A Tale of Two Mathematicians

(It was the best of times. . . for mathematics)

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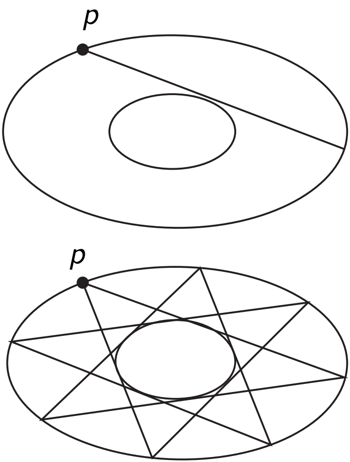
Outline

I. Introduction
II. Poncelet and projective geometry
III. Abel and analysis
IV. Poncelet meets Abel

I. Introduction

This is the story of some of the mathematical work of two mathematicians, Jean Victor Poncelet and Niels Henrik Abel. They were contemporaries in the early 19^{th} century who never met and who were not even aware of each other's work. However, between them Poncelet and Abel laid the cornerstones of the modern field of algebraic geometry, a field that is central to current work in geometry, arithmetic and theoretical physics. In this talk I will try to explain what each of them did, Poncelet in geometry and Abel in analysis, and how the fusion of their work revealed one of the deepest aspects of mathematics. This fusion is captured by an amazing property of playing billiards on a table formed by two ellipses.



The periodicity of the billiard shot is *independent* of the starting point p.

In this talk I will try to make the presentation as accessible as possible. For the geometry part I will draw pictures. There is, however, one caveat. Curves in the plane are represented by solutions to polynomial equations. To have the full set of solutions one must use complex numbers. So I will usually draw the real solutions, leaving it to your imagination that there are other "imaginary" solutions that we shall actually use in geometric arguments but which cannot be easily visualized.

For the analysis part all that will be used is calculus. In fact, the story revolves around integrals that you have been told could not be evaluated. The wonderful fact is that these integrals have a deep structure, going far beyond "evaluation" in the calculus 1A sense. These integrals will also use complex numbers; they will be the usual integrals in calculus taken along paths in the complex plane.

The story line is that the amazing geometric property of the billiards was understood by expressing analytically the operation of shooting a billiard ball, and then Poncelet's theorem becomes translated into a property of integrals of algebraic functions discovered by Abel; in this case the integral is

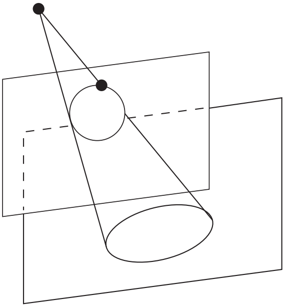
$$\int \frac{dx}{\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}}.$$

Just how the result about the periodicity of the billiard shot becomes translated into a property of this integral is the "deep song" I will hope to tell in this talk.

II. Poncelet and projective geometry

Projective geometry is like ordinary plane geometry with two major differences:

- Ideal points at infinity are added, so that for example any two distinct lines meet in a point (think of the horizon on train tracks).
- Two figures are considered to be the same if they line up when projected from a point.



Here a circle becomes an ellipse when projected.

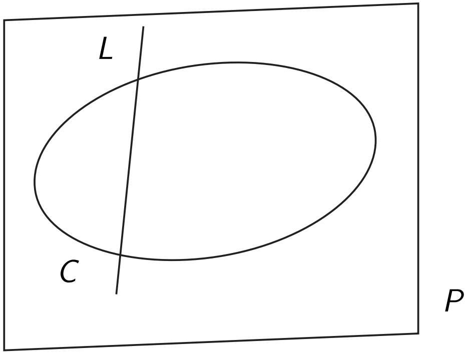
An important aspect of projective geometry is *duality*: Two points in the projective plane P determine a line, and two lines determine a point

Thus the projective plane P has a dual $\check{P}$ where

- the points of $\check{P}$ are the lines in P.

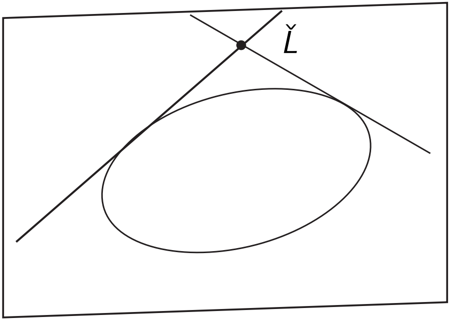
A line in the dual projective plane $\check{P}$ is the set of lines through a point in P.

Much of the classical projective geometry is concerned with *conics*



L meets C
in two points

A conic C has a dual conic $\check{C}$ in $\check{P}$; $\check{C}$ = set of tangent lines to C

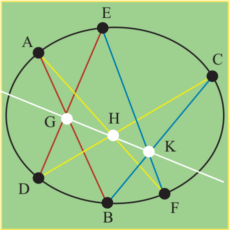


the line $\check{L}$
in $\check{P}$ meets $\check{C}$
in two points

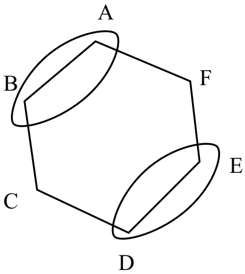
One of the origins of projective geometry arose from the use of perspective in art, from the late middle ages and early renaissance periods. The first major theorem in projective geometry is due to the 16 year old Blaise Pascal (1639), who stated but did not completely prove

Pascal's Theorem

The opposite sides of a hexagon inscribed in a conic meet in three collinear points.



Pascal line GHK of hexagon ABCDEF inscribed in ellipse. The three collinear points are in white.

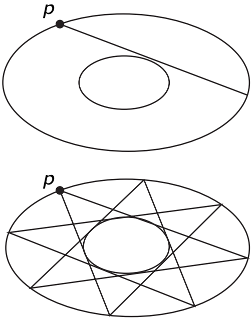


opposite sides identified

Poncelet was a French engineer and mathematician who was one of the main founders of the subject of projective geometry as we know it today. He was captured in one of Napoleon's campaigns against the Russian empire, and while in a Russian jail he spent his time thinking about mathematics. In 1812 he was engrossed in the study of pairs of conics; in particular he discovered an amazing and completely unexpected property of such pairs that became

Poncelet's Theorem

In the figure



If you draw the above figure of a polygon inscribed in one ellipse and circumscribed about the other, this figure closes up for **one** *starting point if, and only if, it closes up for* **any** *starting point.*

In other words, our billiard-like game is periodic everywhere, or nowhere.

This type of problem seems worthy of an engineer but, at least at first glance, more of a mathematical curiosity. But thinking about it, it is difficult not to ask

Why *should such a result be true?*

For Pascal's theorem there is at least a heuristic argument why the result holds. For Poncelet there is none such.

In more detail for Pascal, conics are given by

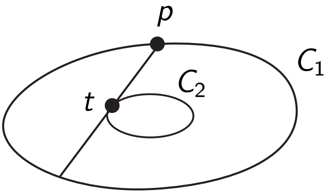
$$ax^2 + 2bxy + cy^2 + ex + fy + g = 0.$$

Scaling to have $g = 1$, there are ∞^5 conics. Thus five general points determine a conic and it is one condition on a sixth point to be on a conic.

Six points determine a hexagon as above, and it is also one condition for the intersection points of the opposite sides to be collinear. Thus, as algebraic geometers say "the dimension counts work out."

If you try the same thing for Poncelet then things go haywire. It is one condition for the billiard shot to be periodic so you would think for each pair of conics there are a finite number of starting points where the figure closes up after n steps. But this is not the case. So what **is** going on here?

One last comment on projective geometry: The basic object in Poncelet's theorem is



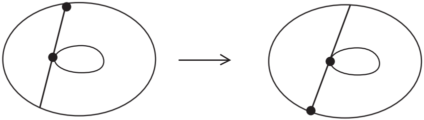
$$(p, t) \in C_1 \times \check{C}_2.$$

Set

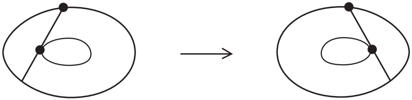
$$E = \{(p, t)\} \subset C_1 \times \check{C}_2.$$

There are two involutions:

$i_1 : E \to E$



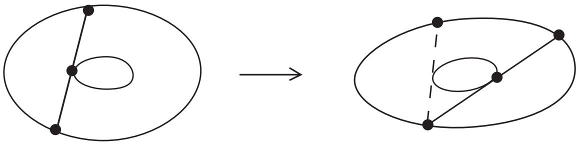
$i_2 : E \to E$



For the compsition

$$j = i_2 \circ i_1$$

we have the basic "pool shot" in Poncelet



The Poncelet condition is therefore expressed as

$$\boxed{j^n(p, t) = (p, t).}$$

III. Abel and analysis

Niels Henrik Abel (1802–1829) was a Norwegian mathematical prodigy, much of whose work was not appreciated or even understood in his short lifetime. As a 19 year old he sent Gauss a paper containing a correct proof of the unsolvability of the quintic, who thought this was the work of a crank and threw the manuscript away.

You will remember from calculus learning a bag of tricks to evaluate integrals. Thus

$$\int x^n dx = \left(\frac{1}{n+1}\right) x^{n+1} \qquad n \neq -1$$

$$\int \frac{dx}{x} = \log x$$

and by partial fractions

$$\int \frac{p(x)}{q(x)}\,dx = r(x) + \sum a_i \log(x - b_i).$$

Even more complicated integrals containing

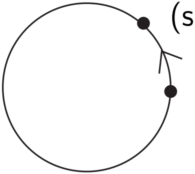
$$\sqrt{ax^2 + bx + c}$$

can, with some work, be evaluated (there are books of tables of these). You will remember the arc length of a circle $x^2 + y^2 = 1$

$$\int \sqrt{dx^2 + dy^2} = \int \frac{dx}{\sqrt{1 - x^2}}$$

Let's write it this way:

$$u = \int^{(\sin u, \cos u)} \frac{dx}{\sqrt{1 - x^*}}$$



that may be used to *define* $\sin u, \cos u$ as those functions that may be used to paramatrize points on a circle by arc length. These functions have wonderful properties such as

$$\sin(u + u') = \sin u \cos u' + \sin u' \cos u$$

that tell you, e.g., how to double the arc length on a circle. We will see below where these come from.

Similar integrals also turn up in elementary mechanics arising from Newton's equation

$$\left(\frac{\dot{u}}{2}\right)^2 = p(u)$$

$$\begin{cases} \dot{u} = \sqrt{p(u)}, \\ du = \sqrt{p(u(t))}dt. \end{cases}$$

However, when we turn to more interesting but more complicated problems integrals of the form

$$\int r\left(x, \sqrt{a(x)}\right) dx, \quad \deg a(x) \geqq 3$$

turn up.

For example, for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

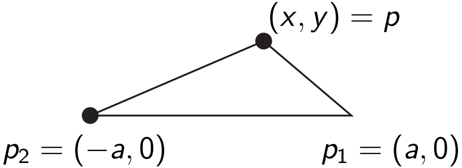
setting $x = a \sin\theta$, $y = b \cos\theta$ and $k^2 = (a^2 - b^2)/a^2$, $a > b$

$$\int \sqrt{dx^2 + dy^2} = \int \frac{a(1 - k^2 \sin^2\theta) d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

and now taking $x = \sin\theta$ we get

$$a \int \frac{(1 - k^2 x^2) dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

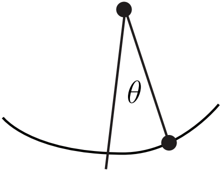
For the lemniscate



where $\overline{pp_1} \cdot \overline{pp_2} =$ constant, the arclength works out to be the nicer integral

$$\int \frac{dx}{\sqrt{1 - x^4}}.$$

In mechanics, the motion of a problem



$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

becomes after using angular velocity as a parameter

$$\int \frac{du}{\sqrt{1 - k^2 \sin u}}$$

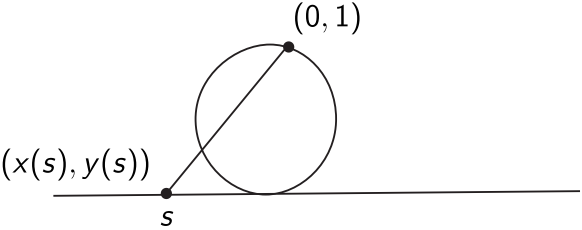
which setting $t = \sin u$ becomes

$$\int \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

The moral is: *Elliptic integrals are ubiquitous*!

Q: *Why can we evaluate integrals containing* $\sqrt{a(x)}$ *when* $\deg a(x) \leqq 2$ *but not when* $\deg a(x) \geqq 3$?

Answer to the easy part: Take the conic $y^2 = a(x)$ and make a linear change of coordinates so that the conic becomes $x^2 + y^2 = 1$. Then we have



$$\begin{cases} x = 4\left(\frac{s}{s^2+4}\right) \\ y = \frac{s^2-4}{s^2+4}. \end{cases}$$

which gives

$$y = \sqrt{\frac{(s^2-4)^2}{(s^2+4)^2}} = \frac{s^2-4}{s^2+4}.$$

By a change of variables

$$\int r\left(x, \sqrt{a(x)}\right) dx = \int R(s)\, ds, \quad R(s) = \text{rational function.}$$

Geometrically we say that *conics may be rationally parameterized by the projective line.*

Enter Abel:

Let $y(x)$ be an algebraic function of x. This means that we have

$$f(x, y(x)) = 0$$

where $f(x, y)$ is a polynomial (say irreducible). An example is

$$y^2 = \sqrt{p(x)}, \quad \text{where the roots } a_i \text{ of } p(x) = 0 \text{ are distinct.}$$

How can we understand

$$\int r(x, y(x))\, dx?$$

An example we shall encounter in Poncelet is

$$\int \frac{dx}{\sqrt{(x - a_1)(x - a_2)(x - a_3)(x - a_4)}} \qquad a_1, a_2, a_3, a_4 \text{ distinct.}$$

Abel's idea:

Instead of the single integral, consider *abelian sums*

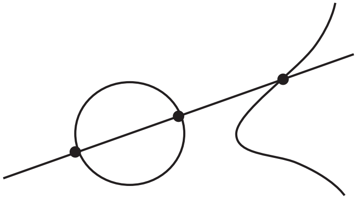
$$u(t) = \sum_i \int^{x_i(t)} r(x, y(x))dx$$

where the $(x_i(t), y_i(t))$ are the intersection points of $f(x, y) = 0$ with a family $g(x, y, t)$ of curves depending on a parameter t.

Abel's theorem: *$u(t)$ is an elementary function of t.*

That is, $u'(t)$ is a rational function of t so that

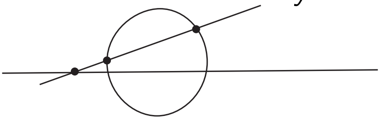
$$u(t) = s(t) + \sum_i b_i \log(t - c_i).$$



$$\begin{cases} y^2 = x^3 + ax + b \\ g(x, y, t) = xt. \end{cases}$$

Moreover, his argument (which could be taught in a calculus or complex function theory class is effective)

Example



When worked out Abel's theorem gives

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \sin x_2 \cos x_1$$

where (x_1, y_1), (x_2, y_2) are the intersection points of the fixed circle with variable line.

Abel's theorem is a very general *addition theorem* for $x(u), y(u)$ defined by

$$u = \int^{(x(u),y(u))} r(x, y(x))\, dx.$$

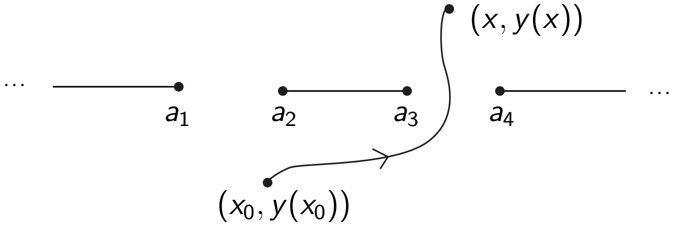
Interlude on complex analysis:

What *do* we mean by

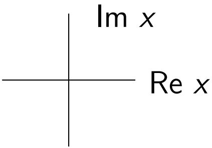
$$\int r(x, y(x))\, dx \ ?$$

Even though Abel came before complex function theory as we now know it, his understanding of the integral is this, illustrated here for the integral we shall be concerned with where

$$y(x) = \sqrt{(x - a_1)(x - a_2)(x - a_3)(x - a_4)}$$



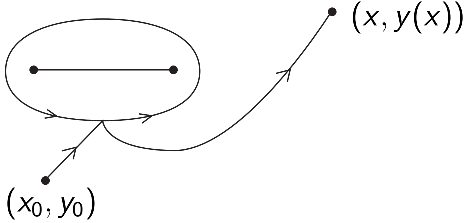
We are working in the complex x plane



and in the slit plane we may choose a single-valued branch of $y(x)$. Then analytic continuation along a path defines $y(x)$. Near $x = \infty$ we set $x' = 1/x$ and then $dx = \frac{-1}{x^2} dx'$ and

$$\frac{dx}{y(x)} = \left(\frac{-1}{(x')^2}\right)(x')^2 \frac{dx'}{\sqrt{(1 - x'a_1)(1 - x'a_2)(1 - x'a_3)(1 - x'a_4)}}$$

Note that the integrand is continuous near $x' = 0$, something we cannot do with $\sqrt{p(x)}$ when $\deg p(x) \leqq 2$. Taking the case of $\int \frac{dx}{\sqrt{1-x_2}}$ we have a similar but simpler picture (only one slit), and here a different choice of path from $(x_0, y(x_0))$ gives a picture



Then

$$\int \frac{dx}{\sqrt{1-x^2}} \text{ changes by } 2\pi$$

so that

$$\int \frac{dx}{\sqrt{1-x^2}} \in \mathbb{C}/2\pi\mathbb{Z}$$

(the integral is the arcsin function). Thus

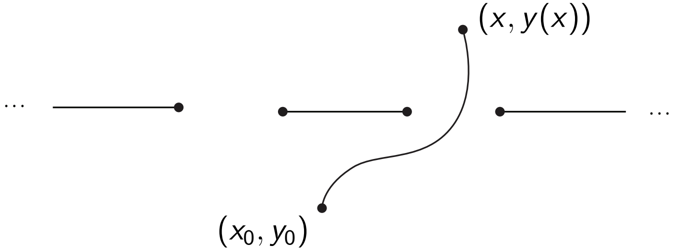
$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} \frac{dx}{\sqrt{1-x^2}} \text{ where } x(u) = \sin u, y(u) = \cos u$$

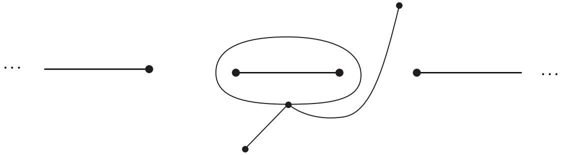
are periodic.

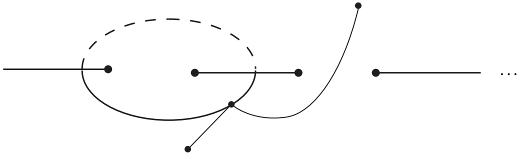
Turning now to the earlier integral where $p(x)$ is of degree four

$$\int \frac{dx}{\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}} = \int \omega.$$

we have the pictures







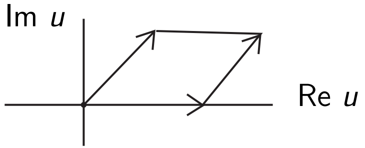
This gives that

$$\implies \int_{(x_0,y_0)}^{(x,y(x))} \in \mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2.$$

This tells us that for

$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} \omega$$

the functions $x(u), y(u)$ are *doubly* periodic functions in the u-plane

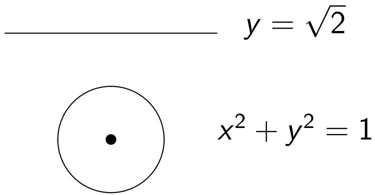


III. Abel meets Poncelet

The basic question is: *What does Poncelet's construction have to do with integrals of algebraic functions and Abel's theorem*? At this point we need to return to the use of complex numbers. Recall that conics are given by quadratic equations

$$ax^2 + 2bxy + cy^2 + ex + fy + 1 = 0.$$

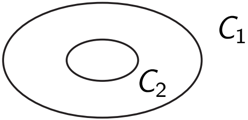
In order to have the right number of solutions to these equations we use complex numbers. Thus the real picture



has no solutions, but over the complexes it has the two solutions

$$\begin{cases} x = \pm\sqrt{-1} = \pm i \\ y = \sqrt{2}. \end{cases}$$

We can only draw the real pictures but we have in mind the whole set of complex solutions to the equations.
Another example is that in the real picture



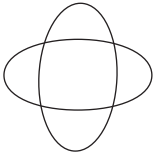
We do not see the complex points of intersection of the two conics. For the pair of ellipses

$$
\begin{aligned}
C_1 : & x^2 + \alpha y^2 = \beta \qquad \beta > 1 \text{ and } \beta/\alpha > 1, \\
C_2 : & x^2 + y^2 = 1
\end{aligned}
$$

the points of intersection are

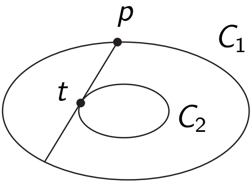
$$
\begin{cases}
x = \pm\sqrt{\frac{\alpha-\beta}{\alpha-1}} \\
y = \pm\sqrt{\frac{1-\beta}{1-\alpha}}.
\end{cases}
$$

Thus using complex numbers we could change coordinates to have for the real picture



We have seen that

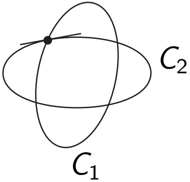
- the basic construction in Poncelet is



  where $(p, t) \in E \subset C_1 \times \check{C}_2$;
- using complex numbers we have $C_1 \cong \mathbb{P}^1$, $\check{C}_2 \cong \mathbb{P}^1$

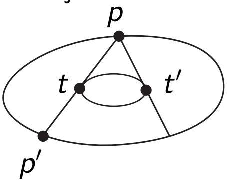
$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 & & \check{\mathbb{P}}^1 \end{tikzcd} \qquad \begin{tikzcd} & (p, t) \arrow[dl] \arrow[dr] & \\ p & & t \end{tikzcd}$$

- each of these is 2:1, and there are *four* branch points



$\implies E \cong \mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$ in two ways

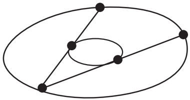
$$i_1(p,t) = (p',t)$$
$$i_2(p,t) = (p,t')$$



- in modern terms, E is realized (in two ways) as an algebraic curve given by an equation $y^2 = \prod_{i=1}^{4}(x - a_i)$ as above. Using the inversion of the elliptical integral given by Abel's theorem E is also realized as $\mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$.

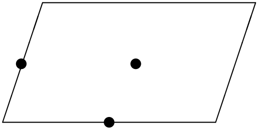
Identifying E with $\mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$ and using that any involution in $\mathbb{C}$ is $u \to -u + \alpha$ we have

$$\begin{cases} i_1(u) = -u + \alpha \\ u_2(u) = -u + \beta \end{cases}$$



$$\begin{aligned} &\implies \quad j(u) = u + (\beta - \alpha) \\ &\implies \quad j^n(u) = u + n(\beta - \alpha) \\ &\implies \quad j^n(u) = u \iff n(\beta - \alpha) \in \mathbb{Z}\pi_1 + \mathbb{Z}\pi_2 \end{aligned}$$

On $\mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$ the Poncelet construction is just adding a fixed vector! The condition for the billiard shot to be periodic is that vector be a division point

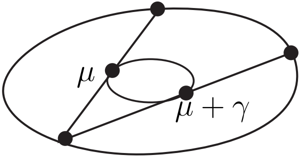


The three points of order 2 in $\mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$ are the dots.

Summary:

Using analysis and Abel's theorem the basic construction in Poncelet's theorem may be realized as represented by $\mathbb{C}/\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2$. When this is done the basic operation in (billiard shot) in Poncelet's theorem becomes translated into

$$u \to u + \gamma, \quad \gamma \in \mathbb{C}$$



The condition that the shot there have period n becomes

$$n\gamma \in \mathbb{Z}\pi_1 + \mathbb{Z}\pi_2,$$

which is *independent* of where we begin.