

### III. Topology and Hodge theory

- ▶ These two topics are closely intertwined and constitute a major aspect of complex algebraic geometry, beginning in the later part of the 19<sup>th</sup> century (Picard, Poincaré, . . . ) into the 1<sup>st</sup> half of the 20<sup>th</sup> century (Lefschetz, Hodge, . . . ) and continuing through today
- ▶ In fact questions about integrals on algebraic surfaces (which are real 4-manifolds) were instrumental in the beginnings of topology — one knew (Darboux, Picard, Poincaré, E. Cartan, . . . ) what differential forms

$$\varphi = a dx + b dy + c dz$$

$$\psi = A dx \wedge dy + B dx \wedge dz + C dy \wedge dz$$

$$\eta = D dx \wedge dy \wedge dz$$

were, and Stokes' theorem

$$\int_U d\omega = \int_{\partial U} \omega$$

shows then when  $d\omega = 0$  that  $\int_{\Gamma} \omega$  was not only invariant under deformation (homotopy) of  $\Gamma$  but also under homology.<sup>1</sup> This led to the notion of *periods*

$$\int_{\Gamma} \omega, \quad d\omega = 0 \text{ and } \Gamma \in H_p(X, \mathbb{Z}).$$

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<sup>1</sup>The exterior derivative  $d$  is uniquely determined (i)  $df = f_x dx + f_y dy + f_z dz$  for a function  $f$ , (ii)  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$  and (iii)  $dx \wedge dy = -dy \wedge dx$  etc.

In the complex case when  $X$  has local holomorphic coordinates  $z = (z_1, \dots, z_n)$

$$\omega = \sum_{I, J} f_{I\bar{J}} dz^I \wedge d\bar{z}^J$$

where  $I = (i_1, \dots, i_p)$ ,  $dz^I = dz^{i_1} \wedge \dots \wedge dz^{i_p}$  etc. and as we saw for algebraic curves the periods reflect the complex structure — this is the start of Hodge theory.

# Outline for the remainder of this lecture

- ▶ Introductory discussion of what an algebraic variety is
- ▶ Statements of the Lefschetz theorems
- ▶ How they arose historically from the study of algebraic functions of two variables (Picard-Lefschetz or PL theory)
- ▶ Origin of the Hodge conjecture (HC)



- ▶ *Complex projective space*  $\mathbb{P}^N$ 
  - ▶ lines through origin in  $\mathbb{C}^{N+1}$
  - ▶  $\mathbb{P}^N = \mathbb{C}^N \cup \mathbb{P}^{N-1}$  ( $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ )
  - ▶ homogeneous coordinates  $[z] = [z_0, \dots, z_N]$
- ▶  $\mathbb{P}^1 =$  Riemann sphere
- ▶  $\mathbb{P}^2 = \mathbb{C}^2 \cup \{\text{lines through the origin}\}$  where  $[z] \leftrightarrow$  line with slope  $z_2/z_1$
- ▶  $\mathbb{P}^N =$  compact complex manifold

## Proof

$$\mathcal{U}_i = \{[z] : z_i \neq 0\} \ni [z]$$

$$\downarrow$$
$$\downarrow$$

$$\mathbb{C}^N \ni (z_0/z_i, \dots, \overset{i}{\wedge} \dots, z_N/z_i)$$

- ▶ Algebraic variety  $X \subset \mathbb{P}^N$  given by  $F_1(z) = \dots = F_m(z) = 0$  where  $F_\alpha(z) =$  homogeneous polynomial.
- ▶ Note that  $\dim_{\mathbb{R}} X = 2 \dim_{\mathbb{C}} X$  and  $X$  is oriented.

## Example

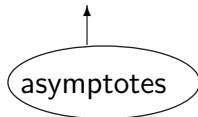
$C$  defined by  $f(x, y) = 0$  in  $\mathbb{C}^2$ . Set

$$x = z_1/z_0, \quad y = z_2/z_0$$

and clear denominators to get

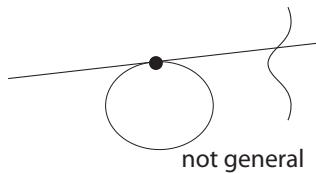
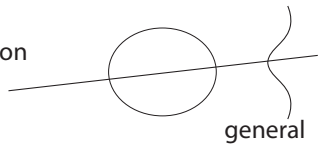
$$\overline{C} = \{F(z) = 0\} \subset \mathbb{P}^2$$

where  $\bar{C} = \left\{ \begin{array}{l} \text{our old} \\ C \subset \mathbb{C}^2 \end{array} \right\} \cup \left\{ \begin{array}{l} \text{points} \\ \text{at } \infty \end{array} \right\} .$

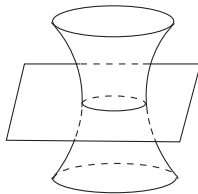


- ▶ suppose  $X^n =$  smooth algebraic variety and  $Y = \mathbb{P}^{N-1} \cap X$  is a general hyperplane section

hyperplane section



quadric surface;  
real picture



**Note:** Equation of the quadric in  $\mathbb{C}^3$  is  $x^2 + y^2 = z^2 + 1$ ;  
equation in  $\mathbb{P}^3$  is  $z_1^2 + z_2^2 = z_3^2 + z_0^2$ ; over  $\mathbb{C}$  this is equivalent  
to  $z_1' z_2' = z_3' z_0'$  where  $z_1' = z_1 + iz_2$ ,  $z_2' = z_1 - iz_2$  etc.



## Lefschetz theorem I

- ▶  $b_{2p+1}(X) \equiv 0 \pmod{2}$  (odd Betti numbers are even)
- ▶  $b_{2p}(X) \geq 1$  (even Betti numbers are positive).

In the second, if  $\dim_{\mathbb{C}} X = n$  and  $H \in H_{2n-2}(Y, \mathbb{Z})$  is the class of the cycle given by  $Y$  then (non-trivially)

$$\underbrace{H \cap \cdots \cap H}_{n-p} \neq 0 \text{ in } H_{2p}(X, \mathbb{Z})$$

## Lefschetz theorem II

$$H_p(Y, \mathbb{Z}) \rightarrow H_p(X, \mathbb{Z}) \text{ is } \begin{cases} \text{isomorphism for} \\ p \leq n-2 \end{cases}$$

onto for  $p = n-1$

## Corollary

$Y$  is connected if  $\dim_{\mathbb{C}} X \geq 2$

**Exercise:**  $f(x, y) =$  irreducible polynomial and  $\{f(x, y) = 0\} = C \subset \mathbb{C}^2$ . Show that  $C$  is connected.

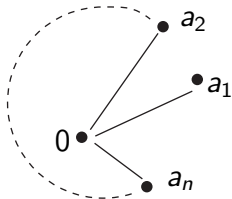


*Geometric idea to study topology of an algebraic variety (idea is one of the most basic in algebraic geometry) — use induction by dimension.*

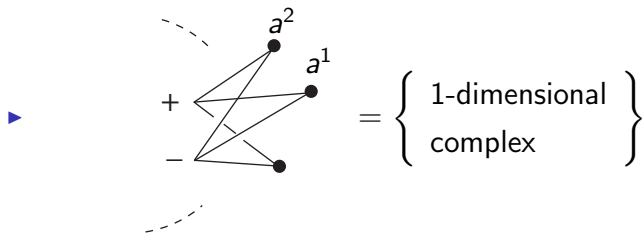
### Example


For  $y^2 = p(x)$  where  $p(x) = \prod_{i=1}^{2g+2} (x - a_i)$

- ▶ first take out the two points over  $x = \infty$
- ▶ next use the picture of the complex  $x$ -plane

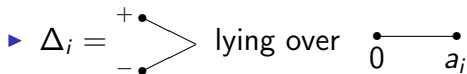


- ▶ retract the slit  $x$ -plane and the part of  $C$  lying over it onto the part lying over the segments



- ▶ on  as we turn around the branch point the two points interchange (local *monodromy*  $T_i$  around  $a_i$ )

▶  $\prod_i T_i = \text{Id}$



▶  $C$  retracts onto the real 1-dimensional complex given by attaching the  $2g + 2$  1-cells  $\Delta_i$  to the two points  $\pm$  lying over 0.

▶  $\Delta_i$  generate the relative homology group

$$H_1(C, \{+, -\}; \mathbb{Z})$$

$$\rightsquigarrow H_1(\overline{C}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

This case is too simple to suggest the general pattern. The next dimension up is due to Picard (1880–2000)

## Example

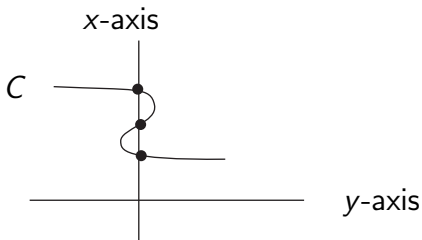
$X$  is the algebraic surface

$$z^2 = f(x, y)$$

where  $C = \{f(x, y) = 0\}$  is a non-singular plane curve. For a general  $y$  we let

$$X_y = \text{curve } z^2 = f(x, y), \quad y \text{ fixed}$$

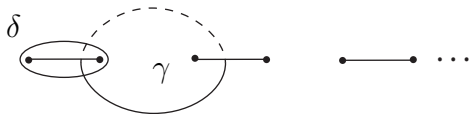
The picture is



$X_y$  is the algebraic curve of the type we have been considering; it is 2:1 covering of the line  $y = \text{constant}$  branched at the points of  $C \cap \{y = \text{constant}\}$

- ▶ smooth for general  $y$
- ▶ singular when the line  $y = \text{constant}$  becomes tangent to  $C$

- ▶ the picture of  $X_y$  is

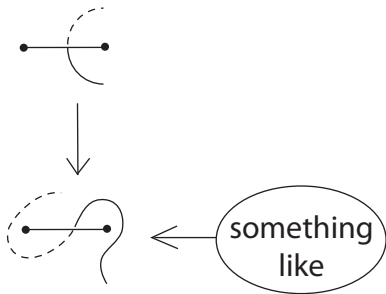


where the branch points and slits will vary with  $y$

- ▶ at a point of tangency two branch points come together and interchange.

▶  $\delta \rightarrow \delta$

▶  $\gamma \rightarrow ?$



## Picard-Lefschetz formula

(PL)

$$\gamma \rightarrow \gamma + \delta$$

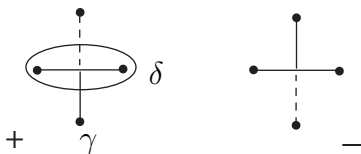
How to show PL? The original argument was analytic and in outline went as follows:

- ▶ locally analytically change coordinates so that the picture is a neighborhood of the origin of the curves

$$C_t = \{u^2 + v^2 = t\}$$

in  $\mathbb{C}^3$  with coordinates  $(u, v, t)$

- ▶ the local picture is





- ▶ set  $t = \sigma^2$  and consider the integrals

$$I_t(\delta) = \int_{\delta} \frac{du}{\sqrt{t - u^2}} = \int_{\delta} \frac{du}{\sqrt{\sigma^2 - u^2}} = \int_{\delta} \frac{du}{\sigma \sqrt{1 - (u/\sigma)^2}}$$

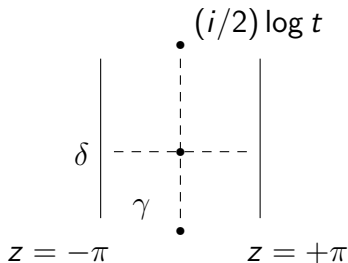
$$I_t(\gamma) = \int_{\gamma} \frac{du}{\sqrt{t - u^2}} = \int_{\gamma} \frac{du}{\sqrt{\sigma^2 - u^2}} = \int_{\gamma} \frac{du}{\sigma \sqrt{1 - (u/\sigma)^2}}$$

- ▶ the curves  $C_t$  are parametrized by

$$z \rightarrow (\sigma \sin z, \sigma \cos z),$$

and a calculation gives

$$\begin{cases} I_t(\delta) = 2\pi \\ I_t(\gamma) = i \log t \end{cases}$$



## Conclusion

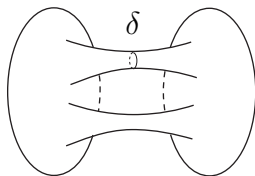
$$\begin{cases} I_{e^{2\pi i t}}(\delta) = I_t(\delta) \\ I_{e^{2\pi i t}}(\gamma) = I_t(\gamma) + I_t(\delta) \end{cases}$$

$$\implies T(\gamma) = \gamma + \delta.$$

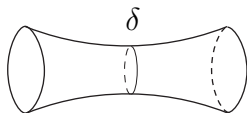


# Topological pictures

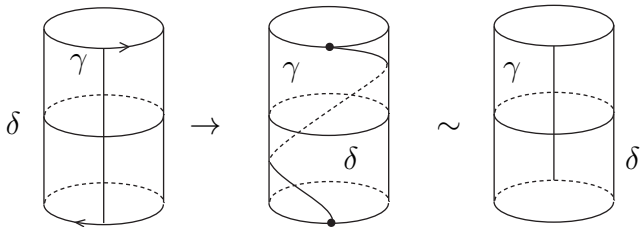
global



local



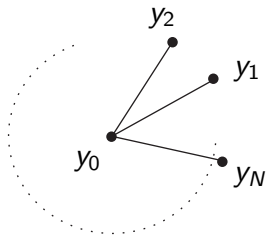
$\delta \rightarrow 0$



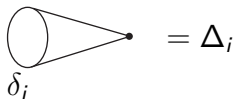
$$\begin{cases} \delta \rightarrow \delta \\ \gamma \rightarrow \gamma + \delta \end{cases}$$

- ▶ few pictures worth 1,000 (10,000?) words
- ▶ heuristic analytic reasoning suggests what the answer should be — then know what to prove.

- ▶  $X^* = X \setminus X_\infty$
- ▶ topological picture of  $X^*$



- ▶ along  $\overline{y_0 y_i}$  we have the locus of the vanishing cycle



$\implies$  ▶  $X^*$  obtained from  $X_0$  by attaching 2-cells  $\Delta_i$

- ▶ In general

$X^*$  obtained from  $X_0$  by attaching  
 $n = \frac{1}{2} (\dim_{\mathbb{R}} X)$  cells

$\implies$  Lefschetz theorems I, II

- ▶ *Single and double integrals*

Returning to  $X$  given by

$$z^2 = f(x, y)$$

there are single integrals (1-forms)

$$\psi = \frac{p(x, y) dx}{z} + \frac{q(x, y) dy}{z}$$

and double integrals (2-forms)

$$\varphi = \frac{r(x, y) dx \wedge dy}{z}$$

The story of the  $\psi$ 's is very interesting but we will only have time to make a few observations. For one such we note that

$$\blacktriangleright \int \psi < \infty \implies d\psi = 0.$$

Proof:

$$\begin{aligned} d\psi &= d\left(\frac{p(x,y)}{z}\right) \wedge dx + d\left(\frac{q(x,y)}{z}\right) \wedge dy \\ &= \frac{r(x,y) dx \wedge dy}{z} \end{aligned}$$

$$\begin{aligned} \implies \frac{1}{4}(d\psi \wedge \overline{d\psi}) &= \left|\frac{r(x,y)}{z}\right|^2 \left(\frac{i}{2}\right) dx \wedge d\bar{x} \wedge \left(\frac{i}{2}\right) dy \wedge d\bar{y} \\ &= \text{volume form on } X \end{aligned}$$

$$0 < \int_X d\psi \wedge \overline{d\psi} = \int_X d(\psi \wedge \overline{d\psi}) = 0 \implies d\psi = 0.$$

- ▶ The space of single integrals is denoted by  $H^{1,0}(X)$  and its dimension  $h^{1,0}(X)$  is called the *irregularity* — reason for the name is that in the early days “most” surfaces seemed to be *regular*, i.e., to have  $h^{1,0}(X) = 0$ .

## Example

For  $z^2 = f(x, y)$  to be irregular the curve  $C$  cannot be smooth, or even have generic singularities, those being where

$$\begin{cases} f_x(x, y) = f_y(x, y) = 0 \\ \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} (x, y) \neq 0 \end{cases}$$

Similarly for a hypersurface

$$F(z_0, z_1, z_2, z_3) = 0$$

in  $\mathbb{P}^3$  it is not easy to write down on  $F$  where  $X$  is irregular.



- Suppose now  $\varphi$  is a regular 2-form; i.e.,

$$\int_{\sigma} \varphi < \infty$$

for any 2-chain  $\sigma$ . We set

$$H^{2,0}(X) = \left\{ \begin{array}{l} \text{space of} \\ \text{regular 2-forms} \end{array} \right\}.$$

The *periods* of  $\psi$  are the

$$\int_{\Gamma} \psi, \quad \Gamma \in H_2(X, \mathbb{Z}).$$

Among the  $\Gamma$ 's are the fundamental classes of algebraic curves  $C \subset X$ ; i.e., the images of

$$H_2(C, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z}).$$

We will discuss these further below.

- ▶ By restriction

$$\psi \rightarrow \psi_y = \frac{p(x, y) dx}{z}$$

we will generally have  $\psi_y \neq 0$  which gives

$$H^{1,0}(X) \hookrightarrow H^{1,0}(X_y).$$

This suggests that we have

$$H^1(X, \mathbb{C}) \hookrightarrow H^1(X_y, \mathbb{C}),$$

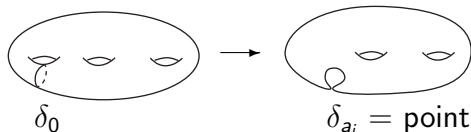
which is true and is what originally suggested the first non-easy case of Lefschetz II — again analysis and topology went hand in hand.

# Another example of the use of analysis to suggest topology:

For a vanishing cycle



traced out by  $\delta_y \in H_1(X_y, \mathbb{Z})$  along the path from 0 to  $a_i$

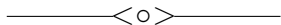


we have

$$\int_{\delta_0} \psi = \int_{\delta_{a_i}} \psi = 0, \quad \psi \in H^{1,0}(X).$$

This led to Picard's argument that

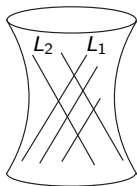
$$\ker\{H_1(X_0, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})\} = \begin{cases} \text{span of the} \\ \text{space of vanishing cycles.} \end{cases}$$



Returning to the discussion of

- ▶ Among the classes in  $H_2(X, \mathbb{Z})$  are those given by the fundamental classes of the algebraic curves  $C$  contained in  $X$ .

Example:



$$\left\{ \begin{array}{l} \text{two families} \\ \text{on lines on a} \\ \text{quadric surface} \\ z_0 z_1 = z_2 z_3 \end{array} \right\}^2$$

$$\rightsquigarrow H_2(X, \mathbb{Z}) \cong \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2]$$

► In general  $C$  is a component of

$$\begin{cases} z^2 = f(x, y) \\ g(x, y, z) = 0 \end{cases}$$

(may take  $g(x, y, z) = g_0(x, y) + g_1(x, y)z$ )

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<sup>2</sup>The lines are  $z_0 = z_2 = 0$ ,  $[z_1, z_3] \in \mathbb{P}^1$  arbitrary and  $z_1 = z_3 = 0$ ,  $[z_0, z_2]$  arbitrary.

$\implies$  On  $X$

$$0 = dg = g_x dx + g_y dy + g_z dz$$

which using  $dz = \left(\frac{1}{2z}\right) (f_x dx + f_y dy)$  gives a relation

$$a dx + b dy \Big|_C = 0$$

$$\implies \psi \Big|_C = 0$$

$$\implies \int_{[C]} \psi = 0.$$

**Conclusion:** *The periods of  $H^{2,0}(X)$  on the homology classes of algebraic curves are equal to zero.*

- ▶ The converse statement is the famous *Lefschetz (1,1) theorem*.
- ▶ The converse to the analogous statement for arbitrary  $X$  is the *Hodge conjecture*.
- ▶ In terms of differential forms of degree 2 on  $X$  there are three types:
  - ▶  $\frac{p(x,y) dx \wedge dy}{z} \leftrightarrow H^{2,0}(X)$
  - ▶ conjugates of these  $\leftrightarrow \overline{H^{2,0}(X)} = H^{0,2}(X)$
  - ▶ those that have a  $dx \wedge d\bar{x}$ ,  $dx \wedge d\bar{y}$ ,  $d\bar{x} \wedge dy$ ,  $dy \wedge d\bar{y}$  which are said to be of type (1,1) and contribute  $H^{1,1}(X)$  to  $H^2(X, \mathbb{C})$ ; it is these that are Poincaré dual to the homology classes carried by the algebraic curves in  $X$ .

## Further topics

- ▶ These involve the *multiplicative structure* on cohomology:  
For  $X$  of dimension  $n$  and  $H \in H^2(X)$  the class of a hyperplane section

$$(*) \quad L^k : H^{n-k}(X) \rightarrow H^{n+k}(X).$$

**Hard Lefschetz theorem:**  $(*)$  is an isomorphism

Lefschetz stated the result but his proof was incomplete.  
Hodge developed Hodge theory to prove  $(*)$ .

- ▶ Define operators  $L, H, \Lambda$  on  $H^*(X)$  by
  - ▶  $L$  as above
  - ▶  $H = (d - n) \text{Id}$  on  $H^d(X)$



Then the commutator

$$[H, L] = 2L.$$

There is a unique  $\mathfrak{sl}_2 = \{L, H, \Lambda\}$  with

$$\begin{cases} [L, \Lambda] = H \\ [H, \Lambda] = -2\Lambda. \end{cases}$$

Decomposing  $H^*(X)$  into irreducible  $\mathfrak{sl}_2$ -modules gives the *Lefschetz decomposition* of cohomology into primitive subspaces — every class is a linear combination of powers of  $L$  applied to primitive classes

$$\begin{cases} L^k \cdot \eta \\ \Lambda \eta = 0. \end{cases}$$

- ▶ Any irreducible  $\mathfrak{sl}_2$ -module is isomorphic to
  - ▶  $V = \text{span}\{x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\}$
  - ▶  $L = \partial_x, \Lambda = \partial_y$
  - ▶ primitive part is generated by  $x^n$ .

**Example:**  $X =$  algebraic surface

$$H^1(X) \xrightarrow{\sim} H^3(X)$$

and

$$H^0(X) \xrightarrow{L} H^2(X) \xrightarrow{L} H^4(X)$$

has

- ▶  $H^2(X)_{\text{prim}} = \ker\{H^2(X) \xrightarrow{L} H^4(X)\}$
- ▶  $H^2(X) = LH^0(X) \oplus H^2(X)_{\text{prim}}$

- ▶ Finally, you may ask: OK, we know a lot about the homology of  $X$  — what about its homotopy?

## Theorem

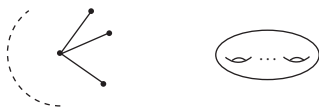
*The rational homotopy type of  $X$  is uniquely determined by  $H^*(X)$ .*

Thus the

- ▶  $\pi_i(X) \otimes \mathbb{Q}$
  - ▶ Massey triple products  $/\mathbb{Q}$ , etc. are all equal to zero
- ⇒ Very strong homotopy-theoretic conditions that  $X$  be topologically a smooth algebraic variety.

## Appendix: Monodromy

- ▶  $C_0$  = smooth algebraic curve over the origin



- ▶ fundamental group  $\pi_1 = \pi_1(\mathbb{C} \setminus \{\text{slits}\})$  acts on  $H_1(C_0, \mathbb{Z})$
- ▶ action of  $\pi_1$  is generated by PL transformation

$$T_i : \gamma \rightarrow \gamma + (\gamma, \delta_i) \delta_i$$

- ▶  $\prod T_i = \text{identity}$
- ▶ action of  $\pi_1$  preserves the intersection form

$$Q : H_1(C_0, \mathbb{Z}) \otimes H_1(C_0, \mathbb{Z}) \rightarrow \mathbb{Z}$$

- ▶ Invariant cycles

$$H_1(C_0, \mathbb{Q})^{\text{inv}} = \text{span}\{\gamma : (\gamma, \delta_i) = 0 \text{ for all } i\}$$

- ▶ Vanishing cycles

$$H_1(C, \mathbb{Q})^{\text{van}} = \text{span}\{\delta_i\}$$

- ▶ If we know that

$$(*) \quad H_1(C_0, \mathbb{Q})^{\text{van}} \cap H_1(C_0, \mathbb{Q})^{\text{inv}} = (0)$$

then

$$Q = \begin{pmatrix} * & * = 0 \\ 0 & * \end{pmatrix}$$

and the monodromy representation is semi-simple

- ▶ Lefschetz stated (\*) but his proof was incomplete — in fact

(\*) is true, but its proof requires analysis

The analysis was provided by Hodge.

- ▶ It is a general fact proved by Deligne in the geometric case and by Schmid in general that *general monodromy representations are always semi-simple*.

The proofs require Hodge theory and are among the most basic properties of the topology of algebraic varieties.

- ▶ The reason Lefschetz wanted to have the result is that

$$(*) \iff \text{Hard Lefschetz}$$

Lefschetz proof of this assertion was correct.