Moduli and Hodge Theory *

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*Talk at UIC (April 5, 2019), and based in part on joint work in progress with Mark Green, Radu Laza and Colleen Robles (GLR). Selected references to works quoted in or related to this talk are given at the end.

Outline

I. *Introduction*

Background and informal statement of the main point of the talk

II. *Hodge theory*

Basic definitions; limiting mixed Hodge structures

III. *Moduli and period mappings*

The canonical minimal completion of the image of a period mapping (work in progress)

IV. *Hodge theory and the moduli space of I-surfaces*

Illustration of how Hodge theory reflects the boundary structure of moduli of regular, general type surfaces X with $p_g(X) = 2$, $K_X^2 = 1$.[†]

[†]The algebro-geometric side is based in significant part on the work of Marco Franciosi, Rita Pardini and Sönke Rollenske (FPR); cf. the references at the end.

I. Introduction

- The construction and study of moduli spaces is of central interest in algebraic geometry.
- Algebraic varieties are built out of three basic types:
  - rationally connected ($\kappa(X) = -\infty$; for curves $g = 0$)
  - Calabi-Yau's, abelian varieties, $\ldots$ ($\kappa(X) = 0$; for curves $g = 1$)
  - general type ($\kappa(X) = \dim X$; for curves $g \geqq 2$).

  Here
  $$\dim H^0(mK_X) \sim m^{\kappa(X)} + \cdots$$
  measures the growth of the dimension of the space of global differential forms $f(z)(dz^1 \wedge \cdots \wedge dz^n)^m$.

- The moduli spaces (if they exist) behave quite differently in the three cases — for X of general type and with a fixed Hilbert polynomial $\overset{m}{\oplus}(\chi(mK_X)) \sim \overset{m}{\oplus} h^0(mK_X)$ Kollár-Shepherd-Barron- Alexeev (KSBA) proved the existence of $\mathcal{M}$ with a canonical projective completion $\overline{\mathcal{M}}$ — for surfaces we need only specify the irregularity $q(X) = h^{1,0}(X) = \left(\frac{1}{2}\right) b_1(X)$, the geometric genus $p_g(X) = h^{2,0}(X)$ and $K_X^2 = c_1^2(X)$. For $q(X) = 0$, $p_g(X) = 2$ and $K_X^2 = 1$ we have an *I-surface* with moduli space $\mathcal{M}_I$.[‡]

[‡]This is the first non-classical (term to be explained) general type surface.

- Examples are
  - the curves with affine equation

$$y^2 = \prod_{i=1}^{2g+2} (x - a_i), \qquad a_i \text{ distinct}$$

    with

$$H^0(K_X) = \left\{ p(x)\frac{dx}{y}, \quad \deg p \leqq q - 1 \right\};$$

    for $g = 2$ this is a general curve;
  - the surfaces with affine equation

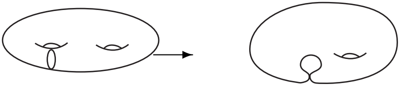
$$z^2 = f(x, y), \quad \deg f(x, y) = 2m$$

    where $f(x, y) = 0$ defines a smooth curve in $\mathbb{P}^2$ and

$$H^0(K_X) = \left\{ p(x, y)\frac{dx \wedge dy}{z}, \ \deg p \leqq m - 3 \right\};$$

    for $m \geqq 4$ this is a general curve in the moduli space.

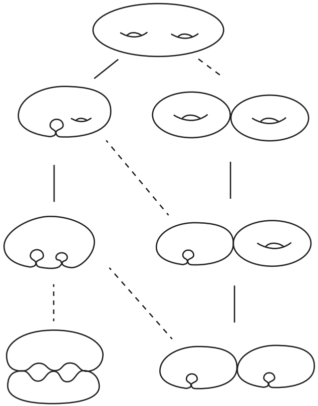
- For $\dim X = n = 1$ we have $\mathcal{M}_g$ with an essentially smooth $\overline{\mathcal{M}}_g$ having a canonical stratification of $\partial\mathcal{M}_g = \overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ — $\overline{\mathcal{M}}_g$ is much studied and very beautiful — the analysis of the classical period matrix of degenerating curves (let $a_1 \to a_2$ in the above example)



provided an early guide to understanding $\partial\mathcal{M}_g$.

- A picture of $\overline{\mathcal{M}}_2$ is

(∗)



This gives the stratification of $\overline{\mathcal{M}}_2$ together with the incidence (degeneration) relations among the strata. (The solid and dotted arrows will be explained later.)

- For $n \geqq 2$ aside from the aforementioned work of Franciosi, Pardini and Rollenske (FPR) I know of no example where a significant part of the structure of $\partial\mathcal{M}$ has been analyzed.[§]
- Associated to a stable curve X as in $(*)$ is a Hodge structure (period matrix) in the case when X is smooth and a limiting mixed Hodge structure (LMHS) in the general case. There is a stratification on the space of Gr(LMHS)'s, and this stratification determines the one pictured in $(*)$. The objective of our work is to be able to use Hodge theory in a similar way to study the moduli space $\mathcal{M}$ for at least some general type surfaces.

[§] Further work by FPR and discussions with them and GLR have led to an extension of some of this work to the next case of H-surfaces ($p_g = 2, q = 0$, and $K_X^2 = 2$).

- The space of (equivalence classes of) LMHS's of a given type may be described using Lie theory. What is needed is

  (i) to connect $\overline{\mathcal{M}}$ to this space via an extended period mapping

  (ii) to then apply this to some interesting examples to determine $\overline{\mathcal{M}}$.

  Following a discussion in Part II of some definitions and properties from Hodge theory, carrying out (i) will be explained in Part III of this talk, and in Part IV we will apply this to the I-surface described above. Informally stated the results will be

  - suitably interpreted the picture $(*)$ seems to carry over very closely as in the curve case
  - there is the added benefit that whereas $\overline{\mathcal{M}}_g$ is smooth, $\overline{\mathcal{M}}_I$ is highly singular along the boundary and the *proof* of (i) suggests how one might desingularize it.

II. Hodge theory

- Associated to an n-dimensional smooth projective variety X is the *Hodge structure* (HS) of *weight* m

$$H^m(X,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}, \quad \overline{H}^{p,q} = H^{q,p}$$

on its cohomology. Here

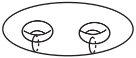
$$H^{m,0}(X) = H^0(\Omega_X^m).$$

Example: For $m = n = 1$ the HS is determined by the period matrix

$$\Omega = \left\| \int_{\gamma_i} \omega_\alpha \right\|$$

$\omega_\alpha \in H^0(\Omega_X^1) \quad (\dim = g)$

$\gamma_i \in H_1(X,\mathbb{Z}) \quad (\cong \mathbb{Z}^{2g})$



Example: For $m = n = 2$ it is determined by a similar period matrix where

$$\dim H^0(\Omega_X^2) = p_g(X), \quad H_2(X, \mathbb{Z})/\text{torsion} \cong \mathbb{Z}^{b_2(X)}.$$

- A *polarized Hodge structure* of weight m (PHS) is (V, Q, F)
  - $$\begin{cases} F^m \subset F^{m-1} \subset \cdots \subset F^0 = V_{\mathbb{C}} & \text{(Hodge filtration)} \\ F^p \oplus \overline{F}^{m-p+1} \xrightarrow{\sim} V_{\mathbb{C}}. \end{cases}$$

  For $V^{p,q} = F^p \cap \overline{F}^q$ the second condition is the same as

$$\begin{cases} V_{\mathbb{C}} = \oplus V^{p,q}, & \overline{V}^{p,q} = V^{q,p} \\ F^p = \bigoplus\limits_{p' \geqq p} V^{p',q} \end{cases}$$

- $Q : V \otimes V \to \mathbb{Q}$, $Q(u, v) = (-1)^m Q(v, u)$ satisfying

$$\begin{cases} Q(F^p, F^{m-p+1}) = 0 & \text{(HR I)} \\ i^{p-q} Q(V^{p,q}, \overline{V}^{p,q}) > 0 & \text{(HR II)} \end{cases}$$

- $H^m(X) = \oplus$ PHS's — in the examples above Q is the intersection form and (HR I) and (HR II) result from

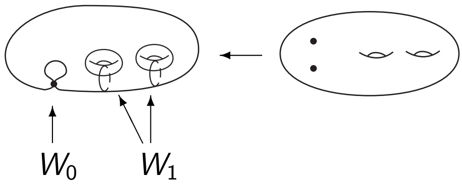
$$\begin{cases} \int_X \omega \wedge \omega' = 0 & \text{(because } \omega \wedge \omega' = 0\text{)} \\ c_n \int_X \omega \wedge \overline{\omega} > 0 & \text{(because } c_n\, \omega \wedge \overline{\omega} > 0\text{)} \end{cases}$$

  where $\omega, \omega' \in H^0(\Omega_X^n)$, $\dim X = n$, and c_n is a constant.
- *Mixed Hodge structure* (MHS) is $(V, W_\bullet, F^\bullet)$
  - $(0) \subset W_0 \subset \cdots \subset W_\ell = V$ (weight filtration)
  - $F^m \subset F^{m-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$ (Hodge filtration)

  where
  - $F^\bullet$ induces a HS of weight k on

$$\mathrm{Gr}_k^W V = W_k / W_{k-1}.$$

Example: $H^m(X)$ where X = complete algebraic variety and the weight filtration on $H^m(X)$ is $W_0 \subset \cdots \subset W_m$.



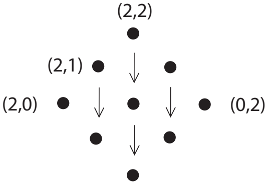
- *Limiting mixed Hodge structure* (LMHS)
  - $N : V \to V$ with $N^{m+1} = 0$

$$W_0(N) \subset \cdots \subset W_{2m}(N)$$

monodromy weight filtration characterized by

$$N : \quad W_\ell(N) \to W_{\ell-2}(N) \text{ and}$$
$$N_k \quad : \mathrm{Gr}^{W(N)}_{m+k} \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{m-k} .$$

- $\begin{cases} (V, W(N), F^{\bullet}_{\lim}) \text{ is a MHS with} \\ N : F^{p}_{\lim} \to F^{p-1}_{\lim}. \end{cases}$
- Gr(LMHS)$\cong \bigoplus_{\ell=0}^{2m} H^{\ell}$ where H^{ℓ} is a HS of weight ℓ — picture is a Hodge diamond. Here $m = 2$ and N is the vertical arrows — the dots are the $H^{p,q}$'s



- We will set $h^{p,q}$ = dimension of the (p, q) dot.
- There will also be a Q in the picture.

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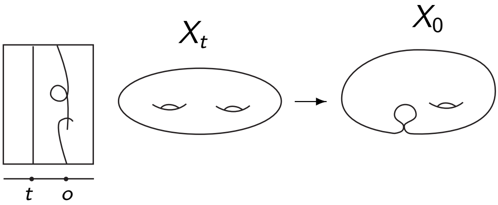
Example:

$$\begin{tikzcd} \mathcal{X}^* \arrow[d] & \subset & \mathcal{X} \arrow[d] \\ \Delta^* & \subset & \Delta \end{tikzcd}$$



- monodromy $T : H^m(X_t) \to H^m(X_t)$

$$\begin{cases} T = T_{ss} T_u \quad \text{(Jordan decomposition)} \\ T_{ss}^k = I,\, T_u = e^N \ \text{ with } \ N^{m+1} = 0 \end{cases}$$

  thus (i) eigenvalues are roots of unity, and (ii) length of Jordan blocks is $\leqq m$.

- the solid lines in the diagram in the introduction represent degenerations with $N \neq 0$.

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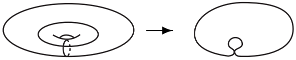
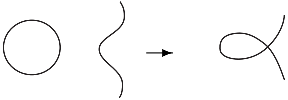
Theorem (Schmid)[¶] *Given $\mathcal{X} \to \Delta$ as above*

$$\lim_{t \to 0} H^m(X_t) = \mathrm{LMHS}.$$

Proof is a combination of

- Lie theory
- complex analysis
- differential geometry

Example:

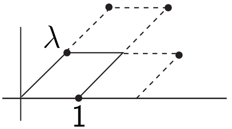
-  topological picture
- 

$$y^2 = x(x-1)(x-t)$$

algebraic picture

[¶]Cf. Cattani-Kaplan-Schmid in the references. The algebraic approach is in C. Peters and J. Steenbrink.

- $X = \mathbb{C}/\Lambda$, $\Lambda = \{1, \lambda\}$

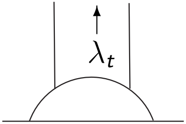


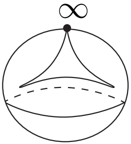
analytic picture

λ determined up to $\lambda \to \frac{a\lambda+b}{c\lambda+d}$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

- $\mathcal{M}_1 \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}$, $\mathcal{H} = \{\lambda : \operatorname{Im}\lambda > 0\}$
- in above example space of PHS's is $\mathcal{H} \subset \mathbb{P}^1$, $V = \left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right)$, $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $F^1 = \left[\begin{smallmatrix} \lambda \\ 1 \end{smallmatrix}\right] \in \mathbb{P}^1$, HR II $\iff \operatorname{Im}\lambda > 0$. As $\lambda \to i\infty$ we have $F^1 \to \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = F^1_{\mathrm{lim}}$.

$$\lambda_t = \frac{\log t}{2\pi i}$$





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How does Lie theory enter?

- *Period domain* $D = \{F^\bullet = \text{flag } \{F^m \subset \cdots \subset F^0 = V_{\mathbb{C}}\}$ in $V_{\mathbb{C}} : (V, Q, F^\bullet) = \text{PHS}\}$
- *compact dual* $\check{D} = \{F^\bullet \text{ is a flag with } Q(F^p, F^{m-p+1}) = 0\}$
- $G = \mathrm{Aut}(V, Q) = \mathbb{Q}$-algebraic group
- $G_{\mathbb{R}}$ acts transitively on D so that

$$\begin{array}{l} D = G_{\mathbb{R}}/H \text{ with } H \text{ compact} \\ \cap \\ \check{D} = G_{\mathbb{C}}/P \text{ with } P \text{ parabolic } \quad \begin{pmatrix} * & * & * & * \\ \circ & * & * & * \\ \circ & \circ & * & * \\ \circ & \circ & \circ & * \end{pmatrix} \end{array}$$

Then $D =$ open $G_{\mathbb{R}}$-orbit in $\check{D}$.

Example:

$m=1$: $D=\mathrm{Sp}(2g,\mathbb{R})/\mathcal{U}(g)=\mathcal{H}_g$ where $g=h^{1,0}$
$m=2$: $D=\mathrm{SO}(2k,\ell)/\mathcal{U}(k)\times\mathrm{SO}(\ell)$ where $k=h^{2,0}$, $\ell=h^{1,1}$

- *Classical case:*

$$\begin{array}{l} D = \text{ Hermitian symmetric domain (HSD)} \\ \| \\ G_R/K, \ K = \text{ maximal compact.} \end{array}$$

Two classical cases are

$m = 1$ (curves, abelian varieties)

$m = 2$ is HSD $\iff$ $k = 1$ (K3's)

thus $h^{2,0} \geqq 2$ is non-classical.
For $n \geqq 3$ and X Calabi-Yau, the D corresponding to $H^n(X)$ is non-classical.

Example: $m = 2$, $D = \mathrm{SO}(2k,\ell)/\mathcal{U}(k) \times \mathrm{SO}(\ell)$ — first non-classical case is when $k = 2$

- $\dim D = 2\ell + 1$
- D has a $G_{\mathbb{R}}$-invariant contact structure

I-surface has $\ell = 28$, $\dim \mathcal{M}_I = 28$ and the image of the period mapping (next topic) is a contact submanifold.

- Period domains have sub-domains corresponding to PHS's with additional structure; e.g.,

$$\begin{array}{ccc} D' & \subset & D \\ \| & & \\ \left\{ \begin{array}{c} \text{reducible PHS's} \\ \text{that are } \oplus\text{'s} \end{array} \right\} & & \end{array}$$

  This is what the dotted lines represent in the diagram in the introduction for $\overline{\mathcal{M}}_2$.

- In general one has Mumford-Tate sub-domains of D, defined to be those PHS's with a given algebra of Hodge tensors.

- *period mappings* arise from holomorphic mappings

$$\Phi : B \to \left\{\begin{matrix}\text{equivalence}\\ \text{classes of}\\ \text{PHS's}\end{matrix}\right\} = \Gamma\backslash D$$

where B is a complex manifold and $\Gamma \subset G_{\mathbb{Z}}$ contains the monodromy group; think of B as the parameter space for a family of smooth algebraic varieties X_b, $b \in B$, whose cohomology groups can be identified with $H^n(X_{b_0})$ for a base point $b_0 \in B$ up to the action of $\pi_1(B, b_0)$ on $H^n(X_{b_0})$.

III. Moduli and period mappings

- Variety Y has *canonical singularities* if for any desingularization $\widetilde{Y} \xrightarrow{f} Y$ we have
$$f^* K_Y = K_{\widetilde{Y}}.$$
Equivalently, if Y is normal, then for $Y^* = Y \backslash Y_{\mathrm{sing}}$ any $\omega \in H^0(K_{Y^*})$ has
$$\int_{Y^*} \omega \wedge \overline{\omega} < \infty.$$
- $\mathcal{M}$ = KSBA moduli space for varieties that are smooth or have canonical singularities.

Question: *What singular varieties X do we add to obtain $\overline{\mathcal{M}}$?*

- Use valuative criterion: Given $\mathcal{X}^* \to \Delta^*$ what X do we use to *uniquely* fill in over the origin to have

$$\begin{tikzcd}
\mathcal{X}^* \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{X} \arrow[d] \\
\Delta^* \arrow[r, phantom, "\subset"] & \Delta
\end{tikzcd}$$

- **Answer (KSBA):** There are two equivalent criterion:
  - X should
    - (a) have semi-log-canonical (slc) singularities (local)
    - (b) K_X should be ample (global)
  - $\mathcal{X}$ should
    - (a′) have canonical singularities (local)
    - (b′) $\omega_{\mathcal{X}/\Delta}$ should be relatively ample (global)

For curves

$$\begin{cases} (\mathrm{a}) = (\mathrm{a}') \iff X \text{ is nodal} \\ (\mathrm{a}) + (\mathrm{b}) = (\mathrm{a}') + (\mathrm{b}') \iff X \text{ is stable.} \end{cases}$$

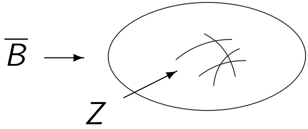
For surfaces there is a list of slc singularities

- normal singularities (the Gorenstein ones are simple elliptic and cusps)
- non-normal (double curve with pinch points and nodes satisfying conditions with respect to the involution)
- they are defined in terms of the $\mathbb{Q}$-Cartier divisor in

$$f^* K_X = K_Y + \sum_i a_i D_i, \qquad a_i \geqq -1$$

(pullback of sections of $K_{X_{\mathrm{reg}}}$ have at most log poles on the exceptional divisor).

- Let B = smooth quasi-projective variety with a smooth, projective completion $\overline{B}$ with $B = \overline{B} \backslash Z$ where $Z = \bigcup Z_i$ is a reduced normal crossing divisor



- Suppose we have a *period mapping* given by

$$\Phi : B \to \Gamma \backslash D, \qquad \Gamma \subset \mathrm{Aut}(V_{\mathbb{Z}}, Q).$$

As noted above the local monodromies around Z_i are quasi-unipotent.

- **Example:** $\mathcal{X} \xrightarrow{\pi} B$ projective family with $\pi^{-1}(b) = X_b$ smooth gives a period mapping where
  - $\Phi(b) =$ PHS on $H^m(X_b)$
  - $\Phi_* : \pi_1(B) \to \Gamma \subset \mathrm{Aut}(X_b)$ is global monodromy.
- *Hodge line bundle* $\Lambda = \det \mathbb{F}^n$ where $\mathbb{F}^n_b = H^0(K_{X_b})$.

  Example: For $\mathcal{X} \xrightarrow{f} B$

$$\Lambda = \det(f_* \omega_{\mathcal{X}/B})$$

- may extend Φ across Z_i where $N_i = 0$ and

$$\Phi : B \to \mathcal{H} \subset \Gamma \backslash D \ \text{ is a proper, holomorphic mapping.}$$

Theorem A1:* *There exists a canonical minimal completion $\overline{\mathcal{H}}$ of $\mathcal{H}$ to which the augmented Hodge line bundle extends as an ample line bundle $\Lambda_e \to \overline{\mathcal{H}}$.*† *Moreover there is an extension of the period mapping to*

$$\Phi_e : \overline{B} \to \overline{\mathcal{H}}.$$

- *What is the boundary $\partial\mathcal{H} = \overline{\mathcal{H}} \backslash \mathcal{H}$?* For $b_0 \in Z$

$$\Phi_e(b_0) = \mathrm{Gr}\left\{ \lim_{b \to b_0} H^m(X_b) \right\}.$$

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*The complete proof of this result is work in progress. The arguments involve geometric constructions arising from the extension data in $\underline{\text{L}}$MHS's (not present for the extension data for just MHS's) and the algebraic geometry of the fibres of Φ_e.

†The augmented Hodge line bundle is $\bigotimes_{p=0}^{[m+1/2]} \det \mathbb{F}^p$. We shall mainly be concerned with the cases $m = 1, 2$.

Example:

$$X_b = \text{ curve } \implies \Phi_e(b_0) = \left\{H^0(X_{b_0}), H^1(\widetilde{X}_{b_0})\right\}$$

where $\widetilde{X}_{b_0}$ = normalization of X_{b_0}.

Example: I-surface example to be discussed below.

Definition: $\overline{\mathcal{H}} = \operatorname{Proj} \Lambda_e$
(depends only on $\mathcal{H}$ and not on $B, \overline{B}$)

Definition: $\overline{\mathcal{H}}$ is the *Satake-Baily-Borel* (SBB) completion of $\mathcal{H}$.

We note that this is a *relative* construction; it depends on $\Phi : B \to \Gamma\backslash D$, in contrast to the classical case where there is a $\overline{\Gamma\backslash D}^{\mathrm{SBB}}$ where Φ extends to $\Phi_e : \overline{B} \to \overline{\Gamma\backslash D}^{\mathrm{SBB}}$ and $\overline{\mathcal{H}}$ is the image.

In the geometric case this implies that $\det(f_* \omega_{\overline{\mathcal{X}}/\overline{B}})$ is free, for which I don't know of an algebraic proof.

- $\mathcal{M}$ = KSBA moduli space, $\overline{B} = \overline{\mathcal{M}}$ is a desingularization.

Theorem A2:[‡] *There is a factorization*

$$\begin{tikzcd} B \arrow[d] & \subset & \overline{B} \arrow[d] \arrow[dr, "\Phi_e"] & \\ \mathcal{M} & \subset & \overline{\mathcal{M}} \arrow[r, dashed] & \overline{\mathcal{H}} \end{tikzcd}$$

Briefly this says

- the period mapping $\mathcal{M} \to \mathcal{H} \subset \Gamma\backslash D$ extends to $\Phi_e : \overline{\mathcal{M}} \to \overline{\mathcal{H}}$; i.e., to a surface corresponding to a boundary point of $\mathcal{M}$ we can *uniquely* associate the associated graded to the LMHS;
- the extended Hodge line bundle on $\overline{B}$ descends to $\overline{\mathcal{M}}$ and there it is free.

[‡]The detailed statement and proof of this result also are a work in progress.

IV. Use of Hodge theory to analyze the moduli space of I-surfaces

A. I-surfaces and their period mappings

- Murphy's law (Vakil) — whatever nasty property a scheme can have already occurs for the moduli spaces of general type surfaces — thus unlike curves one should select "special" surfaces to study — in geometry extremal cases are frequently interesting — Noether's inequality

$$p_g(X) \leqq \frac{K_X^2}{2} + \frac{3}{2}$$

suggests studying surfaces close to extremal — the 1^{st} non-classical case is

Definition: An *I-surface* X is a regular $(q(X) = 0)$ general type surface that satisfies

$$p_g(X) = 2, K_X^2 = 1.$$

- One studies general type surfaces via their pluri-canonical maps

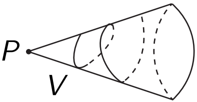
$$(***) \qquad \varphi_{mK_X} : X \dashrightarrow \mathbb{P}H^0(mK_X)^* \cong \mathbb{P}^{P_m-1}$$

  and pluricanonical rings $R(X) = \oplus H^0(mK_X)$.
- Instead of $(***)$ frequently better to use weighted projective spaces corresponding to when we add new generators to $R(X)$ — from

$$P_m(X) = m(m-1)/2 + 3, \qquad m \geqq 2$$

  and Kodaira-Kawamata-Viehweg vanishing one has for the I-surface

$$\begin{aligned} \varphi_{K_X} &: X \dashrightarrow \mathbb{P}^1, \quad |K_X| = \text{ pencil of hyperelliptic curves} \\ \varphi_{2K_X} &: X \to \mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^3 \\ \varphi_{5K_X} &: X \hookrightarrow \mathbb{P}(1,1,2,5) \hookrightarrow \mathbb{P}^{12}. \end{aligned}$$

- Picture/equations
  - 

    $\mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^3$given by
    $(t_0, t_1) \hookrightarrow [t_0^2, t_0 t_1, t_1^2, y]$

    $X =$ 2:1 map branched over P and $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$
  - $z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y)$ (weighted complete intersection) in $\mathbb{P}(1,1,2,5)$
- $\mathcal{M}_I$ is smooth and
  - $\dim \mathcal{M}_I = h^1(T_X) = 28$
  - $\dim D_I = 57 = 2 \dim \mathcal{M}_X + 1$
- $\Phi = \mathcal{M}_I \to \Gamma_I \backslash D_I$ has Φ_* injective (local Torelli)

$$\Downarrow$$

$$\Phi(\mathcal{M}_I) = \text{contact submanifold } \mathcal{H} \hookrightarrow \Gamma_I \backslash D_I$$

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- Γ_I is arithmetic — not known is whether $\Gamma = G_{\mathbb{Z}}$.

B. Stratification of the space of Gr(LMHS)'s

- For curves with $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ we have for LMHS's

$$\begin{tikzcd} {} \arrow[r, dash] & I_0 \arrow[r, dash] \arrow[d, leftrightarrow] & I_1 \arrow[r, dash, dashed] \arrow[d, leftrightarrow] & I_2 \arrow[r, dash] \arrow[d, leftrightarrow] & \cdots \arrow[r, dash] & I_g \arrow[d, leftrightarrow] \\ & \mathcal{H}_g & \mathcal{H}_{g-1} & \mathcal{H}_{g-2} & & \mathcal{H}_0 \end{tikzcd}$$

  - here I_{g-m} corresponds to $[N]$ with $N^2 = 0$, rank $N = m$.

$$\begin{tikzcd} & m & & \\ & \bullet \arrow[dd] & & \mathrm{Gr}_2 \\ g - m \quad \bullet & & \bullet & \mathrm{Gr}_1 \quad (\cong H^1(\widetilde{C})) \\ & \bullet & & \mathrm{Gr}_0 \end{tikzcd}$$

For each boundary component we have the stratification

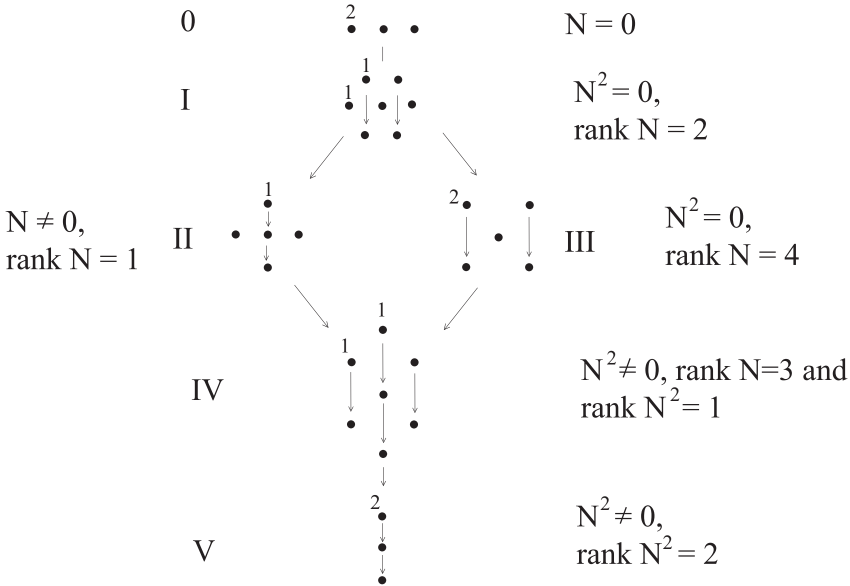
$$H^1 = \oplus H_i^1.$$

The composite of these induces a stratification of $\overline{\mathcal{M}}_g$ by

$$\{\# \ \text{nodes}, \ \ \# \ \text{components}\}.$$

Of course this is just the beginning of the story of $\overline{\mathcal{M}}_g$.

- For surfaces with $p_g = 2$ the classification of Gr(LMHS)'s/$\mathbb{Q}$ is


0
$N = 0$
I
$N^2 = 0$, rank $N = 2$
II
$N \neq 0$, rank $N = 1$
III
$N^2 = 0$, rank $N = 4$
IV
$N^2 \neq 0$, rank $N=3$ and rank $N^2 = 1$
V
$N^2 \neq 0$, rank $N^2 = 2$

- $\begin{cases} \text{For the refined Hodge-theoretic stratification of} \\ \mathrm{Gr}(\mathrm{LMHS}/\mathbb{Z})\text{'s we use } T_{ss} \to \{\text{conjugacy class } [T_{ss}] \\ \text{of } T_s \text{ in } \Gamma\}. \text{ Within each of these strata we use} \\ \text{Mumford-Tate sub-domains appearing} \\ \text{in } \mathrm{Gr}(\mathrm{LMHS})\text{'s in } \overline{\mathcal{M}}_I. \end{cases}$
- We begin by considering the Gorenstein part $\overline{\mathcal{M}}_I^{\mathrm{Gor}} \subset \overline{\mathcal{M}}_I$ — one reason for this is that

  if $X_t \to X$ is a KSBA degeneration of a surface where all the singularities of X are non-Gorenstein, then $N = 0$.

  Hence only Gorenstein singularities can non-trivially contribute to the LMHS/$\mathbb{Q}$.

The following results from coupling the classification in FPR with the analysis of the LMHS's in the various cases.

Theorem B

The Hodge theoretic stratification of $\overline{\mathcal{M}}$ *uniquely determines the stratification of* $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$.

- Rather than display the whole table the following is just the part for simple elliptic singularities (types I_k and III_k) — they have $N^2 = 0$ since for the semi-stable-reduction (SSR) of a degeneration only double curves (and no triple points) occur — all of the other types occur if we include cusp singularities.

- In the following
  - X is irreducible (since K_X is a line bundle with $K_X^2 = 1$ and any component of X will have positive K^2)
  - $d_i =$ degree of elliptic singularity
  - $k = \#$ elliptic singularities — in general, using Hodge theory one may show in general that $k \leqq p_g + 1$
  - $\widetilde{X} =$ minimal desingularization of X — in a SSR given by $\widetilde{\mathfrak{X}} \to \Delta$ the surface $\widetilde{X}$ will appear as one component of the fibre over the origin.

In the following table, in the 1^{st} column subscripts denote the degrees of the elliptic singularities, which are uniquely determined by the $[T_{ss}]$'s — will explain the $\sum(9 - d_i)$ column.

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{III}_{2,2}$	12	rational surface	14	2	16
$\mathrm{III}_{1,2}$	11	rational surface	15	2	17
$\mathrm{III}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

Note that the last column is the sum of the two columns preceding it.

Example: For I_2 the picture is

$$\begin{tikzcd}[column sep=small] & (\widetilde{X}, \widetilde{C}) \arrow[dl] \arrow[dr] & \\ (X_{\mathrm{min}}, C) & & (X, p) \end{tikzcd}$$

Here, $p =$ isolated normal singular point on X, $\widetilde{C} =$ curve on $\widetilde{X}$ that contracts to p — from Hodge theory

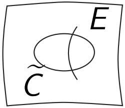
$$2 = p_g(\widetilde{X}) + g(\widetilde{C}) \text{ and } g(\widetilde{C}) = 1, p_g(\widetilde{X}) = 1$$

- $\mathrm{Gr}(\mathrm{LMHS})/\mathbb{Z}$ suggests that $\mathrm{Hg}^1(\widetilde{X})$ has a $\mathbb{Z}^2$ with intersection form

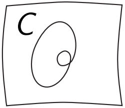
$$\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$

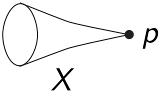
for heuristic reasoning assume basis classes are effective.

- Hodge theory now suggests the picture



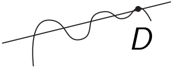
$\widetilde{X} \quad \widetilde{C}^2 = -2,\ E^2 = -1$





$$\begin{cases} X_{\text{min}} = K3 \\ C^2 = 2 \end{cases}$$

$$\implies X_{\text{min}} \xrightarrow{2:1} \mathbb{P}^2 \text{ branched over } D$$



- Reasoning is: Of the two holomorphic 2-forms on X_t, in the limit one acquires a simple pole on $\widetilde{C}$ and the other one ω remains holomorphic. From $K_X^2 = 1$ we infer that $\widetilde{\omega}$ on $\widetilde{X}$ has divisor E so that on X_{min} we have $X_{\mathrm{min}} \cong \mathcal{O}_{X_{\mathrm{min}}}$. Thus

- LMHS has
  $\mathrm{Gr}_2 \cong H^2(X_{\mathsf{min}})_{\mathrm{prim}}$
  $\mathrm{Gr}_3 \cong H^1(\widetilde{C})(-1)$

- $\#$ of PHS's of type $\mathrm{Gr}_3 \oplus \mathrm{Gr}_2 = 1 + 19 = 20$ which suggests
  - codim $= 8$

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- How to get this number? First approximation to the fibre over the origin in a SSR is blowing up p in $\mathcal{X}$ to have

$$\widetilde{X} \cup_{\widetilde{C}} \mathbb{P}^2$$

where $\widetilde{C} \in |\mathcal{O}_{\mathbb{P}^2}(3)|$

- Now have to blow up $9 - (-\widetilde{C}^2) = 7$ points on $\widetilde{C}$ to obtain triviality of the infinitesimal normal bundle as a necessary condition for smoothability. Thus

Fibre over origin in Δ^2 is given by blowing up seven points on $\widetilde{C}$ — this is a del Pezzo.

- Hodge theory suggests where to look — the eight parameters arise from the possible extension data for GR(LMHS) — and following FPR one may now go back and prove things algebraically.
- Finally, *what about the non-Gorenstein singularities?* This has yet to be worked out in detail. Heuristically one might hope to proceed as follows: From the list of normal slc singularities of surfaces these mostly are quotient singularities. For those for which the local monodromy is a non-trivial quotient of the finite group that gives the singularity, one might hope they are detected Hodge-theoretically.

One example of this is the *Wahl singularity* $\frac{1}{4}(1,1)$. It is the quotient of $\mathbb{C}^2$ by $(x,y) \to (\zeta x, \zeta y)$, $\zeta = e^{2\pi i/4}$. If

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \Delta \arrow[r, "2:1"] & \Delta \end{tikzcd}$$

is the minimal SSR, then for I-surfaces it turns out to be the case that

- the period mapping gives $\Phi : \Delta \to D$;
- the point $\Phi(o) \in D$ is a PHS with an extra Hodge class arising from $\mathrm{Hg}^1(\widetilde{X})$, where $\widetilde{X} \to X$ is the minimal desingularization of X.

- The locus of points in $\overline{\mathcal{M}}_I$ where there is a Hodge class of this type is of codimension 1, and these classes contain $\mathbb{P}^1$'s that contract to give Wahl singularities (some details are still being worked out here).

Conclusion

The SBB completion $\overline{\mathcal{H}}$ of the image of moduli under the period mapping gives an invariant that has a rich structure and that provides an important guide to the boundary structure of the moduli space. The fibres of $\widetilde{\overline{\mathcal{M}}} \to \overline{\mathcal{H}}$ are suggested by the extension data associated to a LMHS with fixed associated gradeds.

Thank you

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