g_0 = 0. \quad \square$$

**Step 4:** At this point we have constructed local analytic charts corresponding to a neighborhood  $\bar{u} \subset \bar{B}$  around points of  $\bar{B} \setminus B$  where we have  $\bar{u} \cap B \cong \Delta^{*k} \times \Delta^\ell$  and the VHS is given by a period mapping  $\Delta^{*k} \times \Delta^\ell \rightarrow \Gamma_T \backslash D$ . Here  $\Gamma_T = \{T_1, \dots, T_k\}$  is the *local* monodromy group, but the construction must take into account the *global* monodromy group. In the diagram

$$\begin{array}{ccccc} \tilde{B} & \longleftarrow & \tilde{u} & \xrightarrow{\tilde{\Phi}} & D \\ \downarrow & & \downarrow & & \downarrow \\ B & \subset & u & \xrightarrow{\Phi} & \Gamma \backslash D \end{array}$$

where the tildes are universal covering spaces, we shall first prove

$$(II.16) \quad \text{Only finitely many elements } \gamma \text{ of } \Gamma \text{ outside of } \Gamma_T \text{ satisfy} \\ \gamma \tilde{\Phi}(\tilde{u}) \cap \tilde{\Phi}(\tilde{u}) \neq \emptyset.$$

We shall derive this statement as a consequence of the theorem in [C-D-K].<sup>25</sup> Here we shall use their results in the following form.

We consider the global VHS given by the product

$$\Phi \times \Phi : B \times B \rightarrow (\Gamma \backslash D) \times (\Gamma \backslash D).$$

In terms of local systems, we think of  $\Phi$  as given by a local system  $\mathbb{V}_{\mathbb{Z}} \rightarrow B$  with  $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$  satisfying the usual conditions, and using the bilinear form  $Q$  we identify the fibre  $\mathbb{V}_{p_1} \boxtimes \mathbb{V}_{p_2}$  over  $(p_1, p_2) \in B \times B$  with  $\text{Hom}(\mathbb{V}_{p_1}, \mathbb{V}_{p_2})$ . The condition to have a morphism of Hodge structures

$$v : \mathbb{V}_{p_1} \rightarrow \mathbb{V}_{p_2}$$

preserving the integral structure is then equivalent to have an integral Hodge class  $v \in \text{Hom}(\mathbb{V}_{\mathbb{Z}, p_1}, \mathbb{V}_{\mathbb{Z}, p_2})$ . The equations that under parallel

<sup>25</sup>The reference [C-K2] also contains a clear account and discussion of the proof of the main result in [C-D-K].

transport  $v$  remain a Hodge class in a neighborhood of  $(p_1, p_2)$  in  $B \times B$  define a local analytic subvariety in the neighborhood. The theorem in [C-D-K] has the following consequence:

(II.17) *The condition that there exist an integral Hodge class  $\zeta \in \text{Hom}(\mathbb{V}_{\mathbb{Z}, p_1}, \mathbb{V}_{\mathbb{Z}, p_2})$  and with Hodge length  $\|\zeta\| \leq c$  defines a global algebraic subvariety in  $B \times B$ .*

This implies that the locus in  $B \times B$  where the following is satisfied is an algebraic variety: There is

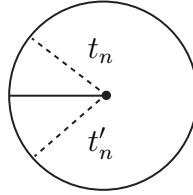
$$v \in \text{Hom}(\mathbb{V}_{1, p_1}, \mathbb{V}_{2, p_2})$$

with the property that there are determinations

$$\mathbb{V}_{1, p_1} \cong V_{\mathbb{Z}}, \mathbb{V}_{2, p_2} \cong V_{\mathbb{Z}}$$

such that  $v$  corresponds to the identity. Here, “determinations” means that we identify  $V_{i, p_i}$  with  $V$  up to the action of the global monodromy group  $\Gamma \subset \text{Aut}(V_{\mathbb{Z}}, Q)$ .<sup>26</sup> The Hodge length of such a  $v$  is equal to the Hodge length of the identity and consequently (II.17) may be applied to give the algebraicity of the locus described above.

We will give a special case of the above that will illustrate the essential idea behind the proof of (II.17). For this we assume that  $\dim B = 1$  and we localize around a point  $p \in \overline{B} \setminus B$  where we have the picture



Here we identify the slit disc with the strip  $|\text{Re } z| \leq 1/2$  in the upper-half-plane and think of  $t_n, t'_n$  as being points  $(\tilde{t}_n, \tilde{t}'_n)$  in this strip with  $\text{Im } \tilde{t}_n, \text{Im } \tilde{t}'_n \rightarrow \infty$ . Then

$$\Phi(t_n) = \Phi(t'_n) \text{ in } \Gamma \setminus D$$

translates into

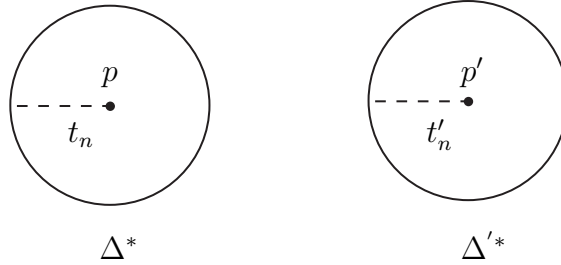
$$\gamma_n \tilde{\Phi}(\tilde{t}_n) = \tilde{\Phi}(\tilde{t}'_n).$$

<sup>26</sup>The issue of whether or not  $\Gamma$  is an arithmetic group does not enter (so that e.g.  $\Gamma$  could be a thin matrix group).

By restricting to the strip  $|\operatorname{Re} z| \leq 1/2$  we have eliminated the local monodromy around the puncture, and so if  $t_n, t'_n$  are *infinite* sequences of points tending to the origin in  $\Delta^*$  and that are identified by  $\Phi$  in  $\Gamma \backslash D$  we conclude that  $\Phi$  is constant. Thus the image of  $\Delta^*$  in  $\Gamma \backslash D$  cannot look like



The idea in the picture is that the  $\diamond$  's should be approaching the origin. A similar argument shows that we cannot localize around two different points  $p, p'$  in  $\overline{B} \backslash B$  to have a picture



where  $\Phi(t_n) = \Phi(t'_n)$  in  $\Gamma \backslash D$ .

For  $B$  of arbitrary dimension, the application of [C-D-K] extends to give the result:

*the identifications of  $\Delta^{*k} \times \Delta^\ell \rightarrow \Gamma \backslash D$  that occur outside of the local monodromy group  $\Gamma_T$  take place along a closed analytic subvariety in  $\Delta^k \times \Delta^\ell$ .*

This analytic variety will have only *finitely* many irreducible components, and from this we may infer (II.17). □

**Remark:** Recall that a *Siegel set* for a subgroup  $\Gamma$  of the arithmetic group  $\Gamma_{\mathbb{Z}} = \operatorname{Aut}(V_{\mathbb{Z}}, Q)$  is given by an open set  $\Sigma \subset D$  such that the set

$$\{\gamma \in \Gamma : \gamma \Sigma \cap \Sigma \neq \emptyset\}$$

is finite. If it were the case that the lift of a VHS  $\Phi : \Delta^{*k} \times \Delta^\ell \rightarrow \Gamma \backslash D$  to  $\tilde{\Phi} : \mathcal{H}^k \times \Delta^\ell \rightarrow D$  maps a Siegel set  $\{|\operatorname{Re} z_i| \leq c, \operatorname{Im} z_i \geq c'\}$  for the

action of  $\mathbb{Z}^k$  on  $\mathcal{H}^k$  given by  $z_i \rightarrow z_i + m_i$ ,  $m_i \in \mathbb{Z}$  to a Siegel set in  $D$ , then the desired result (II.17) would be immediate. We may informally phrase this property as ‘‘Siegel sets for the action of  $\mathbb{Z}^k$  on  $\mathcal{H}^k$  map to Siegel sets for the action of  $\Gamma$  on  $D$ .’’ This property is true for  $k = 1$  as a consequence of [Sc1], but even for nilpotent orbits it fails for  $k \geq 2$ . Thus it seems that some result such as [C-D-K] is needed to be able to prove (II.17).

At this point we have completed the proof of Theorem II.1. Among the principal steps in the argument are

- the use of the relative weight filtration property for a several parameter degeneration of polarized Hodge structures, and
- the use of [C-D-K] to control the global action of monodromy.

### III. CURVATURE PROPERTIES OF THE EXTENDED HODGE BUNDLE (A)

A central ingredient in the proof of Theorem B will be the use of the curvature properties of the extended Hodge bundle. We will give two approaches to these. The first, given in this section, will be inductive on the singular strata of the boundary divisor. Moreover, it will be restricted to the geometric case arising from a family of varieties, one of the points being that in this situation the singularities of the Hodge norms are localizable and visible analytically in a way that is suggestive of the general case. The second argument is given in the next section; it provides a proof of the general result, one in which the relative weight filtration property of several parameter degenerating polarized Hodge structures plays a central role.

We begin by discussing two general properties of currents that will arise.<sup>27</sup> On an  $n$ -dimensional complex manifold  $Y$ , we denote by  $A_c^{p,q}(Y)$  the compactly supported smooth  $(p, q)$  forms. A current  $T$  of type  $(p, q)$  gives a linear function

$$A_c^{n-p, n-q}(Y) \rightarrow \mathbb{C}.$$

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<sup>27</sup>Cf. [De] for a general account and references to the literature.

The currents we shall encounter will be differential forms  $\psi$  whose coefficients will be locally  $L^1$  functions, and the corresponding current  $T_\psi$  is given by

$$T_\psi(\alpha) = \int_Y \psi \wedge \alpha.$$

The differential  $\partial T_\psi(\alpha)$  is defined as usual by

$$\partial T_\psi(\beta) = \pm \int_Y \psi \wedge \partial \beta.$$

Similarly we may define  $\bar{\partial} T_\psi$  and  $\bar{\partial} \partial T_\psi$ .

For the  $\psi$ 's we shall use, we will also be able to define  $\partial\psi$  by applying the formal rules of calculus to the coefficient functions of  $\psi$ . The equality

$$(III.1) \quad \partial T_\psi = T_{\partial\psi}$$

shall mean: first the coefficients  $\partial\psi$  computed formally are locally  $L^1$  functions; and secondly that we have the equation (III.1) of currents. A similar notion holds for  $\bar{\partial}\psi$  and  $\bar{\partial}\partial\psi$ .

**Definition:** We shall say that the current represented by a locally  $L^1$  differential form  $\psi$  has the *property NR* if  $\partial\psi$ ,  $\bar{\partial}\psi$ ,  $\bar{\partial}\partial\psi$  computed formally have  $L^1$  coefficients, and if we have (III.1) for  $\partial\psi$ ,  $\bar{\partial}\psi$  and  $\bar{\partial}\partial\psi$ .

For example, in  $\mathbb{C}$ , we have  $\bar{\partial}\bar{\partial}\log|z| = 0$  formally, while up to a constant the equation of currents

$$\bar{\partial}\bar{\partial}T_{\log|z|} = \delta_0 dz \wedge d\bar{z}$$

holds. On the other hand, again up to a constant

$$\bar{\partial}\bar{\partial}\log(-\log|z|) = \frac{dz \wedge d\bar{z}}{|z|^2(\log|z|)^2}$$

holds both formally and in the sense of currents, so  $\log(-\log|z|)$  has the property NR while  $\log|z|$  does not.

In both these examples the coefficients of the derivatives computed formally are in  $L^1$ ; the difference is that for  $\log|z|$  we pick up a residue

term in  $\partial\bar{\partial}T_{\log|z|}$ , while no such term arises in  $\partial\bar{\partial}T_{\log(-\log|z|)}$ .<sup>28</sup> The term “NR” is meant to suggest “no residues.”<sup>29</sup>

For the second property we first recall that a current  $T$  on  $Y$  has a singular support  $\text{sing } T \subset Y$ , defined to be the smallest closed subset such that on the complement  $Y \setminus \text{sing } T$ , the current  $T$  is represented by a smooth differential form.

More important for our purposes will be the *wave front set*

$$\text{WF}(T) \subset T^*Y.$$

If  $W \subset Y$  is a submanifold, then in general the restriction to  $W$  of a distribution or current  $T$  given on  $Y$  is not defined.<sup>30</sup> However if  $W \subset Y$  is a submanifold whose tangent spaces are transverse to the wave front set in the sense that

$$(III.2) \quad TW \subset \text{WF}(T)^\perp,$$

then the restriction  $T|_W$  is defined.

In this work we will use a modified version of the wave front set, one which will satisfy the condition that the restriction property implied by (III.2) is defined. As an illustration of what will occur, we note as above that the currents we shall be interested in will be constructed from locally  $L^1$ -functions. It may or may not be possible to simply restrict such a function in the usual sense and obtain a well-defined function. As a simple example of what will be done below, on  $\Delta \times \Delta$  with coordinates  $(t, w)$ , the current given by  $1/\log \frac{1}{|t|} + f(w)$  where  $f(w)$  is smooth may be restricted to  $\{0\} \times \Delta$  to give  $f(w)$ .

Returning to our Hodge theoretic situation, we localize to a neighborhood  $\bar{\mathcal{U}}$  of a point of  $Z = \bar{M} \setminus M$  where for  $\mathcal{U} = \bar{\mathcal{U}} \cap M$  we have

$$\mathcal{U} \cong \Delta^{*k} \times \Delta^\ell$$

<sup>28</sup>Note that “ $\partial \log |z|$  computed formally in in  $L^1$ ” means that  $\partial \log |z| \wedge \alpha$  is in  $L^1$  for any  $C^\infty$  form  $\alpha$ .

<sup>29</sup>The property NR implies that the currents defined by  $\psi, \partial\psi, \partial\bar{\partial}\psi$  have vanishing Lelong numbers (cf. [De]).

<sup>30</sup>A good discussion of this with illustrative examples and references may be found under “Wave front sets” in Wikipedia.

with coordinates  $(t, w) = (t_1, \dots, t_k; w_1, \dots, w_\ell)$ . The period mapping (I.1) induces

$$\Phi : \Delta^{*k} \times \Delta^\ell \rightarrow \Gamma_T \backslash D$$

where  $\Gamma_T = \{T_1^{\mathbb{Z}}, \dots, T_k^{\mathbb{Z}}\}$  with  $T_i$  being the unipotent monodromy around the local branch  $t_i = 0$  of  $Z_i$ . We recall our notation: For each subset  $I \subset \{1, \dots, k\}$  we set  $\Delta_I = \{(t, w) : t_i = 0 \text{ for } i \in I\}$  and

$$\Delta_I^* = \{(t, w) : t_i = 0 \text{ for } i \in I, t_j \neq 0 \text{ for } j \in I^c\}$$

where  $I^c$  is the complement of  $I$ .

**Definition:** A positive function  $h$  defined in  $\mathcal{U} \cong \Delta^{*k} \times \Delta^\ell$  is said to have *logarithmic singularities* if

$$h = P \left( \log \frac{1}{|t_1|}, \dots, \log \frac{1}{|t_k|} \right) + R \left( \log \frac{1}{|t_1|}, \dots, \log \frac{1}{|t_k|} \right)$$

where  $P(x_1, \dots, x_k)$  is a homogeneous polynomial whose coefficients are real and are in  $C^\infty(\bar{\mathcal{U}})$  and are positive in the sense that

$$P(x_1, \dots, x_k) > 0 \text{ if all } x_i > 0,$$

$R$  is a real polynomial with  $C^\infty(\bar{\mathcal{U}})$  coefficients and that is lower order than  $P$  in the sense to be explained below, and where the conditions

- (i)  $\log h$  has the property NR;
- (ii) the current  $\Omega_e = (i/2)\bar{\partial}\partial \log h$  is positive and with our extended notion of the wave front set has the property that

$$\text{WF}(\Omega_e) \subset \bigcup_I N_{\Delta_I^*/\bar{\mathcal{U}}}^*$$

where  $N_{\Delta_I^*/\bar{\mathcal{U}}}^*$  is the co-normal bundle of the open stratum  $\Delta_I^*$  in  $\bar{\mathcal{U}}$ .

Because of (i) the current  $\Omega_e$  is defined so that (ii) makes sense. The extended definition of the wave front set that we use will be explained below.

In the remainder of this section we will restrict to the case  $k = 1$  where  $\mathcal{U} \cong \Delta^* \times \Delta^\ell$ . This is essentially the case of 1-parameter degenerations with dependence on parameters. In fact, for notational



simplicity we shall also assume that  $\ell = 1$ , so that we are working in  $\Delta^* \times \Delta$  with coordinates  $(t, w)$ .

The functions  $h$  we shall consider will be of the form

$$(III.3) \quad h = A(t, w) \left( \log \frac{1}{|t|} \right)^m \left( 1 + \frac{B_1(t, w)}{\log \frac{1}{|t|}} + \cdots + \frac{B_m(t, w)}{\left( \log \frac{1}{|t|} \right)^m} \right)$$

where  $A(t, w)$  and the  $B_i(t, w)$  are  $C^\infty$  in  $\Delta \times \Delta$  and  $A(0, w) > 0$ . We note that the expression (III.3) is invariant under holomorphic coordinate changes

$$(III.4) \quad \begin{cases} t' = tf(t, w) & f(t, w) \neq 0 \\ w' = g(t, w) & g_w(0, u) \neq 0. \end{cases}$$

As will be seen below, the motivation for considering functions of this form arises from the periods of holomorphic differentials in a degenerating family of algebraic varieties.

PROPOSITION III.5: *The function (III.3) has logarithmic singularities.*

*Proof.* Denoting by  $C$  the term in parentheses, since  $\log h = \log A + \log(m \log 1/|t|) + \log C$  the only issue concerns the  $\log C$  term. In

$$\partial \bar{\partial} \log C = \frac{\partial C}{C} \wedge \frac{\bar{\partial} C}{C} - \frac{\partial \bar{\partial} C}{C^2}$$

we shall separately examine the singularities in each term. For the first the most singular terms arise from

$$\begin{aligned} \partial \left( \frac{1}{\left( \log \frac{1}{|t|} \right)^a} \right) \wedge \bar{\partial} \left( \frac{1}{\left( \log \frac{1}{|t|} \right)^b} \right), & \quad a, b > 0 \\ \partial \left( \frac{1}{\left( \log \left( \frac{1}{|t|} \right) \right)^a} \right) \wedge \alpha, & \quad a > 0 \text{ and } \alpha \text{ is } C^\infty. \end{aligned}$$

The first of these are of the order

$$\frac{dt \wedge d\bar{t}}{|t|^2 \left( \log \frac{1}{|t|} \right)^c}, \quad c \geq 4,$$

and hence are  $o(\text{PM})$  where PM is the Poincaré metric. The second are of the order

$$\frac{dt}{|t| \left(\log \frac{1}{|t|}\right)^c} \wedge \beta \quad c \geq 2 \text{ and } \beta \text{ is } C^\infty$$

and are again  $o(\text{PM})$ .

The terms  $\partial \log C$  and  $\bar{\partial} \log C$  may be estimated by the last terms just above. For  $\partial \bar{\partial} C / C^2$ , the most singular terms are of the order

$$\partial \bar{\partial} \left( \frac{1}{\left(\log \frac{1}{|t|}\right)^a} \right) \sim \frac{dt \wedge d\bar{t}}{|t|^2 \left(\log \frac{1}{|t|}\right)^{a+2}}, \quad a \geq 1$$

which is again  $o(\text{PM})$ .

Note that the estimates in this argument have no room to spare.  $\square$

We denote by

$$\Omega_e = (i/2) \partial \bar{\partial} \log h$$

the curvature form associated to the function  $h$  in (III.3). Then

$$(III.6) \quad \Omega_e = mA(v, w) \left( \frac{i}{2} \right) \frac{dt \wedge d\bar{t}}{|t|^2 (\log |t|)^2} + o(\text{PM})$$

and assuming that  $m > 0$  it is positive with

$$\text{sing } \Omega_e = \{0\} \times \Delta.$$

It defines a closed, positive  $(1, 1)$  current on  $\Delta \times \Delta$  (cf. [C-K-S1] and [Kol2]). As for  $\text{WF}(\Omega_e)$ , the terms in  $\Omega_e$  not containing a  $dt$  or  $d\bar{t}$  are of the form

$$\frac{\gamma}{(\log |t|)^a}$$

where  $\gamma$  is a smooth  $(1, 1)$  form and  $a > 0$ . Thus although it is *not* the case that  $\text{WF}(\Omega_e) = N_{\{0\} \times \Delta / \bar{u}}^*$  is the co-normal bundle of  $\{0\} \times \Delta$  in  $\Delta \times \Delta$  in the usual sense, the restriction

$$(III.7) \quad \Omega_e|_{\{0\} \times \Delta}$$

is a well-defined smooth  $(1, 1)$  form. In fact, the above calculation shows that to define (III.7) we use the prescription

- (i) in the formula for  $\partial \bar{\partial} \log h$  first set  $dt = d\bar{t} = 0$ ;

(ii) then in the remaining terms take the limit as  $t \rightarrow 0$  (i.e., set  $1/\log \frac{1}{|t|} = 0$ ).<sup>31</sup>

The point in (ii) is that after (i) is done, in what remains  $\log 1/|t|$  only appears in the denominator and with positive powers. We note that the above prescription is invariant under coordinate changes (III.4).

The calculation in the proof of the proposition gives

$$(III.8) \quad \Omega_e|_{\{0\} \times \Delta} = (t/2)\partial\bar{\partial} \log A(0, w).$$

The somewhat subtle point here is that if we successively set  $dt = d\bar{t} = 0$  and then  $\log 1/|t| = \infty$ , *no* terms other than  $\partial\bar{\partial} \log A(0, w)$  remain.

We now apply the above to a weight  $n$  variation of Hodge structure over  $\Delta^* \times \Delta$ . Denote the canonically extended Hodge bundle by  $F_e \rightarrow \Delta \times \Delta$  and let  $\sigma(t, w)$  be a nowhere vanishing holomorphic section of this bundle. We assume that  $m$  is maximal with  $\sigma \in W_{n+m}(N) \cap F_e$ , and denote by  $\sigma_m(w)$  the projection of  $\sigma(0, w)$  in  $\text{Gr}_{n+m}^{W(N)} F_e$ . Then  $\sigma_m(w)$  is a non-zero section of  $\text{Gr}_{n+m}^{W(N)} V \cap F_e^n$  over  $\{0\} \times \Delta$ .

PROPOSITION III.9: *The Hodge norm  $\|\sigma(t, w)\|^2$  is of the form (III.3), and*

$$\partial\bar{\partial} \log \|\sigma(t, w)\|^2|_{\{0\} \times \Delta} = \partial\bar{\partial} \log \|\sigma_m(w)\|^2.$$

COROLLARY: *If  $\Omega_e$  is the Chern form of the extended Hodge line bundle  $\Lambda_e \rightarrow \Delta \times \Delta$ , and if  $\Omega_{\{0\} \times \Delta}$  is the Chern form of the graded to the associated variation of mixed Hodge structure along  $\{0\} \times \Delta$ , then the restriction  $\Omega_e|_{\{0\} \times \Delta}$  is defined and*

$$\Omega_e|_{\{0\} \times \Delta} = \Omega_{\{0\} \times \Delta}.$$

The corollary follows from the proposition by taking  $\sigma$  to be a generating section of the line bundle  $\Lambda_e \rightarrow \Delta \times \Delta$ .

We shall prove the proposition in the weight  $n = 2$  geometric case of a family  $\mathcal{X}^* \xrightarrow{\pi} \Delta^* \times \Delta$  of smooth surfaces where  $\sigma(t, w)$  is a section

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<sup>31</sup>This same prescription will be used in the several parameter case in the next section. The issue will be to have the property (ii), specifically to show that the limit as  $t = (t_1, \dots, t_k) \rightarrow (0, \dots, 0)$  exists which will require subtle properties of several variable degenerating Hodge structures.

of  $\pi_*\omega_{\mathcal{X}/\Delta^*\times\Delta}$  given by a family

$$\psi(t, w) \in H^0\left(\Omega_{X(t, w)}^2\right)$$

of holomorphic 2-forms along the smooth fibres  $X_{(t, w)} = \pi^{-1}(t, w)$ .<sup>32</sup> By base change and semi-stable reduction we may assume that we have a smooth completion  $\mathcal{X} \xrightarrow{\pi} \Delta \times \Delta$  of the family where the singular fibres  $X_{(0, w)}$  have normal crossings. The local models are

- $X_{(0, w)}$  is smooth and the mapping  $\pi$  is locally given by  $(x_1, x_2, x_3, w) \rightarrow (x_1, w)$ ; i.e.,  $t = x_1$ ;
- $X_{(0, w)}$  has a smooth double curve and the mapping  $\pi$  is given by  $(x_1, x_2, x_3, w) \rightarrow (x_1x_2, w)$ ; i.e.,  $t = x_1x_2$ ;
- $X_{(0, w)}$  has a double curve with triple points and the mapping  $\pi$  is locally given by  $(x_1, x_2, x_3, w) \rightarrow (x_1x_2x_3, w)$ ; i.e.,  $t = x_1x_2x_3$ .

By a standard property of the canonical extension, the 2-forms giving sections of  $\pi_*\omega_{\mathcal{X}/\Delta\times\Delta}$  are locally Poincaré residues

$$\psi(t, w) = \text{Res} \left( \frac{g(x_1, x_2, x_3, w) dx_1 \wedge dx_2 \wedge dx_3}{f(x_1, x_2, x_3, w)} \right)$$

where  $g$  is holomorphic and  $f$  is given by

$$\begin{cases} f = x_1 - t \\ f = x_1x_2 - t \\ f = x_1x_2x_3 - t \end{cases}$$

in the three cases listed above. The properties of the extension  $\psi(0, w)$  to a section of  $F_e \rightarrow \{0\} \times \Delta$  relative to the weight fibration are, in reverse order to the cases listed above,

- $\psi(0, w)$  induces a non-zero section in  $\text{Gr}_4^{W(N)}$  if, and only if, the double residues of  $\psi(0, w)$  at the triple points are not all zero; i.e., if

$$g(0, 0, 0, w) \neq 0;$$

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<sup>32</sup>As discussed at the end of Section VI similar algebro-geometric considerations suggested the general form of the singular Hodge metrics and their curvatures for a degenerating variation of Hodge structure.

- the double residues of  $\Psi(0, w)$  are zero; then  $\psi(0, w) \in W_3(N)$  and  $\psi(0, w)$  induces a non-zero section in  $\text{Gr}_3^{W(N)}$  if the single residues of  $\psi(0, w)$  along the double curve are non-zero; i.e., if

$$g(0, 0, 0, w) = 0 \text{ but } g(x_1, 0, x_3, w) \neq 0;$$

- the double and single residues of  $\psi(0, w)$  are zero; then  $\psi(0, w)$  induces a non-zero section of  $\text{Gr}_2^{W(N)}$  and  $\psi(0, w)$  is a holomorphic 2-form on the desingularization  $\tilde{X}_{(0,w)}$  of  $X_{(0,w)}$ .

The Hodge norm is, up to a constant, the  $L^2$ -norm

$$\|\psi(t, w)\|^2 = \int_{X(t,w)} \psi(t, w) \wedge \overline{\psi(t, w)}$$

of the holomorphic 2-forms  $\psi(t, w)$ . Then  $\|\psi(t, w)\|^2$  has an expansion in terms of powers of  $\log \frac{1}{|t|}$ , and the local contributions to the expansion in each of the above cases are respectively

- $\|\psi(t, w)\|^2 = |g(0, 0, 0, w)|^2 \left(\log \frac{1}{|t|}\right)^2 + B_1(t, w) \log \frac{1}{|t|} + B_2(t, w);$
- $\|\psi(t, w)\|^2 = \left(\int |g(x_1, 0, 0, w)|^2 dx_1 \wedge d\bar{x}_1\right) \log \frac{1}{|t|} + C(t, w);$
- $\|\psi(t, w)\|^2 = \int |g(0, x_1, x_2, x_3)|^2 dx_2 \wedge d\bar{x}_2 \wedge dx_3 \wedge d\bar{x}_3$

where  $B_1, B_2, C$  are smooth functions. This establishes the first part of the proposition: namely, that the Hodge norms are of the form (III.3).

For the second part we will discuss the above three cases. In the first case,  $\sigma_4(w)$  is a section of  $\text{Gr}_4^{W(N)}$ (LMHS), which is a family of polarized Hodge-Tate structures along  $\{0\} \times \Delta$ . The period domain is 0-dimensional and its curvature form, which is

$$(i/2)\bar{\partial}\partial \log A(0, w)$$

where  $A(0, w) = |h(w)|^2$  with  $h(w)$  holomorphic, is zero.<sup>33</sup> However, of interest is to observe that the polarizing form on  $\text{Gr}_4^{W(N)}$ (LMHS) is by definition

$$Q(N^2u, \bar{v}).$$

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<sup>33</sup>More precisely, one has a family of Hodge metrics on a single Hodge structure (this one being Hodge-Tate). This defines a Hermitian line bundle on the parameter space, and the associated curvature form is zero.

On the other hand

$$h(w) = \sum \text{double residues of } \psi(0, w)$$

where the sum is over a subset of the double residues at the triple points of  $X_{(0,w)}$ . The identifications of the polarizing form on  $\text{Gr}_4^{W(N)}(\text{LMHS})$  with  $|h(w)|^2$  will be discussed below.

In the second case,  $\sigma_3(w)$  is a section of  $\text{Gr}_3^{W(N)}(\text{LMHS})$ , which is a Tate twist of a variation of Hodge structure of weight one. Geometrically, the double residues of  $\psi(0, w)$  are zero and the single residues induce holomorphic 1-forms  $\text{Res } \psi(0, w)$  on the normalization  $\tilde{D}_w$  of the double curve of  $X_{(0,w)}$ . In this case there are two potential polarizing forms

- (i)  $Q(Nu, v)$  on  $\text{Gr}_3^{W(N)}(\text{LMHS})$  (Hodge-theoretic one);
- (ii)  $\int_{\tilde{D}_w} \text{Res } \psi(0, w) \wedge \overline{\text{Res } \psi(0, w)}$  (algebraic-geometric one).

We will see in the appendix to this section that, up to a constant,

$$(i) = (ii).$$

In other words

- (III.10) *On  $\text{Gr}_3^{W(N)}(\text{LMHS})$  the polarizing form arising from the limiting mixed Hodge structure coincides with the natural polarizing form on sub-Hodge structures of  $H^{1,0}(\tilde{D}_w)$ .*

Finally, in the third case the 2-form  $\psi(0, w)$  is holomorphic on the desingularization  $\tilde{X}_{(0,w)}$  and the polarizing form is just the usual one given by

$$\int_{\tilde{X}_{(0,w)}} \psi(0, w) \wedge \overline{\psi(0, w)}.$$

At this point we may complete the argument for Theorem C in the introduction in the special case where we consider only the weight  $n = 2$  case, and we restrict to the geometric situation where the period mapping (I.1) arises from a projective family  $\mathcal{X}^* \rightarrow \Delta^{*k} \times \Delta^\ell$  of smooth algebraic surfaces.

We first consider the case of a 1-parameter degeneration, and in the corollary to Proposition III.9 we will suffice to consider the case  $k = 1$ ,

$\ell = 1$  and here, we take a section

$$\sigma(t, w) = \psi_1(t, w) \wedge \cdots \wedge \psi_{p_g}(t, w)$$

of  $\det F_e$  where the  $\psi_i(t, w)$  give a framing of the canonically extended Hodge vector bundle  $F_a \rightarrow \Delta \times \Delta$  that is adapted to the weight filtration

$$W(N) \cap F_e.$$

As previously noted, that means that we filter the sections of  $F_e \rightarrow \Delta \times \Delta$  by their logarithmic growth along  $\{0\} \times \Delta$ . Setting  $h^0 = \dim I^{0,0}$  and  $h^{1,0} = \dim I^{1,0}$ , where we recall the  $I^{p,q}$  are the Hodge decomposition of  $\text{Gr}(\text{LMHS})$  along  $\{0\} \times \Delta$ , the calculation in the proof of the proposition gives that up to a constant

$$\Omega_e = (2h^0 + h^{1,0})\text{PM} + \text{LOT}$$

where  $\text{PM} = \frac{dt \wedge d\bar{t}}{|t|^2 (\log \frac{1}{|t|})^2}$  is the Poincaré metric and “LOT” are lower order terms.<sup>34</sup> Moreover the restriction

$$\Omega_e|_{\{0\} \times \Delta}$$

of the current  $\Omega_e$  is defined and there it coincides with the Chern form of the Hodge line bundle for the VHS over  $\{0\} \times \Delta$  given by the associated graded to the LMHS defined there.<sup>35</sup>

We now consider the case of a period mapping (I.1) for general  $k$  and  $\ell$ , and we will argue that

(III.11) *The general case may be reduced to it by a succession of 1-parameter degenerations.*

Before turning to the argument, we remark an in many ways more satisfactory proof of Theorem C will be given in the next section. There the analysis of the behavior of the curvature form in sectors in  $\Delta^{*k} \times \Delta^\ell$  will be given. For example, when  $k = 2$  the sectors will be  $|t_1/t_2| > c_1$ ,  $|t_2/t_1| > c_2$ ,  $c_3 < |t_1/t_2| < c'_3$ . For  $k \geq 3$  they are somewhat more subtle.

<sup>34</sup>Lower order terms means that the ratio LOT/PM tends to zero as  $t \rightarrow 0$ .

<sup>35</sup>This required the *non*-vanishing of  $A(0, w)$  in (III.3).

Turning to (III.11), the argument will be based on the results in [C-K-S1] detailing the structure of how several parameter variations of Hodge structure degenerate. Taking the case  $k = 2$  we shall explain the meaning of the equation

$$(III.12) \quad \lim_{t_2 \rightarrow 0} \left( \lim_{t_1 \rightarrow 0} \Phi(t_1, t_2, w) \right)^{\text{Gr}} = \lim_{t_1, t_2 \rightarrow 0} \Phi(t_1, t_2, w).$$

First, for a 1-parameter degeneration  $\Phi = \Delta^* \times \Delta^\ell \rightarrow \Gamma_T \backslash D$  by

$$(III.13) \quad \lim_{t \rightarrow 0} \Phi(t, w)$$

we mean the variation of mixed Hodge structure given by the limiting mixed Hodge structures along  $\{0\} \times \Delta^\ell$ . In (III.12) this serves to define the terms inside the large parentheses on the LHS. The result is a variation of polarizable mixed Hodge structures along  $\{0\} \times \Delta^* \times \Delta$ . The ‘‘Gr’’ on the outside means that we take the associated graded to the mixed Hodge structures. Applying (III.13) to the resulting VHS along  $\{0\} \times \Delta^* \times \Delta^\ell$  gives a variation of mixed Hodge structure along  $\{0\} \times \{0\} \times \Delta^\ell$ , and then we take the associated graded to obtain a variation of Hodge structure along this locus.

Turing the the right-hand side of (III.12), by [C-K-S1] the limit as  $t_1, t_2 \rightarrow 0$  defines a variation of mixed Hodge structure along  $\{0\} \times \{0\} \times \Delta^\ell$ , and we then take the variation of Hodge structure given by the associated graded to these mixed Hodge structures. The equation (III.12) means equality of the two variations of Hodge structure along  $\{0\} \times \{0\} \times \Delta^\ell$ .

Lurking behind the above words are subtle properties of several variable degenerations of Hodge structure. Specifically they include

- (i) the independence of  $\lambda$  of the weight filtration  $W(N_\lambda)$  for  $N_\lambda = \sum_{i=1}^k \lambda_i N_i$ ,  $\lambda_i > 0$ ;<sup>36</sup>

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<sup>36</sup>For 2-parameter degenerations it is obvious that the weight filtration is the same along ones that are not tangent to the axes. It is not obvious that this is true when the parameter arc is tangent to an axes, or that the weight filtration is invariant under base changes.



- (ii) the relative weight filtration property of the  $W(N_I)$  where  $N_I = \sum_{i \in I} N_i$  for a subset  $I \subset \{1, \dots, k\}$ ;<sup>37</sup> and
  - (iii) the asymptotic analysis of the period mapping in sectors in  $\Delta^{*k}$ .
- All of these will enter into the proof of Theorem C to be given in the next section.

We conclude this section by discussing what some of the issues are that arise in trying to extend the above geometric argument to the several parameter case.

The setting is a projective family  $\mathcal{X}^* \xrightarrow{\pi} \Delta^{*k} \times \Delta^\ell$  of smooth varieties  $X_{(t,w)} = \pi^{-1}(t, w)$  where  $(t, w) = (t_1, \dots, t_k; w_1, \dots, w_\ell)$  are coordinates in  $\Delta^{*k} \times \Delta^\ell$ . According to Abramovicz-Karu [A-K], after successive modifications and base changes the above family may be completed to

$$\mathcal{X} \xrightarrow{\pi} \Delta^k \otimes \Delta^\ell$$

where  $\mathcal{X}$  is smooth and the singular fibres  $X_w = \pi^{-1}(0, w)$  are locally a product of reduced normal crossing varieties. For the purposes of illustration we take the case  $k = 2, \ell = 1$  of a degenerating family of surfaces. The strata of  $X_w$  together with local coordinates on  $\mathcal{X}$  and the mapping  $\pi$  are

$$\begin{aligned} X_w^{[1]}(x_1, x_2, x_3, x_4) &\rightarrow (t_1 = x_3, t_2 = x_4), \\ X_w^{[2,1]}(x_1, x_2, x_3, x_4) &\rightarrow (t_1 = x_1x_2, t_2 = x_4), \\ X_w^{[3,1]}(x_1, x_2, x_3, x_4) &\rightarrow (t_1 = x_1x_2x_3, t_2 = x_4), \\ X_w^{[2,2]}(x_1, x_2, x_3, x_4) &\rightarrow (t_1 = x_1x_2, t_2 = x_3x_4) \end{aligned}$$

and similarly for  $X_w^{[1,2]}, X_w^{[1,3]}$ . The sections  $\psi(t, w)$  of the direct image of the relative dualizing sheaf are locally double Poincaré residues of 4-forms where the two functions in the denominator are the defining equations of the graph of  $\pi$ . For example, for  $X_w^{[2,2]}$

$$\psi(t, w) = \text{Res Res} \left( \frac{f(x_1, x_2, x_3, x_4) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{(x_1x_2 - t_1)(x_3x_4 - t_2)} \right).$$

---

<sup>37</sup>This is a subtle and very special property, recalled in Section IV below, of the relation between the weight filtrations constructed from a pair of commuting nilpotent operators on a vector space. The proofs of (i) and (ii) require Hodge theory, including the second Hodge-Riemann bilinear relation.

The highest order terms in the expansion of the Hodge norm

$$\|\psi(t, w)\|^2 = \int_{X(t, w)} \psi(t, w) \wedge \overline{\psi(t, w)}$$

are of the form

$$A_1(w) \left( \log \frac{1}{|t_1|} \right)^2 + B(w) \left( \log \frac{1}{|t_1|} \right) \left( \log \frac{1}{|t_2|} \right) + A_2(w) \left( \log \frac{1}{|t_2|} \right)^2.$$

The lower order terms are of the form

$$C_1(w) \log \frac{1}{|t_1|} + C_2(w) \log \frac{1}{|t_2|} + D(w).$$

When we compute  $\partial\bar{\partial} \log \|\psi(t, w)\|^2$  and set

$$dt_1 = d\bar{t}_1 = dt_2 = d\bar{t}_2 = 0$$

it is possible that we could be left with a term like

$$\frac{\log 1/|t_1| + \log 1/|t_2|}{\left( \log \frac{1}{|t_1|} \right) \left( \log \frac{1}{|t_2|} \right)},$$

which does not have a limit as  $t_1, t_2 \rightarrow 0$ . Consequently we need some control of what can appear in  $\partial\bar{\partial} \log \|\psi(t, w)\|^2$ . Now  $\log 1/|t_1|$  corresponds to  $N_1$ ,  $\log 1/|t_2|$  to  $N_2$  and  $\log 1/|t_1| + \log 1/|t_2|$  to  $N_1 + N_2$ . Thus it is necessary to examine more deeply how the relative weight filtrations interact; this is the relative weight filtration property and the issue of how it enters leads us into the next section.

**Appendix to §III.** In order to prove (III.10) we will describe the limiting mixed Hodge structure and its polarization for a family of surfaces  $\mathcal{X} \xrightarrow{\pi} \Delta$  with central fibre

$$X = \bigcup_{i \in I} X_i, \quad I = \text{ordered index set}$$

a reduced normal crossing divisor in a smooth 3-fold  $\mathcal{X}$ .<sup>38</sup> The usual notations

$$\begin{aligned} X^{[1]} &= \coprod_i X_i, \\ X^{[2]} &= \coprod_{i<j} X_i \cap X_j, \\ X^{[3]} &= \coprod_{i<j<k} X_i \cap X_j \cap X_k \end{aligned}$$

will be used for the desingularized strata of  $X$ . The groups that appear in the complex whose cohomology gives the associated graded to the LMHS are

$$H^a(X^{[b]})(-c), \quad 0 \leq c \leq b-1.$$

We set

$$I^m = \mathrm{Gr}_m^{W(N)}(\mathrm{LMHS}), \quad 0 \leq m \leq 4.$$

The  $I^m = \bigoplus_{p+q=m} I^{p,q}$  are the  $E_2$ -terms of a spectral sequence, where the  $E_1$ -terms and differential  $d_1, E_1 \rightarrow E_1$  will now be described in dual pairs.

For  $\mathrm{Gr}_4^{W(N)}$  and  $\mathrm{Gr}_0^{W(N)}$ , denoting respectively Gysin and restriction maps by  $G$  and  $R$  we have the dual complexes

$$\begin{aligned} \text{(III.A.1)} \quad \text{(a)} \quad & H^0(X^{[3]})(-2) \xrightarrow{G} H^2(X^{[2]})(-1) \xrightarrow{G} H^4(X^{[1]}), \\ \text{(b)} \quad & H^0(X^{[1]}) \xrightarrow{R} H^0(X^{[2]}) \xrightarrow{R} H^0(X^{[3]}) \end{aligned}$$

and where initial and terminal cohomology groups are

$$\begin{aligned} \text{(III.A.2)} \quad I^4 &= I^{2,2} \\ &= \text{kernel of } G \text{ in the initial term of the first sequence,} \end{aligned}$$

$$\begin{aligned} \text{(III.A.3)} \quad I^0 &= I^{0,0} \\ &= \text{co-kernel of the second term in the second sequence,} \end{aligned}$$

and where

$$N^2 : I^{2,2} \rightarrow I^{0,0}$$

---

<sup>38</sup>A general reference for this discussion is Chapter 11 in [P-S]. Here we will use the setting and notations developed in [G-G1].

is the “identity” under the composite map

$$\ker G \rightarrow H^0(X^{[3]})(-2) \rightarrow H^0(X^{[3]}) \rightarrow \operatorname{coker} R.$$

Here “identity” means the usual identity mapping where we ignore Tate twists.

Next  $\operatorname{Gr}_2^{W(N)}$  is the cohomology in the middle of the complex

$$(III.A.4) \quad \begin{array}{ccccc} & & H^2(X^{[1]}) & & \\ & G' \nearrow & & \searrow R' & \\ H^0(X^{[2]})(-1) & & \oplus & & H^2(X^{[2]}) \\ & \searrow R & & \nearrow G & \\ & & H^0(X^{[3]})(-1) & & \end{array}$$

As noted in [G-G1], it is a consequence of the Friedman condition in [Fr] for smoothability to 1<sup>st</sup> order of the abstract normal crossing variety  $X$  that the above is actually a complex; i.e., that the composition  $(R' \oplus G) \circ (G' \oplus R) = 0$ . We will explain this in more detail at the end of this appendix.

The monodromy maps are induced by

$$\begin{array}{ccccc} \ker G & \subset & H^0(X^{[3]})(-2) & & \text{in (III.A.1)(a)} \\ \downarrow N & & \downarrow \text{identity} & & \\ \operatorname{coker} R \cap \ker G & \subset & H^0(X^{[3]})(-1) & & \text{in (III.A.4)} \\ \downarrow N & & \downarrow \text{identity} & & \\ \operatorname{coker} R & \subset & H^0(X^{[3]}) & & \text{in (III.A.1)(b),} \end{array}$$

and the iteration  $N^2$  is (III.A.2).

For the odd weights for  $\operatorname{Gr}^{W(N)}$ (LMHS) the analogue of (III.A.1) is the pair of dual complexes

$$(III.A.5) \quad \begin{array}{ll} \text{(a)} & H^1(X^{[2]})(-1) \xrightarrow{G} H^3(X^{[1]}) \\ \text{(b)} & H^1(X^{[2]}) \xrightarrow{R} H^1(X^{[2]}) \end{array}$$

and

$$\begin{aligned} I^3 &= \text{kernel of } G \text{ in the first sequence} \\ I^1 &= \text{co-kernel of } R \text{ in the second sequence.} \end{aligned}$$

Monodromy is given by

$$\ker G \subset H^1(X^{[2]})(-1) \xrightarrow{\text{“identity”}} H^1(X^{[2]}) \rightarrow \text{coker } R.$$

Replacing  $X$  by  $X_w$  we then have the description

$F_a^2 \cap \text{Gr}_4^{W(N)} = I^{2,2} \subset H^1(X_w^{[2]})(-1)$  is represented by the double residues of forms  $\psi(0, w)$ .

$F_a^2 \cap \text{Gr}_3^{W(N)} = I^{2,1} \subset H^0(\Omega_{X_w^{[2]}}^1)(-1)$  is represented by single residues of forms  $\psi(0, w)$  whose double residues are zero.

$F_a^2 \cap \text{Gr}_2^{W(N)} = I^{2,0} \subset H^0(\Omega_{X^{[1]}}^2)$  is represented by the holomorphic 2-forms  $\psi(0, w)$  whose both double and single residues vanish.

We now turn to the issue of polarizations. There are two polarizing forms on the groups

$$I^{2,k} = F_a^2 \cap \text{Gr}_{2+k}^{W(N)}(\text{LMHS}), \quad k = 2, 1.$$

One is the Hodge-theoretic one arising from

$$\tilde{Q}(u, \bar{v}) = Q(N^k u, \bar{v}).$$

The other is the geometric one obtained by

- first taking limits, we realize the elements in  $I^{2,k}$  as singular differential forms on  $X_{(0,w)}$ ;
- then by taking sequential residues of these forms we obtain holomorphic differentials on the desingularized strata  $X_w^{[1+k]}$  of  $X_{(0,w)}$ ;
- finally we take the usual polarizing forms  $\int \alpha \wedge \bar{\beta}$  of holomorphic forms on smooth varieties.<sup>39</sup>

The claim is that, up to constants,

(III.A.6) *The Hodge-theoretic and geometric polarizing forms coincide.*

---

<sup>39</sup>For 0-dimensional varieties this is just the usual product of complex numbers.

We shall give the argument for this in the critical case  $k = 1$ . The situation is this:

- we have a family  $X_t$  of smooth surfaces specializing to a singular surface  $X_0$  that has a double curve  $D_0 \subset X_0$ ;
- $\psi_t$  are holomorphic 2-forms in  $H^0(\Omega_{X_t}^2)$  that specialize to  $\psi_0 \in H^0(\Omega_{\tilde{X}_0}^2(\tilde{D}_0))$ , which is a 2-form on the normalization  $\tilde{X}_0$  of  $X_0$  having a log pole on the inverse image  $\tilde{D}_0$  of the double curve on  $X_0$ .

By what we have seen above, there is an expansion

$$\int_{X_t} \psi_t \wedge \bar{\psi}_t = C \log \frac{1}{|t|} + \text{LOT}.$$

On the other hand we have the 1-form  $\text{Res}(\tilde{\psi}_0) =: \psi_0 \in H^0(\Omega_{\tilde{D}_0}^1)$ , and the assertion is that up to a universal constant

$$\int_{\tilde{D}_0} \text{Res} \psi_0 \wedge \overline{\text{Res} \psi_0} = C$$

for the same constant  $C$  as in the preceding equation. By localizing along  $\tilde{D}_0$  and iterating the integral, this essentially amounts to the following 1-variable result: In  $\mathbb{C}^2$  we consider the analytic curve  $C_t$  given by

$$xy = t,$$

and on  $C_t$  we take the Poincaré residue

$$\varphi_t = \text{Res} \left( \frac{g(x, y) dx \wedge dy}{xy - t} \right).$$

Then locally

$$\int_{C_t} \varphi_t \wedge \bar{\varphi}_t = |g(0, 0)|^2 \log \frac{1}{|t|} + \text{LOT}.$$

We conclude this appendix with a brief discussion of some of how parts of [Fr] apply to complexes constructed from an abstract normal crossing divisor  $X = \cup X_i$  to give conditions on complexes constructed from the cohomology group  $H^a(X^{[b]})(-c)$  to be the  $E_1$ -term of a spectral sequence whose abutment is a limiting mixed Hodge structure. If  $D = \coprod_{i < j} D_{ij}$  is the double locus of  $X$ , then as in [Fr] in terms of  $X$

above there is defined the *infinitesimal normal bundle*  $\mathcal{O}_D(X)$ , and a necessary condition for the smoothability of  $X$  is

$$(III.A.7) \quad \mathcal{O}_D(X) \cong \mathcal{O}_D.$$

If  $X$  is smoothable to be the central fibre in  $\mathcal{X} \rightarrow \Delta$ , then  $\mathcal{O}_D(X) = \mathcal{O}_D \otimes \mathcal{O}_X(X)$ . The cohomological implications of (III.A.7) then give conditions that diagrams such as (III.A.4) actually be complexes whose cohomology is then the associated graded to a limiting mixed Hodge structure.<sup>40</sup> In other words, the condition (III.A.7) is sufficient to construct as in [P-S] the spectral sequence that would arise from  $\mathcal{X} \rightarrow \Delta$ . To keep the notation as simple as possible we shall do the case where  $X = X_1 \cup X_2 \cup X_3$  where  $X_i$  is locally given by  $x_i = 0$  in  $\mathbb{C}^3$ . If  $X$  is smoothable so that along the double locus the smoothing is given by  $x_1x_2 = t$ , then the relation

$$dt = x_2dx_1 + x_1dx_2$$

translates away from the triple points into

$$\mathcal{O}_{D_{12}}(X_1) \otimes \mathcal{O}_{D_{12}}(X_2) \cong \mathcal{O}_{D_{12}}.$$

For a smoothable triple point given by  $x_1x_2x_3 = t$  we have

$$dt = x_2x_3dx_1 + x_1x_3dx_2 + x_1x_2dx_3,$$

which at  $x_1x_2 = 0$ ,  $x_3 = 0$  gives

$$\mathcal{O}_{D_{12}}(X_2) \otimes \mathcal{O}_{D_{12}}(X_2) \otimes \mathcal{O}_{D_{12}}(p) \cong \mathcal{O}_{D_{12}}.$$

From this we obtain the *triple point formula*

$$(III.A.8) \quad D_{12}^2|_{X_1} + D_{12}^2|_{X_2} + 1 = 0$$

where  $D_{12}^2|_{X_i}$  is the self intersection of  $D_{12}$  in  $X_i$ .

---

<sup>40</sup>This discussion may be extended to the case when  $X$  is locally a product of normal crossing divisors (such as arise from stable nodal curves), and also to the several parameter case where  $X$  is locally a product of normal crossing divisors such as arise in the semi-stable reduction constructed in [A-K]. The details and applications of this will appear elsewhere.

We now explain how (III.A.8) enters into (III.A.4). In

$$\begin{array}{ccc}
 & & H^2(X^{[1]}) \\
 & \nearrow^{G'} & \\
 H^0(X^{[2]})(-1) & & \oplus \\
 & \searrow_R & \\
 & & H^0(X^{[3]})(-1)
 \end{array}$$

the map is

$$\text{(III.A.9)} \quad \begin{array}{ccc}
 & & [D_{12}]|_{X_1} - [D_{12}]|_{X_2} \\
 & \nearrow & \\
 1_{D_{12}} & & \oplus \\
 & \searrow & \\
 & & 1_p
 \end{array}$$

where  $[D_{12}]|_{X_i}$  is the class of  $D_{12}$  in  $H^2(X_i)$ . For

$$\begin{array}{ccc}
 H^2(X^{[1]}) & & \\
 & \searrow_{R'} & \\
 & & H^2(X^{[2]}) = H^2(X_{12}) \oplus H^2(X_{13}) \oplus H^2(X_{23}) \\
 & \nearrow_G & \\
 H^0(X^{[3]})(-1) & & 
 \end{array}$$

the maps are induced by

(III.A.10)

$$[D_{12}]|_{X_1} - [D_{12}]|_{X_2} \rightarrow \left( D_{12}^2|_{X_2} + D_{21}^2|_{X_1} \right) [X_{12}] \oplus (-[X_{13}]) \oplus (-[X_{23}])$$

where as above  $D_{12}^2|_{X_2}$  is the self-intersection of  $D_{12}$  in  $X_2$  and similarly for  $D_{21}^2|_{X_1}$ , and where  $[X_{ij}]$  is the fundamental class of  $X_{ij}$ . The points here are

- If  $C$  is a smooth, irreducible curve on a surface  $Y$ , then the restriction  $H^2(Y) \rightarrow H^2(C)$  maps the class  $[C] \in H^2(Y)$  of  $C$  to the self-intersection number  $C^2$  times the generator of  $H^2(C)$ ; this accounts for the first term in (III.A.10).



- If  $C, C'$  are smooth, irreducible curves in  $Y$  meeting a point, then  $H^2(Y) \rightarrow H^2(C')$  maps  $[C]$  to a generator of  $H^2(C')$ ; this accounts for the last two terms in (III.A.10).

Using the above to compute the maps  $G', R', G, R$  in (III.A.4) we may draw the conclusion

*The triple point formula for each pair of components of  $X$  implies that (III.A.4) is a complex.*

#### IV. CURVATURE PROPERTIES OF THE EXTENDED HODGE BUNDLE (B)

In this section we will give a proof of Theorem C as stated in the introduction. The argument will be given in several steps and will be independent from that in the preceding section and from the results in [C-K-S1] and [Kol2] about Chern forms in the literature.

**Step 1:** We shall first establish the setting and notations. We assume given a variation of Hodge structure of weight  $n$

$$\Phi : \Delta^{*k} \times \Delta^\ell \rightarrow \Gamma_T \backslash D$$

where  $\Gamma_T = \{T_1^{\mathbb{Z}}, \dots, T_k^{\mathbb{Z}}\}$  is the group generated by the unipotent monodromies  $T_i$  around  $t_i = 0$ . The  $\Delta^\ell$ -factors are parameters that play no essential role; for notational simplicity we shall assume that  $\ell = 0$ .

As above, we set

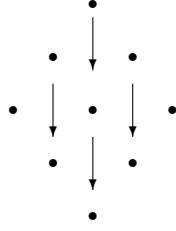
$$\begin{aligned} \Delta_I &= \{t = (t_1, \dots, t_k) : t_i = 0 \text{ for } i \in I\} \\ &\cup \\ \Delta_I^* &= \{t : t_i = 0 \text{ for } i \in I, t_j \neq 0 \text{ for } j \in I^c\}. \end{aligned}$$

The variation of Hodge structure  $\Phi$  induces variations of polarized mixed Hodge structure  $\lim_{t \rightarrow \Delta_I^*} \Phi(t)$  on the open strata  $\Delta_I^*$ , and we denote by

$$\Phi_I : \Delta_I^* \rightarrow \Gamma_I \backslash D_I$$

the variations of Hodge structure obtained by taking the primitive parts of the associated graded to  $\lim_{t \rightarrow \Delta_I^*} \Phi(t)$ .

There are two pictures of limiting mixed Hodge structures that will be convenient to use. The more common one is the Hodge diamond picture



arising from the Deligne composition  $\bigoplus_{0 \leq p+q \leq n} I^{p,q}$  of the canonically associated  $\mathbb{R}$ -split mixed Hodge structure. Here the vertical arrows are the monodromy operators

$$N_I =: \sum_{i \in I} N_i$$

associated to  $\lim_{t \rightarrow \Delta_I^*} \Phi(t)$ .<sup>41</sup>

The other is the picture that displays the  $N_I$ -strings that arise from the canonical representation of  $\mathfrak{sl}_2$  on

$$V_{\text{gr}, I} =: \text{associated graded to } (V, W(N_I)).$$

Our  $V_{\text{gr}, I}$  there is a semi-simple operator

$$Y_I : \text{Gr}_m^{W(N_I)} V \rightarrow \text{Gr}_m^{W(N_I)} V$$

which on each piece is a multiple of the identity and which satisfies  $[Y_I, N_I] = -2N_I$ .<sup>42</sup>

<sup>41</sup>Sometimes this diagram is rotated to be in the first quadrant in the  $(p, q)$ -plane.

<sup>42</sup>Here we are using that if we have a graded vector space  $A = \bigoplus A_m$  and a nilpotent operator  $J : A \rightarrow A$  where  $J(A_m) \subset A_{m-2}$ , then there is an  $\mathfrak{sl}_2$  action on  $A$  with  $J$  as nil-negative element. Which multiple of the identity on  $A_m$  depends on where one centers the weight filtration associated to a nilpotent operator.

Then there is a canonical  $\mathfrak{sl}_2$   $\{N_I, Y_I, N_I^+\}$  with  $N_I$  as the nil-negative element. The  $N_I$ -string picture is

$$\begin{array}{c} H_I^0(-n) \xrightarrow{N_I} H_I^0(-(n-1)) \xrightarrow{N_I} \cdots \xrightarrow{N_I} H_I^0 \\ H_I^1(-(n-1)) \xrightarrow{N_I} \cdots \xrightarrow{N_I} H_I^1 \\ \vdots \\ H_I^n \end{array}$$

where  $H^{n-i}$  is a Hodge structure of weight  $n-i$  and  $H^{n-i}(-(n-i))$  is the primitive part appearing at the top of the corresponding  $N_I$ -string. It will be convenient to abbreviate this by setting

$$V_{\text{gr}, I} = \bigoplus_i H_I^{n-i} \otimes \mathcal{U}_i$$

where  $\mathcal{U}_i = \text{Sym}^i \mathcal{U}$  with  $\mathcal{U}$  being the standard representation of  $\mathfrak{sl}_2$ .

We recall that by the Hard Leftschetz Property of  $W(N_I)$ ,

$$N_I^i : H_I^{n-i}(-i) \xrightarrow{\sim} H_I^{n-i}$$

is an isomorphism, and the polarization on  $H_I^{n-i}$  is defined by

$$Q_I(u, \bar{v}) = Q(N_I^i u, \bar{v}).$$

We recall our notations  $F_e \rightarrow \Delta^k$  for the canonically extended Hodge vector bundle and  $\Lambda_e \rightarrow \Delta^k$  for the Hodge line bundle where  $\Lambda_e = \det F_e$ . Over  $\Delta_I^*$  we have the Hodge line bundle  $\Lambda_I \rightarrow \Delta_I^*$ . The bundles  $\Lambda_e, \Lambda_I$  have metrics arising from the polarizing forms, and we denote by  $\Omega_e, \Omega_I$  the corresponding Chern forms. Theorem C may be formulated as

$$(IV.1) \quad \begin{array}{l} \text{Modulo } dt_i, d\bar{t}_i \text{ for } i \in I, \\ \lim_{t \rightarrow \Delta_I^*} \Omega(t) = \Omega_I. \end{array}$$

Implicit in this statement is that the limit exists.

**Step 2:** We will reduce to the case of a nilpotent orbit. Setting  $\ell(t_j) = (1/2\pi i) \log t_j$  the variation of Hodge structure is

$$\Phi(t) = \exp \left( \sum_j \ell(t_j) N_j \right) \cdot u(t) \cdot F_0$$

where  $u : \Delta^k \rightarrow G_{\mathbb{C}}$  is a holomorphic mapping. The metric is

$$\begin{aligned} (v, w) &= Q \left( \exp \left( \sum_j \ell(t_j) N_j \right) u(t) v, \overline{\exp \left( \sum_j \ell(t_j) N_j \right) u(t) w} \right) \\ &= Q \left( \overline{u(t)^{-1}} \exp \left( \sum_j \left( \ell(t_j) - \overline{\ell(t_j)} \right) N_j \right) u(t) v, \bar{w} \right) \\ &= Q \left( \exp \left[ \overline{u(t)^{-1}} \left( \sum_j \left( \frac{1}{2\pi i} \right) \log |t_j|^2 \right) u(t) N_j \right] v, \bar{w} \right). \end{aligned}$$

Letting  $\text{Gr}^{W(N_I)} F_e$  be the graded vector bundle associated to the filtration on  $F_I \rightarrow \Delta_I^*$  induced by  $W(N_I)$ , then using the identification

$$\text{Gr}^{W(N_I)} F_e = \bigoplus_i H^{n-i,0}(-i)$$

if  $v \in H^{n-i,0}(-i)$  and  $w \in H^{n-i,0}$  we have

$$(v, w) = \left( \frac{1}{i!} \right) Q \left( \overline{u(t)^{-1}} \left( \sum_j \left( \frac{1}{2\pi i} \right) \log |t_j|^2 N_j \Big|_{H^{n-i,0}(-i)} \right)^i u(t) v, \bar{w} \right).$$

If  $H$  is the metric on  $\Lambda_e$ , then setting  $x_j = \log 1/|t_j|$  up to non-zero constants

$$\begin{aligned} H &= \prod_{i=0}^n \det \left( \overline{u(t)^{-1}} \left( \sum_j x_j N_j \right) \Big|_{H^{n-i,0}(-i)} u(t) \right)^i \\ &= \left( (\det \overline{u(t)})^{-1} \det u(t) \right)^n \left( \prod_{i=0}^n \det \left( \sum_j x_j N_j \Big|_{H^{n-i,0}(-i)} \right)^i \right).^{43} \end{aligned}$$

When we take  $\partial\bar{\partial} \log H$  the first factor is a  $C^\infty$  form on  $\Delta^k$ . From this we infer that it is equivalent to prove (IV.1) for a nilpotent orbit.

**Step 3:** We use the notations

- $N_I = \sum_{i \in I} x_i N_i$ ,  $N_{I^c} = \sum_{j \in I^c} x_j N_j$ ;
- $N = \sum_i x_i N_i = N_I + N_{I^c}$ .

<sup>43</sup>The meaning of taking the determinant will be explained in Step 3. Basically it is the induced mapping on the top exterior powers of two vector spaces that are identified by a linear mapping.

Then for the nilpotent orbit the metric in the Hodge bundle is

$$(IV.2) \quad P = \prod_{i=0}^n \det \left( \left( N|_{H^{n-i,0}(-i)} \right)^i \right).$$

Here we are taking a basis for  $H^{n-i,0}(-i)$  and using the corresponding basis for  $H^{n-i,0}$  using the identification given by  $N_I$ . Since the fibre of the Hodge line bundle is  $\bigoplus^i \wedge^{h^{n-i,0}} H^{n-i,0}(-i)$ , we see that Hodge metric has the indicated form. It is a homogeneous polynomial in the  $x_i$  of degree  $\sum_{i=0}^n i h^{n-i,0}$ . We set

$$(IV.3) \quad S_I = \prod_{i=0}^n \det \left( \left( N_I|_{H_I^{n-i,0}(-i)} \right)^i \right).$$

We will show that there is a factorization

$$(IV.4) \quad P = S_I P_I + R,$$

where  $P_I$  is the Hodge metric in the Hodge line bundle corresponding to  $\Phi_I$  and  $R_I$  is a remainder term to be dealt with later. If  $R_I = 0$ , then

$$\begin{aligned} \partial\bar{\partial} \log P &= \partial\bar{\partial} \log S_I + \partial\bar{\partial} \log P_I \\ &\equiv \partial\bar{\partial} \log P_I \text{ modulo } dt_i, d\bar{t}_i \text{ for } i \in I \end{aligned}$$

which, without having to take limits, gives the desired result (IV.4).

From  $[N_I, N_{I^c}] = 0$  we have that  $N_{I^c}$  acts on  $\text{Gr}^{W(N_I)} V$  and hence on  $\text{Gr}^{W(N_I)} \mathfrak{g}$ . Since  $N_{I^c}$  decreases weights on  $\text{Gr}^{W(N_I)} V$

$$N_{I^c} = \bigoplus_{m \leq 0} N_{I^c, m}$$

where  $N_{I^c, m} = \text{Gr}_m^{W(N_I)} N_{I^c}$ . We may write

$$V_{\text{gr}} \cong \bigoplus H_{i,j}^{n-i-j} \otimes \mathcal{U}_i \otimes \mathcal{U}'_j.$$

$V_{\text{gr}}$  is notation for the associated graded to a bi-filtration on  $V$  that we do not need to specify in detail, and where

- $H_{i,j}^{n-i-j}$  is a polarized Hodge structure whose weight  $n-i-j$  depends on  $I$ ;
- $\mathcal{U}_i = S^i \mathcal{U}$  where  $\mathcal{U}$  is the standard representation of  $\mathfrak{sl}_2$  in which  $N_I$  is the nil-negative element;

- $\mathcal{U}'_j = S^j \mathcal{U}$  where  $\mathcal{U}$  is the standard representation of  $\mathfrak{sl}_2$  in which  $N_{I^c}$  is the nil-negative element.

The action of powers of  $N_I$  and  $N_{I^c}$  on  $V \cong V_{\text{gr}}$  are derived from the maps in the commutative square

$$\begin{array}{ccc} H_{i,j}^{n-i-j}(-i-j) & \xrightarrow{N_I^i} & H_{i,j}^{n-i-j}(-j) \\ N_{I^c,o}^j \downarrow & & \downarrow N_{I^c,o}^j \\ H_{i,j}^{n-i-j}(-i) & \xrightarrow{N_I^i} & H_{i,j}^{n-i-j}. \end{array}$$

Then  $H$  given in (IV.2) becomes the product of terms

$$\det \left( N_I^i N_{I^c,o}^j \Big|_{H_{i,j}^{n-i-j,0}(-i-j)} \right)$$

plus terms involving  $N_{I^c,m}$  for  $m < 0$ . This gives that

$$(IV.5) \quad P = \prod_{i,j} \det \left( N_I^i \Big|_{H_{i,j}^{n-i-j,0}(-i-j)} \right) \det \left( N_{I^c,o}^j \Big|_{H_{i,j}^{n-i-j,0}(-i-j)} \right) + R_I$$

where  $R_I$  consists of terms involving the  $N_{I^c,m}$  for  $m < 0$ .

From (IV.3) we see that we need to compare

$$V_{\text{gr},I} \cong \oplus H_I^{n-i} \otimes \mathcal{U}_i \quad \text{and} \quad V_{\text{gr}} \cong H_{i,j}^{n-i-j} \otimes \mathcal{U}_i \otimes \mathcal{U}'_j.$$

From the definition of the  $H_{i,j}^{n-i-j}$  we have

$$H_I^{n-i,0} \cong \oplus H_{i,j}^{n-i-j,0}$$

so that in (IV.4)

$$S_I = \prod_{i=1}^n \det \left( N_I^i \Big|_{H_I^{n-i,0}(-i)} \right).$$

As a consequence of the relative weight filtration property (II.3) we have

$$H^{n-m,0} \cong \oplus_{i+j=m} H_{i,j}^{n-i-j,0}.$$

The metric  $P_I$  in the Hodge bundle  $\Lambda_I \rightarrow \Delta_I^*$  arises from the forms  $Q(N_{I^c}^m v, \bar{w})$  where  $v, w \in \text{Gr}^{W(N_I)} V$ . Again using the relative weight filtration property, we see that these terms are the same as those that

appear in the second factor in (IV.5) that factor being

$$P_I = \prod_{i,j} \left( \det N_{I^c,o}^j \Big|_{H_{i,j}^{n-i-j,o}(-i-j)} \right).$$

This establishes (IV.4) and completes Step 3.

**Step 4:** We use the notation

$$h_I^{n-i,o} = \dim H_I^{n-i,o}.$$

For a monomial  $M = x_1^{d_1} \cdots x_k^{d_k}$  we set

$$\deg_I(M) = \sum_{i \in I} d_i.$$

The following is a key step in the argument:

**PROPOSITION IV.6:** *For  $P$  given by (IV.2) and any  $I \subset \{1, \dots, k\}$  we have*

$$(a) \deg_I M \leq nh_I^o + (n-1)h_I^{1,o} + \cdots + h_I^{n-1,o} = \sum_{i=1}^n ih_I^{n-i,o}.$$

*If  $\pi$  is a permutation of  $\{1, \dots, k\}$  and*

$$d_{\pi,i} = \sum_j j \left( h_{\{\pi(1), \dots, \pi(i)\}}^{n-j,o} - h_{\{\pi(1), \dots, \pi(i-1)\}}^{n-j,o} \right)$$

*then*

$$(b) M_\pi = x_{\pi(1)}^{d_{\pi,1}} \cdots x_{\pi(k)}^{d_{\pi,k}} = x_1^{d_{\pi,\pi^{-1}(1)}} \cdots x_k^{d_{\pi,\pi^{-1}(k)}}$$

*appears with a non-zero coefficient in  $D$ .*

The statement (a) says that

*The monomials  $M$  appearing with non-zero coefficient in  $P$  are in the convex hull of the  $k!$  monomials  $\{M_\pi\}$ .*

This means that the exponents of  $M$  in  $\mathbb{Z}^k$  are in the convex hull of those of the  $\{M_\pi\}$ .

We recall our notation

$$\begin{aligned} N_{I^c,o} &= 0\text{-graded piece of } N_{I^c} \text{ in } \text{Gr}^{W(N_I)} \mathfrak{g}, \\ V_{\text{gr},I} &\cong \bigoplus_{i=0}^n H_I^{n-i} \otimes \mathcal{U}_i, \\ V_{\text{gr}} &\cong \bigoplus_{i=0}^n \bigoplus_{a=0}^i H_{a,i-a}^{n-i} \otimes \mathcal{U}_a \otimes \mathcal{U}'_{i-a} \end{aligned}$$

where  $\mathcal{U}_i$  is the standard representation of  $\mathfrak{sl}_2$  with nil-negative element  $N_I$  and  $\mathcal{U}'_j$  is the same using  $N_{I^c,o}$ . Note that we use the “ $o$ ” rather than “ $0$ ” in  $N_{I^c,o}$ . We emphasize that the  $H_{a,i-a}^{n-i}$  depend on  $I$ .

The map  $N^i : H^{n-i,o}(-i) \xrightarrow{\sim} H^{n-i,o}$  induces a map

$$\bigoplus_{a=0}^i H_{a,i-a}^{n-i,o}(-i) \rightarrow \bigoplus_{a=0}^i H_{a,i-a}^{n-i,o}.$$

By construction the  $N_I$  weights of vectors in  $H_{a,i-a}^{n-i}$  are all equal to  $a$ .

Thus

- $\wedge^{h^{n-i,o}} \left( \bigoplus_{a=0}^i H_{a,i-a}^{n-i,o}(-i) \right)$  has weight  $\sum_{a=0}^i ah_{a,i-a}^{n-i,o}$ ;
- $\wedge^{h^{n-i,o}} \left( \bigoplus_{a=0}^i H_{a,i-a}^{n-i,o} \right)$  has weight  $-\sum_{a=0}^i ah_{a,i-a}^{n-i,o}$ .

Consequently, any monomial in  $\det \left( N^i|_{H^{n-i,o}(-i)} \right)$  drops weight by  $2 \sum_{a=0}^i ah_{a,i-a}^{n-i,o}$ . Now

$$\det \left( \left( N|_{H^{n-i,i}(-i)} \right)^i \right) = \det \left( \left( (N_I + N_{I^c,o})|_{H^{n-i,o}(-i)} \right)^i \right) \\ + \text{terms involving } N_{I^c,\text{neg}}$$

where  $N_{I^c,\text{neg}} = \sum_{m < 0} N_{I^c,m}$ . Any term involving  $N_{I^c,\text{neg}}$  will have weight that satisfies

$$2 \deg_I M + d = 2 \sum_{a=0}^i ah_{a,i-a}^{n-i,o}$$

where  $d > 0$  is the total negative weight of  $N_{I^c,\text{neg}}$ . In particular,

$$\deg_I M < \sum_{i=0}^a ah_{a,i-a}^{n-i,o}.$$

For  $P = \prod_{i=0}^n \left( \det \left( N|_{H^{n-i,o}(-i)} \right)^i \right)$  we obtain

$$(IV.7) \quad \left\{ \begin{array}{l} P = \prod_{i=0}^n \left( \det \left( (N_I + N_{I^c,o})|_{H^{n-i,o}(-i)} \right)^i \right) + \\ \text{a linear combination of monomials satisfying} \\ \deg_I M < \sum_{i=0}^n \sum_{a=0}^i ah_{a,i-a}^{n-i,o}. \end{array} \right.$$



Using the bookkeeping formula

$$h_I^{n-a,o} = \sum_{i=a}^n h_{a,i-a}^{n-i,o}$$

we have

$$\sum_{i=0}^n \sum_{a=0}^i ah_{a,i-a}^{n-i,o} = \sum_{a=0}^n \sum_{i=a}^n h_{a,i-a}^{n-i,o} = \sum_{a=0}^n h_I^{n-a,o}.$$

Together with (IV.7) this gives

$$P = \prod_{i=0}^n \left( \det \left( N_I + N_{I^c} |_{H^{n-i,o}(-i)} \right)^i \right) + \text{a correction term}$$

where the correction term has  $\deg_I < \sum_{a=0}^n ah_I^{n-a,o}$ . Combining with (IV.5) above gives  $P = S_I P_I$ . This establishes (a) in the proposition.

A parallel argument shows that for  $I \cap J = \emptyset$ ,

$$\begin{aligned} S_{I \cup J} &= \prod_{i=0}^n \det \left( \left( N_I + N_{J,o} |_{H_{I \cup J}^{n-i,o}} \right)^i \right) \\ &+ \text{a collection of terms with } \deg_I < \sum_{n=0}^n ah_I^{n-a,o}. \end{aligned}$$

By the definition of  $H_{I \cup J}^{n-i}$

$$\det \left( \left( N_I + N_{J,o} |_{H_{I \cup J}^{n-i,o}(-i)} \right)^i \right) \neq 0$$

and

$$\deg_I \left( \det \left( \left( N_I + N_{J,o} |_{I \cup J}^{n-i,o}(-i) \right)^i \right) \right) = \sum_{n=0}^n ah_I^{n-a,o}.$$

It is automatically the case that

$$\deg_{I \cup J}(\text{all terms of } S_{I \cup J}) = \sum_{a=0}^n ah_{I \cup J}^{n-a,o}.$$

Thus

$$\begin{aligned}
& \deg_J \det \left( \left( N_I + N_{J,o} \Big|_{H_{I \cup J}^{n-i,o}(-i)} \right)^i \right) \\
&= \deg_{I \cup J} \det \left( \left( N_I + N_{J,o} \Big|_{H_{I \cup J}^{n-i,o}(-i)} \right)^i \right) \\
&\quad - \deg_I \det \left( \left( N_I + N_{J,o} \Big|_{H_{I \cup J}^{n-i,o}(-i)} \right)^i \right) \\
&= \sum_{a=0}^n a (h_{I \cup J}^{n-a,o} - h_I^{n-a,o}).
\end{aligned}$$

Proceeding inductively on  $\{\pi(1), \pi(2)\} \subset \cdots \subset \{\pi(1), \dots, \pi(n)\}$  we obtain that if  $N_{\{\pi(1), \dots, \pi(r)\}, 0}$  is a weight 0 piece of  $N_{\{\pi(1), \dots, \pi(r)\}}$  with respect to  $\text{Gr}^{W(N)_{\{\pi(1), \dots, \pi(r-1)\}, 0}}$ , then

$$\prod_{a=0}^n \det \left( \left( N_{\{\pi(1)\}} + N_{\{\pi(1), \pi(r)\}, 0} + \cdots + N_{\{\pi(1), \dots, \pi(r)\}, 0} \Big|_{H^{n-i,o}(-i)} \right)^i \right)$$

is a non-zero multiple of

$$x_{\pi(1)}^{d_1} x_{\pi(r)}^{d_2} \cdots x_{\pi(r)}^{d_r} \text{ where } d_i = \sum_a a \left( h_{\{\pi(1), \dots, \pi(r)\}}^{n-a,o} - h_{\{\pi(1), \dots, \pi(r)\}}^{n-a,o} \right).$$

This is our  $M_\pi$ . Tracking the correction terms,

$$D = \sum_{\pi} c_\pi M_\pi + \text{terms strictly in the convex hull of the } M_\pi$$

where  $c_\pi$  for all  $\pi$ .

This proves (b) in Theorem C.  $\square$

**Step 5:** We now have

$$P = S_I P_I + R,$$

$R$  homogeneous of degree  $= \deg P$  and satisfies (a) in Proposition IV.6

Working modulo  $dt_1, d\bar{t}$  for  $i \in I^c$ ,

$$\begin{aligned}
-\partial\bar{\partial} \log P &= \frac{\partial P \cap \bar{\partial} P - P \partial\bar{\partial} P}{P^2} \\
&= \frac{(S_I \partial P_I + \partial R) + (S_I \bar{\partial} P_I + \bar{\partial} R) - D_I^2 P_I \partial\bar{\partial} P_I - S_I P_I \partial\bar{\partial} R}{D_I^2 P_I^2} \\
&= \frac{\partial P_I \wedge \bar{\partial} P_I - P_I \partial\bar{\partial} P_I}{P_I^2} + \text{correction term} \\
&= -\partial\bar{\partial} \log P_I + \text{correction term modulo } dt_1, dt_r \text{ for } i \in I^c
\end{aligned}$$

where

$$\text{correction term} = \begin{pmatrix} S_I(\partial P_I \cap \bar{\partial} R + \partial R \cap \bar{\partial} P_I) + \partial R \cap \bar{\partial} R \\ -S_I P_I \partial \bar{\partial} R \end{pmatrix} / S_I^2 P_I^2.$$

Since we are working mod  $dt_i, d\bar{t}_i$  for  $i \in I$ ,  $\partial R, \bar{\partial} R, \partial \bar{\partial} R$  have the same properties that  $R$  has. In conclusion:

$$\text{correction term} = \text{Correction 1} + \text{Correction 2}.$$

$$\text{Correction 1: } \frac{\partial P_I \cap \bar{\partial} R + \partial R \cap \bar{\partial} P_I - P_I \partial \bar{\partial} R}{S_I P_I}.$$

$$\text{Correction 2: } \frac{\partial R \cap \bar{\partial} R}{S_I^2}.$$

The numerator of Correction 1 has  $\deg_I < \deg_{I, S_I}$ , and all monomials satisfy (a) in Proposition IV.6. The numerator of Correction 2 has terms in the  $I$ -variables that are a product of monomials  $M_1 M_2$  where each  $M_i$  has  $\deg_I \geq \deg_I S_I$  and also satisfies (a).

What we need to show is:

$$\text{(IV.8) } \begin{array}{l} \text{Given a monomial } M \text{ in the } I\text{-variables satisfying (1)} \\ \deg_I M < \deg_I D, \text{ and (2) } M \text{ satisfies (a), then} \\ \lim_{t \rightarrow \Delta_I} M/S_I = 0. \end{array}$$

Note that (1) and (2) for  $M$  implies that for some monomial  $M'$ ,  $\deg_I(M'M) = \deg_I S_I$  and  $M'M$  lies in the convex hull of the argument  $M_\pi$ 's for  $S_I$ .

Note that  $t \rightarrow \Delta_I$  is the same as all  $x_i \rightarrow \infty$  for  $i \in I$ .

### Step 6:

CLAIM: *If  $M'M$  is as above, then*

$$\frac{M'M}{S_I} \text{ is bounded as } x_i \rightarrow \infty \text{ for all } i \in I.$$

**Proof of Claim:** Because the numerator and denominator are homogeneous of the same degree, the ratio is the same for  $(x_1, \dots, x_n)$  and  $(\lambda x_1, \dots, \lambda x_n)$ ,  $\lambda > 0$ . For simplicity, re-index so that  $I = \{1, \dots, d\}$ . Let  $x_\nu = (x_{\nu_1}, \dots, x_{\nu_d})$  be a set of points in  $\{x_i > 0, i \in J\}$  such that

$$\lim_{\nu \rightarrow \infty} \frac{M'M(x_\nu)}{S_I(x_\nu)} = \infty.$$

Consider a successive set of subsequences such that for all  $i, j$ , we have one of three possibilities:

- (i)  $\lim_{\nu \rightarrow \infty} x_{\nu_i}/x_{\nu_j} = \infty$ ;
- (ii)  $x_{\nu_i}/x_{\nu_j}$  is bounded above and below;
- (iii)  $\lim_{\nu \rightarrow \infty} x_{\nu_i}/x_{\nu_j} = 0$ .

Now replace the sequence by the following subsequence: Let  $I_{ij}, \dots, I_r$  be the partition of  $I$  such that  $i = j \iff$  (ii) holds for  $i, j$ . Order them so that (i) holds for  $i, j \leftarrow I_{\nu_1}, j \in I_{\nu_2}, j \in I_{\nu_2}$  and  $m_1 < m_2$ . We may find a  $C > 0$  such that  $\frac{1}{C} \leq x_{\nu_i}/x_{\nu_j} \leq C$  if  $i, j$  are in same  $I_\nu$  and, for any  $B > 0$ ,

$$x_{\nu_i}/x_{\nu_j} > B^{m_2 - m_1} \text{ if } i \in I_{m_1}, j \in I_{m_2}, \quad \nu \text{ sufficiently large.}$$

By compactness, we may pick a subsequence so that

$$\lim_{\nu \rightarrow \infty} (x_{\nu_i}/x_{\nu_j}) = C_{ij} \text{ if } i, j \in \text{same } I_m.$$

Introduce variables  $y_1, \dots, y_r$  and let

$$x_i = a_i y_r \text{ if } i \in I_r, \quad a_i/a_j = C_{i,j}, \quad a_r > 0.$$

We may restrict our cone by taking

$$\tilde{N}_m = \sum_{i \in J_r} a_i N_i.$$

This reduces us to the case  $|I_r| = 1$  for all  $r$ , i.e.,

$$\lim_{\nu \rightarrow \infty} x_{\nu_i}/x_{\nu_j} = \infty \text{ if } i < j.$$

It follows that for any  $B$ ,

$$x_{\nu_i}/x_{\nu_j} > B^{j-i} \text{ for } \nu \gg 0.$$

Now

$$\frac{x_{\nu_1}^{k_1} x_{\nu_2}^{k_2} \dots x_{\nu_\alpha}^{k_\alpha}}{x_{\nu_1}^{k_1} \dots x_{\nu_j}^{k_j}} \rightarrow 0 \text{ if } \begin{array}{l} k_1 + \dots + k_\alpha = \ell_1 + \dots + \ell_\alpha \text{ and} \\ k_1 < \ell_1 \text{ or } k_1 = \ell_1, k_2 < \ell_2 \text{ or} \\ k_1 = \ell_1, \dots, k_{\alpha-1} < \ell_{\alpha-1} \end{array}$$

Consequently  $S_I = CM_{\{1,2,\dots,\alpha\}} +$  terms of slower growth as  $\nu \rightarrow \infty$  and where  $C > 0$ , i.e.,

$$(M_{\{1,2,\dots,\alpha\}}/\text{others terms})(x_\nu) > B.$$

Since  $M'M$  belongs to the convex hull of the  $M_\pi$ ,  $(M'M/M_{\{1,2,\dots,\alpha\}})(x_0)$  is bounded as  $\nu \rightarrow \infty$ . This proves (IV.8).

As a corollary of (IV.8), under the same hypothesis,

$$\lim_{x_1 \rightarrow 0 \text{ for } i \in I} M'(x) = \infty.$$

We then have

$$\lim_{x_i \rightarrow 0 \text{ for } i \in j} (M/S_I)(x) = 0;$$

i.e.,

$$\lim_{t \rightarrow \Delta_I} (M/S_I) = 0.$$

This is what we needed to show; i.e., that

$$\lim_{t \rightarrow \Delta_I} (-\partial\bar{\partial} \log P) = -\partial\bar{\partial} \log P_I \text{ modulo } dt_1, d\bar{t}_r \text{ for } i \in I^c$$

and that the limit exists.

## V. PROOF THAT THE EXTENDED HODGE LINE BUNDLE IS AMPLE

In this section we will give a proof of Theorem B. We recall the statement: Given a period mapping  $\Phi : B \rightarrow \Gamma \backslash D$  and a smooth projective completion  $\bar{B}$  of  $B$  such that the local monodromies around the irreducible branches  $Z_i$  of the reduced normal crossing divisor  $Z = \bar{B} \backslash B$  are unipotent, the image  $M = \Phi(B) \subset \Gamma \backslash D$  is a complex analytic variety, and in Section II we have constructed a completion  $\bar{M}$  of  $M$  such that the period mapping extends to give a diagram

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & M \subset \Gamma \backslash D \\ \cap & & \downarrow \\ \bar{B} & \xrightarrow{\Phi_e} & \bar{M}. \end{array}$$

The Hodge line bundle over  $\Gamma \backslash D$  induces  $\Lambda \rightarrow M$  and there is an extension  $\Lambda_e \rightarrow \bar{M}$  such that  $\Phi_e^*(\Lambda_e)$  is the canonically extended Hodge bundle on  $\bar{B}$ . The result to be established is

$$\Lambda_e \rightarrow \bar{M} \text{ is ample.}$$

The proof will be given in two steps, the first of which is a general result — not related to Hodge theory — and is an extension to singular

varieties of the classical Kodaira theorem. The second step will extend the proof of the first to the case where the metric and curvature have singularities of the type described in Sections III and IV above.

**Step one:** In this step it will be convenient to change our notation to better reflect the general nature of the result being proved. Thus we assume

- $X$  is a compact, complex analytic variety;
- $L \rightarrow X$  is a holomorphic line bundle having a Hermitian metric whose Chern form  $\Omega$  is positive in the Zariski tangent spaces to  $X$ .

In fact, we assume that  $X$  is covered by open neighborhoods  $U$  which are realized as analytic subvarieties in  $\mathbb{C}^N$ , and that the restrictions  $L|_U \cong \mathcal{O}_U$  are trivialized and the metric in  $L|_U$  is the restriction to  $U$  of a positive smooth function defined on an open set in  $\mathbb{C}^N$ .

- $\tilde{X} \xrightarrow{\pi} X$  is a desingularization of  $X$  with connected fibres; we set  $\tilde{L} = \pi^*L$ .

In fact, although it is probably not necessary, we also assume that  $\tilde{X}$  is a smooth projective variety, as this will be the case for our application to Hodge theory. With the above notations and assumptions we will show that

**THEOREM V.1:**  $L \rightarrow X$  is ample.

For the definition of ample we shall use

*for any coherent analytic sheaf  $\mathcal{F} \rightarrow X$ , we have  $h^q(\mathcal{F} \otimes L^m) = 0$   
for  $q > 0$  and  $m \geq m_0(\mathcal{F})$ .*

Finally we shall relax the first assumption above in that we allow  $X$  to be a *complex analytic scheme*; i.e., we do not assume that the analytic space  $(X, \mathcal{O}_X)$  is reduced. This is necessary as the proof will be given by induction on  $\dim X$ , and even if  $X$  itself is reduced we shall see that in the intermediate steps of the argument the analytic varieties that arise may not be reduced. The second assumption above should then be that  $L_{\text{red}} \rightarrow X_{\text{red}}$  has a Hermitian metric with a positive Chern form as described there. The argument in Step one is an adaptation of

the standard one in algebraic geometry; e.g., the one on pages 31ff. in [K-M].

We begin my noting that

$$(V.2) \quad L \rightarrow X \text{ is strictly nef.}$$

Strictly nef means that for any analytic curve  $C \subset X$  we have

$$L \cdot C = \deg \left( L|_{C_{\text{red}}} \right) > 0.$$

In fact, for any  $k$ -dimensional analytic subvariety  $Z \subset X$  we have

$$L^k \cdot Z := c_1(L)^k [Z_{\text{red}}] > 0.$$

The reason for this is that for  $\tilde{C} = \pi^{-1}(C) \subset \tilde{X}$  we first have

$$c_1(L)[C_{\text{red}}] = c_1(\tilde{L})[\tilde{C}_{\text{red}}].$$

Then  $\tilde{\Omega} = \pi^*(\Omega)$  is non-negative  $(1, 1)$  form on  $\tilde{X}$  with the property that for  $\xi \in T\tilde{X}$ ,

$$\tilde{\Omega}(\xi) = 0 \iff \pi_*(\xi) = 0.$$

It follows that

$$c_1(\tilde{L})[\tilde{C}_{\text{red}}] = \int_{\tilde{C}_{\text{red}}} \tilde{\Omega} > 0.$$

As we will see, it is really the properties (V.2) and (V.3) below that are needed for the argument.

A second property is

$$(V.3) \quad L \rightarrow X \text{ is big.}$$

For us, this is a direct consequence of results of Demailly [De], specifically his holomorphic Morse inequalities. It also follows from the work of Siu [Si] on the Grauert-Riemenschneider conjecture. If  $\dim X = d$ , then since  $\tilde{\Omega} \geq 0$  and  $\tilde{\Omega}^d > 0$  on a Zariski open set in  $\tilde{X}$ , it follows from the Riemann-Roch theorem that the Euler characteristic

$$\chi[\tilde{L}^m] = cm^d + \dots, \quad c > 0.$$

By Demailly (loc. cit.)

$$h^q(\tilde{L}^m) = o(m^d), \quad q > 0$$

which gives that  $\tilde{L} \rightarrow \tilde{X}$  is big.

We next have

$$0 \rightarrow \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{F} \rightarrow 0$$

where  $\mathcal{F}$  is supported on a proper subvariety of  $X$ . Since  $\tilde{L}$  is trivial on the connected fibres of  $\tilde{X} \xrightarrow{\pi} X$ , we have

$$0 \rightarrow L^m \rightarrow \pi_* \tilde{L}^m \rightarrow \mathcal{F} \otimes L^m \rightarrow 0$$

which using

$$H^0(\tilde{X}, \tilde{L}^m) \cong H^0(X, \pi_* \tilde{L}^m)$$

and the big-ness of  $\tilde{L} \rightarrow \tilde{X}$  gives (V.3).

The key part of the proof is to show that

$$(V.4) \quad L \rightarrow X \text{ is free.}$$

Assuming this, in case  $X$  is reduced the linear systems  $|mL|$  for  $m \gg 0$  give holomorphic — not just meromorphic — maps

$$\varphi_m : X \rightarrow \mathbb{P}^{N_m}$$

with

$$\varphi_m^* \mathcal{O}_{\mathbb{P}^{N_m}}(1) = L^m.$$

Because of (V.2) no positive-dimensional subvariety of  $X$  is contracted by  $\varphi_m$ , so that  $\varphi_m$  is a finite map and this gives the result in this case.

Still assuming that we have V.4 in the reduced case, to give the argument when  $X$  may not be reduced we proceed by induction on  $\dim X$ . If  $\dim X = 1$  the result follows from

$$\deg \left( L|_{X_{\alpha, \text{red}}} \right) > 0$$

where  $X_{\alpha}$  are the irreducible components of  $X$ . If  $\dim X$  is arbitrary, assuming as we may that  $X$  is irreducible, in the exact sequence

$$(V.5) \quad 0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_{\text{red}}} \rightarrow 0$$

the sheaf  $\mathcal{F}$  has a filtration whose associated graded sheaves  $\text{Gr}^{\bullet} \mathcal{F}$  are  $\mathcal{O}_{X_{\text{red}}}$ -modules. Tensoring (V.5) with  $L^m$  and using the result  $h^1(\mathcal{O}_{X_{\text{red}}}, \text{Gr}^{\bullet} \mathcal{F} \otimes L^m) = 0$  for  $m \gg 0$  in the reduced case leads, by the usual spectral sequence argument, to  $h^1(X, \mathcal{F} \otimes L^m) = 0$  for  $m \gg 0$ .



To finally prove (V.4) in general when  $X$  may not be reduced we use the classical argument. Replacing  $L$  by a high power, we use (V.2) to give that there exists a possibly non-reduced effective divisor  $Y \in |L|$ . By the induction assumption,  $L_Y = \mathcal{O}_Y(L)$  is ample. From the cohomology sequence of

$$0 \rightarrow L_X^{m-1} \rightarrow L_X^m \rightarrow L_Y^m \rightarrow 0$$

and  $h^1(L_Y^m) = 0$  for  $m \gg 0$ , we obtain

$$H^1(L_X^{m-1}) \rightarrow H^1(L_X^m) \rightarrow 0, \quad m \geq m_0.$$

Thus the  $h^1(L^m)$  are non-increasing for  $m \geq m_0$ , and for  $m \geq m_1$  we will have

$$H^1(L_X^{m-1}) \xrightarrow{\sim} H^1(L_X^m).$$

This gives

$$H^0(L_X^m) \rightarrow H^0(L_Y^m) \rightarrow 0.$$

Then since  $L_Y \rightarrow Y$  is free the same will be true for  $L_X \rightarrow X$ .

**Step two:** The proof of Theorem B now may be completed by combining the argument just given with Theorem C as stated in the introduction. We first note that for any irreducible curve  $C \subset \overline{M}$  we have

$$(V.6) \quad \deg(\Lambda_e|_C) = \int_C \Omega_e > 0.$$

Indeed, for some index set  $I$  the intersection  $C^* =: C \cap Z_I^*$  will be a Zariski open set in  $C$ . From the analysis of the singularities of  $\Omega_e$  given in Section III (cf. Proposition III.9), it follows that the integral in (V.6) is defined. By the construction of  $\overline{M}$ , the image  $\Phi_I(C^*) \subset \Gamma_I \setminus D_I$  is a (possibly non-complete) curve, and consequently the integral is positive.

It remains to show that if  $\dim M = d$ , then the integral

$$(V.7) \quad \int_{\overline{M}} \Omega_e^d > 0$$

is defined and is positive. This result is proved in [C-K-S1] with important amplifications in [Kol2]. It also follows from the calculation in

Sections III, IV above. In a neighborhood of a point in a codimension one stratum  $Z_i^*$  the calculations in Section III easily give the result. In general one needs to consider possible cross-terms of the form

$$\frac{dt_i \wedge d\bar{t}_i}{|t_i|^2 (\log |t_i|)^2} \wedge \frac{dt_j \wedge \alpha}{|t_j| \left( \log \frac{1}{|t_j|} \right)^a}$$

where  $\alpha$  is either  $C^\infty$  or has a  $dt_j/|t_j| \left( \log \frac{1}{|t_j|} \right)^b$  term. The first case is easy, since  $d\bar{t}_j$  must appear elsewhere in  $\Omega_e^d$ . In the second case, since both  $a, b$  are positive any such term will not cause the integral to diverge.

Once we have (V.6) and (V.7) the same argument as in step one may be used to give a proof of Theorem B.  $\square$

## VI. CURVATURE PROPERTIES OF THE HODGE VECTOR BUNDLE

This section will be divided into three parts, listed just below, followed by an appendix.

- (i) Generalities on Hermitian vector bundles;
- (ii) Proof of Theorem D;
- (iii) General issues concerning the Chern forms.

At the end we shall give some general comments, partly of a historical nature, about the positivity of the bundles arising in Hodge theory during the early and middle developments of the topic; these developments are all that are needed for the present work.

### (i) *Generalities on Hermitian vector bundles*

Given a holomorphic vector bundle  $E \rightarrow Y$  over a complex manifold, to a Hermitian metric in the bundle there is canonically associated its Chern connection  $D: A^\circ(E) \rightarrow A^1(E)$  with curvature  $\Theta_E \in A^{1,1}(\text{Hom}(E, E))$  and curvature form

$$\Theta_E(e, \xi) = \langle (\Theta_E(e), e), \xi \wedge \bar{\xi} \rangle.$$

To interpret the curvature we shall use the associated projective bundle  $\pi: \mathbb{P}E \rightarrow Y$  with tautological line bundle  $\mathcal{O}_{\mathbb{P}E}(1)$ . Here we are

using the standard convention

$$(\mathbb{P}E)_y = \mathbb{P}E_y^* = \pi^{-1}(y)$$

and so that the fibres of  $\mathcal{O}_{\mathbb{P}E}(1)$  along  $\pi^{-1}(y)$  are the 1-dimensional quotients of  $E_y$ . As is well known, for the direct images we have

$$(VI.1) \quad \pi_* \mathcal{O}_{\mathbb{P}E}(m) = S^m E, \quad m \geq 0.$$

We will denote points of  $\mathbb{P}E$  by  $(y, [e^*])$  where  $0 \neq e^* \in E_y^*$  and  $[e^*]$  is the line spanned by  $e^*$ . Then

$$\mathcal{O}_{\mathbb{P}E}(1)_{(y, [e^*])} = \mathbb{C}e^*.$$

Using the conjugate linear isomorphism  $E_y^* \cong E_y$  given by the Hermitian metric in  $E \rightarrow Y$ , there is an induced metric in  $\mathcal{O}_{\mathbb{P}E}(1)$ . We denote by  $\Omega_E$  its Chern form, and note that

$$\Omega_E|_{(\mathbb{P}E)_y} \text{ is the Fubini-Study form on } \mathbb{P}E_y^*.$$

At each point of  $\mathbb{P}E$  the vertical space

$$V_{(y, [e^*])} = \ker \pi_* : T_{(y, [e^*])}\mathbb{P}E \rightarrow T_y Y$$

is defined, and using that  $\Omega_E|_{T_{[e^*]}\mathbb{P}E_y^*}$  is a positive (1,1) form we may define the horizontal space

$$H_{(y, [e^*])} = (V_{(y, [e^*])})^\perp \subset T_{(y, [e^*])}\mathbb{P}E$$

where  $(\ )^\perp$  is relative to  $\Omega_E$ . We then have for  $c \in E_y$  and  $\xi \in T_y Y$

$$(VI.2) \quad \Theta_E(e, \xi) = \Omega_E|_{H_{(y, [e^*])}}(\xi).$$

This means:  $e^*$  corresponds to  $e$  using  $E_y^* \cong E_y$  via the metric, and on the right-hand side  $\xi \in H_{(y, [e^*])}$  under the isomorphism

$$\pi_* : H_{(y, [e^*])} \cong T_y E.$$

The RHS of (VI.2) is the value of  $\Omega_E$  on  $\xi \wedge \bar{\xi}$ . In other words,

$$(VI.3) \quad \begin{aligned} & \textit{The Chern form of } \mathcal{O}_{\mathbb{P}E}(1) \textit{ is equal to the Fubini-Study} \\ & \textit{form on the fibres of } \mathbb{P}E \rightarrow Y, \textit{ and on the horizontal} \\ & \textit{spaces it is identified with the curvature form of } E \rightarrow Y. \end{aligned}$$

As an application of this and as a check on signs, we shall prove the

PROPOSITION VI.4: *If  $Y$  is compact and  $E \rightarrow Y$  is positive, then*

$$H^0(Y, E^*) = 0.$$

*Proof.* From (VI.1) we have that  $H^0(Y, E^*) \cong H^0(Y, \mathcal{O}_{\mathbb{P}E^*}(1))$ , so it will suffice to show that this group is zero. If  $\sigma \in H^0(Y, E^*)$ , then at a maximum point of  $\|\sigma\|^2$  we have

$$(i/2)\partial\bar{\partial}\log\|\sigma\|^2 \leq 0.$$

In the fibration  $\mathbb{P}E^* \rightarrow Y$  the (1,1) form  $(i/2)\partial\bar{\partial}\log\|\sigma\|^2$  is minus the Fubini-Study form in the vertical tangent space, and using the identification described above it is  $-\Theta_{E^*}(\sigma, \xi)$  in the horizontal tangent space. Our assumption gives  $-\Theta_{E^*}(\sigma, \xi) > 0$ , which is a contradiction.  $\square$

As another check on signs we have the following special case of a result of Bloch-Gieseker ([B-G]).

PROPOSITION VI.5: *If  $Y$  is compact and  $E \rightarrow Y$  is positive with rank  $E \geq \dim Y$ , then any section  $\sigma \in H^0(Y, E)$  has a zero.*

*Proof.* We assume that  $\sigma$  has no zero and go to a minimum  $y_0$  of  $\log\|\sigma\|^2$  where we have

$$(i/2)\partial\bar{\partial}\log\|\sigma\|^2(y_0) \geq 0.$$

This time the term on the left is the restriction to the graph of  $\sigma$  of the (1,1) form

$$(\text{Fubini-Study form}) + \Theta_E(\sigma(y_0), \bullet).$$

The first term is positive of rank equal to  $\dim E_y - 1$ , while the second is negative of rank  $\dim T_{y_0}Y > \dim E_y - 1$ . Thus  $(i/2)\partial\bar{\partial}\log\|\sigma\|^2$  must have at least one negative eigenvalue, which gives a contradiction.  $\square$

Finally, for use below we have the

PROPOSITION VI.6: *If  $Y$  is compact Kähler and  $\sigma \in H^0(Y, E)$  is a section with  $\Theta_E(\sigma) = 0$ , then  $D\sigma = 0$ .*

*Proof.* We have

$$\partial\bar{\partial}(\sigma, \sigma) = \partial(\sigma, D\sigma) = (D\sigma, D\sigma) + (\sigma, \Theta_E(\sigma)) = (D\sigma, D\sigma).$$

Then if  $\omega$  is the Kähler form and  $\dim Y = d$ ,

$$0 = \int_X \omega^{d-1} \wedge (i/2) \partial\bar{\partial} \|\sigma\|^2 = \int_Y \omega^{d-1} (i/2) (D\sigma, D\sigma) \geq 0$$

where equality holds only if  $D\sigma = 0$ .  $\square$

(ii) *Proof of Theorem D*

The first step in the argument is to show that the curvature  $\Theta_F$  has a very special form that we now explain. Suppose given complex vector spaces  $T, F, E$  and a linear map

$$A : T \rightarrow \text{Hom}(F, E).$$

Assume that  $T$  is the  $(1,0)$  part of the complexification of a real vector space and that each of  $F, E$  have Hermitian metrics. We are thinking of the case

$$(VI.7a) \quad T = T_b B, \quad F = F_b, \quad E = (F^{n-1}/F^n)_b$$

and where

$$(VI.7b) \quad A = \Phi_{*,n}$$

is the first piece of the differential of a period mapping. (To avoid notational clutter we drop reference to the point  $b \in B$ .)

Given the above data we define

$$(VI.8) \quad \Theta = A \wedge {}^t \bar{A} \in \text{Hom}(F, F) \otimes T^* \otimes \bar{T}^*$$

where  ${}^t \bar{A}$  is defined relative to the Hermitian structures on  $F$  and  $E$ . We note that

$$\Theta + {}^t \Theta = 0.$$

We may then define the Chern forms  $c_q(\Theta)$  by taking the characteristic polynomial of  $(1/2\pi i)\Theta$ . In the examples (VI.7a), (VI.7b) above we have from [G1] that  $\Theta = \Theta_F$ . In fact, for  $e \in F_b$  and  $\xi \in T_b B$  the curvature form for the Hodge vector bundle is given by

$$(VI.9) \quad \Theta_F(e, \xi) = \|\Phi_{*,n}(\xi)\|^2$$

so that  $\langle A, \xi \rangle$  is  $\Phi_{*,n}(\xi)$ . We will give a proof of this in the lemma just below.

The point is that when the curvature has the special form (VI.8), the Chern forms are non-negative and their vanishing has a linear algebra interpretation. For this we have the following

LEMMA VI.10: *The linear mapping  $A$  induces*

$$\wedge^q A : \wedge^q T \rightarrow \wedge^q F^* \otimes S^q E$$

*and up to a universal constant*

$$c_q(\Theta) = \|\wedge^q A\|^2.$$

*Proof.* The notation means

$$\|\wedge^q A\|^2 = (\wedge^q A, \wedge^q A)$$

where in the inner product we use the Hermitian metrics on  $F^*$  and  $E$ , and we identify

$$\wedge^q T^* \otimes \overline{\wedge^q T^*} \cong (q, q)\text{-part of } \wedge^{2q} (T^* \oplus \overline{T^*}).$$

Then letting  $A^*$  denote the adjoint of  $A$  we have

$$\wedge^q \Theta = \wedge^q A \otimes \wedge^q A^* = \wedge^q A \otimes (\wedge^q A)^*$$

and

$$c_q(\Theta) = \text{Tr } \wedge^q (\Theta) = (\wedge^q A, \wedge^q A). \quad \square$$

In matrix terms, if

$$A = \dim F \times \dim T \text{ matrix with entries in } E$$

then

$$\wedge^q A = \left\{ \begin{array}{l} \text{matrix whose entries are the } q \times q \\ \text{minors of } A \text{ where the entries of } E \\ \text{are multiplied as polynomials} \end{array} \right\}.$$

It follows that again up to a universal constant

$$c_q(\Theta_F) = \sum_{\alpha} \Psi_{\alpha} \wedge \overline{\Psi}_{\alpha}$$

where the  $\Psi_{\alpha}$  are  $(q, 0)$  forms. In particular, any monomial  $c_I(\Theta_F) \geq 0$ .

We note that

$$(VI.11) \quad \begin{array}{l} \text{the vanishing of the matrix } \wedge^q \Phi_{*,n} \text{ is **not** the same as} \\ \text{rank } \Phi_{*,n} < q. \end{array}$$

In fact, as follows from (I.2)

$$(VI.12) \quad \text{rank } \Phi_{*,n} < q \iff c_1(\Theta_F)^q = 0.$$

In the geometric case when we have

$$\Phi_{*,n} : T_b B \rightarrow \text{Hom}(H^0(\Omega_{X_b}^n), H^1(\Omega_{X_b}^{n-1}))$$

and the algebro-geometric interpretation of (VI.12) is standard; e.g.,  $\Phi_{*,n}$  injective is equivalent to local Torelli holding for the  $H^{n,0}$ -part of the Hodge structure.

In contrast, the algebro-geometric map corresponding to (VI.11) is

$$\wedge^q T_b B \otimes \wedge^q H^{n,0}(X_b) \rightarrow \text{Sym}^q H^{n-1,1}(X_b),$$

which thus far does not seem to fall into the standard theory.

We conclude this subsection with a result that pertains to the question at the end of the introduction.

**PROPOSITION VI.13:** *If  $\Phi : B \rightarrow \Gamma \backslash D$  has no trivial factors, and if  $h^{n,0} \leq \dim B$  and  $H^0(\overline{B}, F_e) \neq 0$ , then*

$$c_{h^{n,0}}(F) \neq 0.$$

*Proof.* We will first prove the result when  $B = \overline{B}$ . The general case will then be done following a discussion of the singularities of  $\Theta_{F_e}$ .

We let  $\sigma \in H^0(F, B)$  and assume that  $c_{h^{n,0}}(F) = 0$ . Then  $\sigma$  is everywhere non-zero, and we may go to a minimum of  $\|\sigma\|^2$ . From Proposition (VI.6) we have  $D\sigma = 0$ , which using (VI.9) implies that the norm  $\|\sigma\|$  is constant and

$$\nabla \sigma = 0$$

where  $\nabla$  is the Gauss-Manin connection. Using standard arguments ([G2]) we may conclude that the variation of Hodge structure has a trivial factor.

If  $B \neq \overline{B}$ , the arguments given in Section III may be adapted to show that the proof of Proposition VI.9 still goes through. The point

is the equality of the distributional and formal derivatives that arise in integrating by parts.  $\square$

Finally for this subsection, we remark that if  $L \rightarrow \overline{B}$  is a positive line bundle, then for any  $\epsilon > 0$  and even though the metric is singular  $F_e \otimes L^\epsilon$  is a positive vector bundle and  $\mathcal{O}_{\mathbb{P}E_e}(1) \otimes \pi^* L^\epsilon$  is an ample line bundle. It seems plausible that this will lead to a proof of Viehweg's results about the weak positivity of the direct images of relative dualizing sheaves in algebraic fibres spaces.<sup>44</sup>

(iii) *General issues concerning the Chern forms*

We consider a variation of Hodge structure given by a period mapping (II.2) satisfying the conditions stated there. We are interested in the behavior of the curvature form  $\Theta_F(e, \xi)$  and Chern forms  $c_q(\Theta_F)$  on  $\overline{B}$ . Their basic properties are:

(VI.14) *For any monomial  $c_A(\Theta_F)$ , we have that*

- (i)  *$c_A(\Theta_F)$  defines a closed, positive current whose entries are  $L^1$  functions;*
- (ii) *the wave front set  $\text{WF}(c_A(\Theta_F))$  is defined and*

$$\text{WF}(c_A(\Theta_F)) \subseteq \bigcup_I N_{Z_I^*/\overline{B}}^*$$

- (iii) *because of (ii) the restrictions  $c_A(\Theta_F)|_{Z_I^*}$  are defined and are smooth forms on  $Z_I^*$  which satisfy*

$$c_A(\Theta_F)|_{Z_I^*} = c_A\left(\Theta_{F_e|_{Z_I^*}}\right).$$

In order to establish these properties we offer the following would observations:

**A:** If  $h$  is the Hodge metric relative to a local holomorphic frame around a point of  $Z = \overline{B} \setminus B$ , then we may calculate  $\theta = h^{-1}\partial\theta$  and  $\Theta = \bar{\partial}\theta$  either as currents or formally by calculus, and when this is done the results coincide (the property NR in Section III; as noted there, this implies that the Lelong numbers of the currents  $\theta$  and  $\Theta$  are zero.).

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<sup>44</sup>For a treatment of fractional powers of line bundles we refer to [De].



**B:** We may restrict the forms  $c_A(\Theta_F)$  by the same process as in Section IV; i.e.,

- first along  $Z_I^*$  formally set  $dt_i = d\bar{t}_i = 0$ ;
- then set  $\log 1/|t_i| = \infty$ ; i.e., the limits as  $t \rightarrow Z_I^*$  of what is left after the first step exist.

Proof analysis gives that  $A$  can be done by the methods in [C-K-S1] and [Kol2] or those in Section IV above, and also that  $B$  can be carried out also using the methods in Section IV. The details will be further discussed elsewhere.

We also note that one  $\mathbb{P}F_e \xrightarrow{\pi} \bar{B}$  if we consider the Chern form  $\Omega$  of  $\mathcal{O}_{\mathbb{P}F_e}(1)$ , then

**C:**  $\Omega$  defines a closed (1,1) current on  $\mathbb{P}F_e$  that has properties analogous to (i)–(iii) above;

**D:** the push-forward  $\pi_*\Omega$  in the sense of currents exists and may be used to define the Chern forms  $c_q(\Theta_F)$  via the Grothendieck relation. The details of this will also be discussed elsewhere.

We conclude with a question motivated by the results in [V1] and [V2]:

**Question VI.15:** If  $E \rightarrow X$  is a vector bundle over a compact, complex manifold that has a Hermitian metric with positive semi-definite curvature form

$$\Theta_E(e, \xi) \geq 0,$$

and if the Chern form

$$c_1(E) = c(\det E) = \text{Tr} \left( \left( \frac{1}{2\pi i} \right) \Theta_E \right) > 0$$

is positive definite, then is the Kodaira dimension

$$\kappa(E) \geq \dim X?$$

For period mappings we have the related

**Question:** If  $\Lambda \rightarrow B$  has positive Chern form then is

$$\kappa(F) \geq \dim B?$$

Both questions may be raised when the stated conditions hold on a Zariski open set.

Finally one may pose the less specific

**Question:** Under the above assumptions, are there positivity conditions on the Chern monomials that will increase the estimate on the Kodaira dimension?

We will conclude this section with some historical comments on the evolution of our understanding of the positivity of the Hodge bundles up to the period when the papers [V1], [V2] and [Kol2] appeared; these works are the ones most relevant to the current work. Early computations of the curvature of homogeneous vector bundles over flag domains appeared in the 1960s and showed that their behavior in the Hermitian symmetric and non-Hermitian symmetric cases were quite different (cf. [G-S1]). The computations of the curvature for the Hodge bundles and tangent bundles over period domains given in [G2] and [G-S1], [G-S2] then made the point that when restricted to integral manifolds of the infinitesimal period relation the curvatures had positivity properties analogous to those in the classical Hermitian symmetric domain setting. In particular the Hodge vector bundle was semi-positive and the Hodge line bundle had positive curvature if the differential of the period mapping had suitable injectivity properties. The paper [Fu] was an important development here. As discussed above, for a variation of Hodge structure the Hodge vector bundle is almost never positive and the understanding of just how positive it actually is under geometrical assumptions on the differential of the period mapping seems to be still incomplete.

The next stage was understanding the behavior of the Hodge metrics and curvature forms when a variation of Hodge structure degenerates along a normal crossing divisor. Here the works [Sc1], [C-K-S1] and [Kol2] played a major role; it is worth noting that both [C-K-S1] and [Kol2] were in part motivated by algebro-geometric questions related to the Iitaka conjecture and where the singularities of the Chern form of

the Hodge line bundle entered in a crucial way (cf. [Ka1], [Ka2], [Ka3], [U1], [U2], [V1], [V2] and the references cited in those works).

As a historical note, the indication that the curvature forms had Poincaré metric singularities may have originally come from the regularity of the Gauss-Manin connection (cf. [De]). Namely, the result that the periods of algebraic differential forms had logarithmic singularities in a degenerating family  $\{X_t\}$  of algebraic varieties implies that the Hodge norms, given up to a constant by

$$\|\psi_t\|^2 = \int_{X_t} \psi_t \wedge \bar{\psi}_t$$

where  $\psi_t$  is a holomorphic  $n$ -form on  $X_t$  for  $t \neq 0$ , are of the form

$$\|\psi_t\|^2 = \left( \log \frac{1}{|t|} \right)^m h_0(t)$$

where  $h_0(t)$  is positive and where it together with  $\partial \log h_0$  and  $\partial \bar{\partial} \log h_0$  are bounded (cf. the discussion in Section III above). Then up to a constant

$$\partial \bar{\partial} \log \|\psi_t\|^2 = m \partial \bar{\partial} \log(-\log |t|) + \text{LOT}$$

where  $\partial \bar{\partial}(-\log |t|) = \frac{dt \wedge d\bar{t}}{|t|^2 (\log |t|)^2}$  is the Poincaré metric and LOT are lower order terms. If instead the Hodge norms had singularities like

$$\|\psi_t\|^2 = \left( \frac{1}{|t|^\alpha} \right) h_0(t), \quad \alpha > 0$$

then the curvature form would have had terms like

$$\partial \bar{\partial} \log |t| = \delta_0 dt \wedge d\bar{t}$$

and therefore one would have picked up contributions given by currents with non-zero Lelong numbers in the Chern classes of the Hodge bundles. Thus the difference between  $\partial \bar{\partial} \log(-\log |t|)$  and  $\partial \bar{\partial} \log |t|$ , i.e., the difference between periods having logarithmic singularities and having poles, suggests the mild singularity behavior of the curvatures of these bundles.

The main result of this work is the existence and ampleness of the extended Hodge line bundle

$$\Lambda_e \rightarrow \bar{M}.$$

It may be viewed as an extension of the earlier positivity-type results, a main point being that  $\text{Proj}(\Lambda_e)$  exists and may be exactly described as the general analogue of the Satake-Baily-Borel compactification in the classical cases.<sup>45</sup> From an exterior differential system perspective, the maximal integral varieties  $\Omega_e = 0$  of the extended Chern form define a foliation of  $\bar{B}$  by complex subvarieties whose quotient by contracting the connected leaves of the foliation exactly captures identifying the limiting mixed Hodge structures that have the same associated graded. The main issue here is to make sense out of the exterior equation  $\Omega_e = 0$  when  $\Omega_e$  has singularities.

Finally we note that the curvatures of the Hodge bundles and their singularities is currently an extremely active and interesting topic; we refer to [P] for a recent survey paper with references to some of the current work in this area.

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<sup>45</sup>Classical refers to curves, K3's, Shimura varieties etc. where the period domain is Hermitian symmetric.

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