

Hodge Theory and Moduli*

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*Much of these talks is based on joint work with Mark Green, Radu Laza, and Colleen Robles. The algebro-geometric aspects of the main example are based on the work of Marco Franciosi, Rita Pardini and Sönke Rollenske ([FPR] in the references), and on discussions that the four of us have had with them related to a possible joint project.

Outline

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I.A. Introductory comments

- ▶ Algebraic geometry is frequently seen as a very interesting and beautiful subject, but one that is also very difficult to get into; this is partly due to its breadth, as traditionally algebra, topology, analysis, differential geometry, Lie theory — and more recently combinatorics, logic, . . . — are used to study it. I have tried to make these notes accessible to a general audience by illustrating topics with elementary examples and informal geometric and heuristic arguments, and with occasional side comments for experts in the subject.

Although algebra, both commutative and homological, are the central tools in algebraic geometry, their use in many current areas of research (birational geometry and the minimal model program) is frequently highly technical and will not be extensively discussed in these lectures. On the other hand, partly because Hodge theory involves analysis, Lie theory and differential geometry as well as highly developed cohomological methods, it is perhaps less prevalent in the more algebraic publications in the field. One objective of these talks is to illustrate how, in partnership with the more algebraic and homological methods, Hodge theory may be used to study interesting and important geometric questions.

In summary the theme of these talks is to discuss how complex analysis, differential geometry and Lie theory may be used to study a basic problem in algebraic geometry.

- ▶ An affine algebraic variety is given by the solution space over the complex numbers to polynomial equations

$$(*) \quad f_i(x_1, \dots, x_n) = 0 \quad i = 1, \dots, n.$$

The “elementary” examples are

linear spaces $ax + by + c = 0$



conics $ax^2 + 2bxy + cy^2 + ex + fy + g = 0$

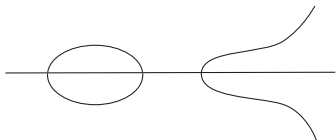


quadrics $Q(x) = \sum_{i,j=1}^n a_{ij}x_i x_j + \sum_{i=1}^n b_i x_i + c = 0, \quad a_{ij} = a_{ji}$



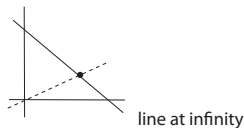
The first non-elementary examples are cubics

$$y^2 = 4x^3 + ax + b$$



The first two elementary examples and the non-elementary example are algebraic curves. One of course considers higher dimensional varieties; surfaces, threefolds, . . . We will be primarily concerned with curves and surfaces.

- ▶ The above examples are all affine algebraic varieties in \mathbb{C}^2 or \mathbb{C}^n . In general one adds to an affine variety the asymptotes or “points at infinity” to obtain the projective space $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}_\infty^{n-1}$. The picture of the projective plane \mathbb{P}^2 is

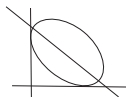


Then the picture in \mathbb{P}^2 of the parabola in \mathbb{C}^2 given by

$$xy = 1$$

A 2D coordinate system with a horizontal axis and a vertical axis. Two hyperbolic branches are shown: one in the first quadrant and one in the third quadrant, both approaching the axes as asymptotes.

is something like



- ▶ The equations of the completion in \mathbb{P}^n of an affine variety given in \mathbb{C}^n by (*) are obtained by homogenizing: set $x_i = z_i/z_0$ and clear denominators to obtain

$$f_i(z_0, z_1, \dots, z_n) = 0$$

where $f_i(\lambda z_0, \lambda z_1, \dots, \lambda z_n) = \lambda^{d_i} f_i(z_0, z_1, \dots, z_n)$, $d_i = \deg f_i$. Points in \mathbb{P}^n then have homogeneous coordinates $[z_0, z_1, \dots, z_n]$. The parabola above becomes

$$z_1 z_2 = z_0^2.$$

- ▶ Later on we will consider varieties in weighted projective spaces $\mathbb{P}(a_0, a_1, \dots, a_n)$ where $\lambda \in \mathbb{C}^*$ acts on z_i by $\lambda(z_i) = z_i^{a_i}$, and in the quotient $\mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$ the algebraic varieties are defined by weighted homogeneous polynomials.

Example: $\mathbb{P}(1, 1, 2)$ is embedded in \mathbb{P}^3 by

$$[x_0, x_1, y] \rightarrow [x_0^2, x_0x_1, x_1^2, y].$$

The image is

$$z_0z_2 = z_1^2 \text{ which is } \img alt="A diagram of a cone with a black dot at its vertex, representing the image of the embedding." data-bbox="600 295 695 371"/>$$

with the singular point $[0, 0, 0, 1]$.

- ▶ Two algebraic varieties are considered to be *equivalent* if there is a “change of variables” that transforms one into the other. Thus if the discriminant $b^2 - ac \neq 0$ all conics in \mathbb{P}^2 are equivalent to the circle

$$z_1^2 + z_2^2 = z_0^2.$$

Initially changes of variables were linear transformations (including projections); later on rational changes of variables became allowed.

- ▶ Historically algebraic varieties arose from two sources: projective geometry (lines and linear spaces, conics and higher dimensional quadrics), and from complex analysis. In complex analysis the issue was to understand the integrals

$$\int r(x, y(x)) dx, \quad f(x, y(x)) = 0$$

of algebraic functions, and the “functions” defined by inverting the integrals

$$w = \int^{x(w)} r(x, y(x)) dx.$$

Here $f(x, y)$ is a polynomial and $r(x, y)$ is a rational function; $y(x)$ is a multi-valued “algebraic function” defined by $f(x, y(x)) = 0$. The integral depends on choosing a path γ of integration and a branch $y(x)$ of $f(x, y(x)) = 0$ along γ . Thus

$$w = \int^{\sin w} \frac{dx}{\sqrt{1-x^2}} = \int^{\sin w} \frac{dx}{y(x)}$$

where $x^2 + y^2 = 1$ and $\sqrt{dx^2 + dy^2} = dx/y$ gives the parametrization of the circle by arclength in terms of “elementary” functions (trigonometric functions and logarithms).

However parametrizing the ellipse by arclength led to integrals such as

$$w = \int^{p(w)} \frac{dx}{\sqrt{4x^3 + ax + b}}$$

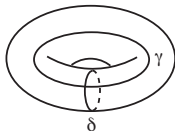
which gave non-elementary functions and led Euler, Legendre, Abel, Gauss, Jacobi, Riemann, . . . to the beginnings of the rich and deep interplay between analysis and algebraic geometry. This evolved into modern *Hodge theory*, and it is this interface between analysis and algebraic geometry that is a main theme of these talks.

- ▶ Part of the richness of the subject of algebraic geometry are the multiple perspectives that may be used in its study:
 - ▶ *geometric*;
 - ▶ *algebraic* — e.g., as we will briefly discuss, in birational geometry the algebraic classification of certain classes of singularities



of algebraic varieties plays a central role;

- ▶ *analytic*; we have mentioned complex analysis and the integrals of algebraic functions;
- ▶ *topological*

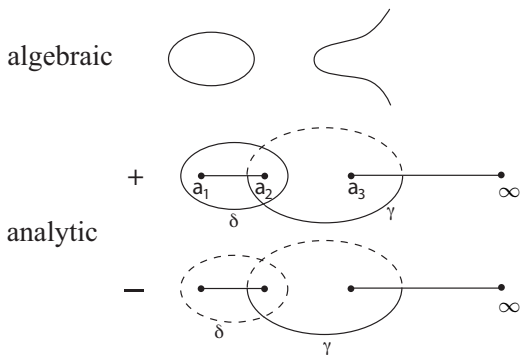


Example: The compact complex manifold associated to the algebraic curve

$$X = \{y^2 = (x - a_1)(x - a_2) \cdots (x - a_{2g+1})\}$$

where the a_i are distinct is a compact Riemann surface of genus g , which can be studied from all of these perspectives.

For $g = 1$ as in the cubic above, the pictures are



where a_1, a_2, a_3 are the roots of the cubic equation in the RHS of the above equation.

The integral

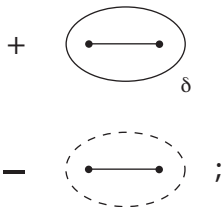
$$\int^w \frac{dx}{y}$$

is determined up to the periods

$$\pi_1 = \int_{\delta} \frac{dx}{y}, \quad \pi_2 = \int_{\gamma} \frac{dx}{y};$$

one may show that $\pi_1 \neq 0$ and $\text{Im}(\pi_2/\pi_1) > 0$.

Incidentally for the circle $y^2 = 1 - x^2$ the picture is

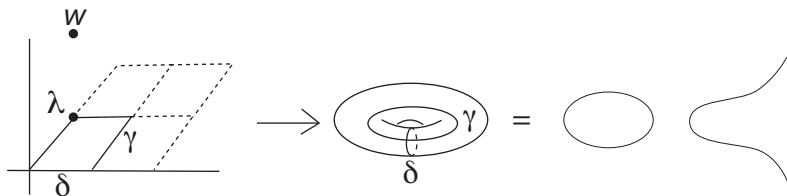


there is only one cycle δ . In this case there is only one period

$$\int_{\delta} \frac{dx}{y}.$$

For another analytic perspective the function $p(w)$ given by inverting the elliptic integral is a doubly periodic entire function (doubly due to the periods $\int_{\delta} dx/y$ and $\int_{\gamma} dx/y$)[†] that leads to the parametrization of the curve X given by $y^2 = 4x^3 + ax + b$

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & X \\ \Psi & & \Psi \\ w & \longrightarrow & (p(w), p'(w)) \end{array}$$



[†]For the circle above there is only one period and inverting the integral gives singly periodic trigonometric functions.

The $p'(w)$ arises from

$$dw = d \int^{p(w)} \frac{dx}{y} = \frac{p'(w)dw}{y(w)}.$$

Here the ratio of the periods

$$\lambda = \pi_2/\pi_1$$

has $\text{Im } \lambda > 0$ and is determined up to

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d}$$

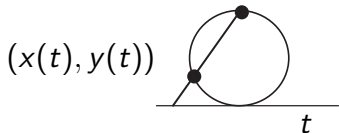
with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ reflecting the choices of the basis $\delta, \gamma \in H_1(X, \mathbb{Z})$ with $(\delta, \gamma) = 1$.

I.B Classification problem; first case of relation between moduli and Hodge theory

- ▶ A central problem in algebraic geometry is to classify the equivalence classes of algebraic varieties. For curves there are two types of parameters:
 - ▶ discrete (e.g., the genus $g = (1/2) (b_1(X) = 1^{\text{st}} \text{ Betti number})$ for smooth algebraic curves)
 - ▶ continuous (*moduli*; the smooth curves of genus g form a $(3g - 3 + \rho)$ -dimensional family \mathcal{M}_g , where $\rho = \dim \text{Aut}(X)$).

Example (curves):

- $g = 0$ ▶ there is only one equivalence class; for example using the birational map given by stereographic projection all non-singular conics are projectively equivalent and are birationally equivalent to a line \mathbb{P}^1



$x(t), y(t)$ are rational functions of t .

$g = 1$ ▶ $\dim \mathcal{M}_1 = 1$, with

$$j = \frac{1728a^3}{\Delta}, \quad \Delta = a^3 - 27b^2$$

giving a 1-1 map $\mathcal{M}_1 \rightarrow \mathbb{C}$; thus the curves

$$\begin{cases} x_0^3 + x_1^3 + x_2^3 = x_0x_1x_2 & (\text{in } \mathbb{P}^2) \\ Q_1(x_0, x_1, x_2, x_3) = Q_2(x_0, x_1, x_2, x_3) = 0 & (\text{in } \mathbb{P}^3) \end{cases}$$

are in each case equivalent to

$$y^2 = 4x^3 + ax + b \text{ for a unique value of } j.$$

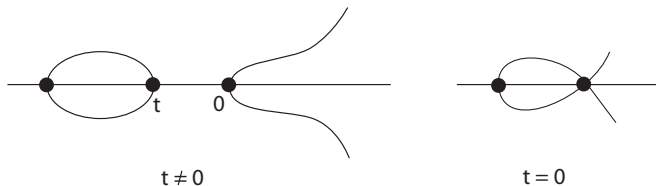
- ▶ In general one hopes that
 - a moduli space \mathcal{M} will be an algebraic variety, generally *not* complete (compact)
 - there will be a canonical completion $\overline{\mathcal{M}}$ corresponding to adding *certain* singular varieties.

In these talks we will assume (i) and will be primarily concerned with (ii). *What does (ii) mean?*

We imagine a family of plane curves

$$X_t = \{f(x, y, t) = 0\}, \quad t \in \Delta$$

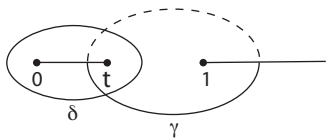
that are smooth for $t \in \Delta^*$ but may be singular for $t = 0$. A picture like



gives such a family

Applying coordinate changes depending on t can give a family \tilde{X}_t such that \tilde{X}_t is equivalent to X_t for $t \neq 0$ but \tilde{X}_0 is quite different for $t = 0$. *How can we say what a **canonical** choice for X_0 should be?*

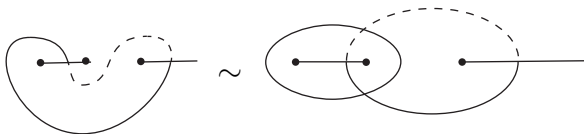
Historically one suggested answer to this question was provided by Hodge theory; i.e., considering the *period matrix* associated to the curve. For the example $y^2 = x(x-t)(x-1)$



$$\int_{\delta} dx/y = \pi_1$$

$$\int_{\gamma} dx/y = \pi_2$$

as t turns around 0 for γ we get a picture something like



implying that in homology

$$\begin{cases} \delta \rightarrow \delta \\ \gamma \rightarrow \gamma + \delta. \end{cases}$$

Using elementary complex analysis one may show that

- ▶ $\pi_1(t)$ is non-zero and holomorphic for $t \in \Delta$;
- ▶ $\pi_2(t) = \pi_1(t) \frac{\log t}{2\pi i} + (\text{holomorphic function of } t \in \Delta).$

In general for any family X_t , $t \in \Delta^*$, of smooth genus 1 curves we will have periods π_1, π_2 as above where

$$\lambda = \pi_2/\pi_1, \quad \text{Im } \lambda > 0$$

and we are thinking of $\lambda = \lambda(t)$ as a point in $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ where $\mathbb{H} = \{w \in \mathbb{C} : \text{Im } w > 0\}$ is the upper half plane. The periods are locally holomorphic functions of $t \in \Delta^*$, and as t turns around the origin the cycles δ, γ will undergo a *monodromy transformation*

$$\left\{ \begin{array}{l} \left(\begin{array}{c} \pi_2 \\ \pi_1 \end{array} \right) \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{array}{c} \pi_2 \\ \pi_1 \end{array} \right), \text{ where} \\ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ is the monodromy matrix.} \end{array} \right.$$

Now setting $w(z) = \lambda(t)$ where $e^{2\pi iz} = t$ gives a diagram

$$\begin{array}{ccccc} z \in & \mathbb{H} & \xrightarrow{w} & \mathbb{H} & \\ \downarrow & \downarrow & & \downarrow & \\ e^{2\pi iz} = & t \in \Delta^* & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} & \end{array}$$

where

$$w(z+1) = Tw(z)$$

Lemma 1: *The eigenvalues of T satisfy $|\mu| = 1$*

Since the characteristic polynomial of T has integral coefficients, by a result from analytic number theory

$$\mu = e^{2\pi ip/q}$$

is a root of unity. Replacing t by t^q gives that T is unipotent, and we may then assume that

$$T = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad m \in \mathbb{Z}^+.$$

Lemma 2: Given a holomorphic mapping $w : \mathbb{H} \rightarrow \mathbb{H}$ satisfying

$$w(z + 1) = w(z) + m,$$

it follows that

$$w(z) = mz + u(e^{2\pi iz})$$

where u is bounded as $\text{Im } z \rightarrow \infty$.

Taking $m = 1$ for simplicity this gives

$$\lambda(t) = \frac{\log t}{2\pi i} + u(t),$$

where $u(t)$ is holomorphic in Δ .

Both of these lemmas are proved by complex analysis arguments using the Schwarz lemma in the form

w is distance decreasing in the hyperbolic (Poincaré) metric

and the observation that

the length of the circles $|t| = \epsilon$ tends to zero as $t \rightarrow 0$.

Sketch of the proof of Lemma 1: Let $z_n \in \mathbb{H}$ be a sequence with $\operatorname{Re} z_n = 0$, $\operatorname{Im} z_n \rightarrow \infty$ and set

$$w_n = w(z_n).$$

Then

$$w(z_n + 1) = Tw(z_n) = Tw_n.$$

The hyperbolic distance $d(z_n, z_n + 1) \rightarrow 0$, and by the distance decreasing property of w we have

$$d(w_n, Tw_n) \rightarrow 0.$$

Now $w_n = A_n \cdot w$ for some fixed $w \in \mathbb{H}$ and $A_n \in \mathrm{SL}_2(\mathbb{R})$, and from

$$d(w, A_n^{-1} T A_n w) \rightarrow 0$$

a little argument shows that by passing to a subsequence we will have

$$A_n^{-1} T A_n \rightarrow H = \{\text{isotropy group of } w\}.$$

Since H is compact, all its eigenvalues have absolute value 1 and this implies the same for T .

Conclusion: *The periods of an arbitrary family of $g = 1$ algebraic curves over Δ^* have the same asymptotic behavior as a family acquiring nodal singularities given locally analytically by*

$$x^2 = y^2 + tf(x, y).$$

The pictures are

analytic



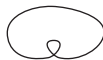
local picture of



topological

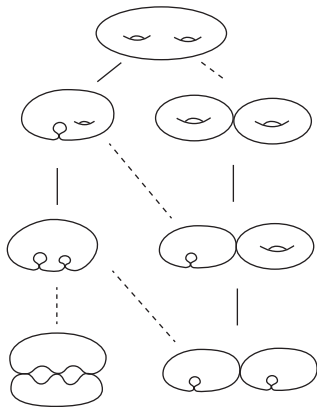


local picture of



This analysis of the asymptotics of the period matrix (Hodge structure) extends to that of algebraic curves of any genus $g \geq 2$ and provided an early suggestion as to what $\overline{\mathcal{M}}_g$ should be.

The picture of $\overline{\mathcal{M}}_2$ is



This gives the stratification of $\overline{\mathcal{M}}_2$ together with the incidence (degeneration) relations among the strata. (The solid and dotted arrows will be explained later.)

We will see that this stratification is captured by the Hodge structures and their limits. The objective of these talks is to discuss how this picture might be extended to some completed moduli spaces of varieties of general type (analogues of curves of genus $g \geq 2$) and to illustrate how this works for the first non-classical algebraic surfaces (called I -surfaces and which have the invariants $p_g(X) = 2$, $q(X) = 0$, $K_X^2 = 1$).

- ▶ To jump ahead and anticipate the main points to be made; with notations and terminology to be explained there are first the general results (some of which are work in progress)
 - ▶ for each class of surfaces of general type the moduli space \mathcal{M} exists and has a canonical completion $\overline{\mathcal{M}}$;

- ▶ there is a period mapping

$$\mathcal{M} \xrightarrow{\Phi} \mathcal{P} \subset \Gamma \backslash D$$

that associates to each surface X the Hodge structure on $H^2(X, \mathbb{Z})$;

- ▶ there is a canonical minimal completion $\overline{\mathcal{P}}$ of \mathcal{P} to projective variety, and the period mapping extends to

$$\overline{\mathcal{M}} \xrightarrow{\Phi_e} \overline{\mathcal{P}}.$$

Then there are specific results for the I -surface mentioned above

▶ Equations/picture

- ▶ $z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y)$ (weighted degree 10 hypersurface in $\mathbb{P}(1, 1, 2, 5)$)

- ▶ $\left\{ \begin{array}{l} \text{Diagram of a cone } V \text{ with vertex } P \\ \mathbb{P}(1, 1, 2) \hookrightarrow \mathbb{P}^3 \text{ given by} \\ (t_0, t_1) \hookrightarrow [t_0^2, t_0 t_1, t_1^2, y] \end{array} \right.$
- $\left\{ \begin{array}{l} X = 2:1 \text{ map to } \mathbb{P}(1, 1, 2) \text{ branched over } P \\ \text{and } V \in |\mathcal{O}_{\mathbb{P}^3}(5)| \end{array} \right.$

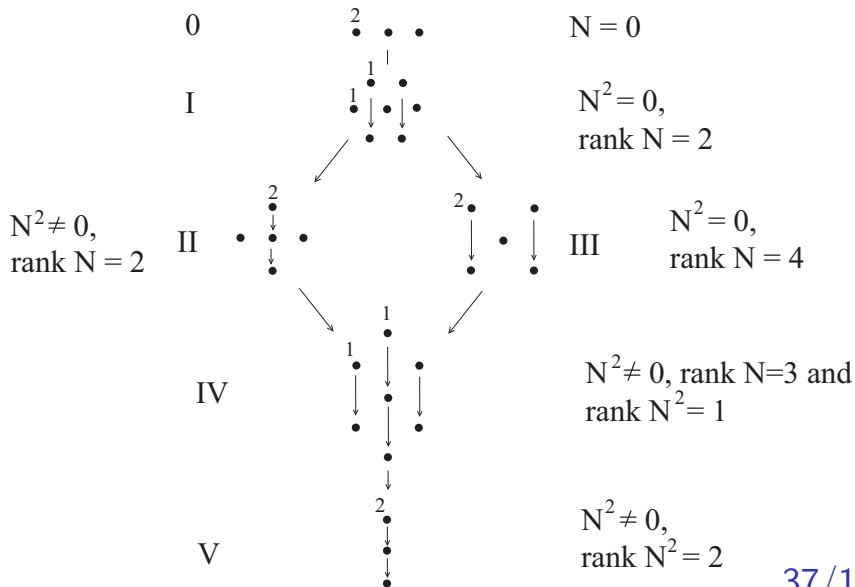
- ▶ $\pi_1(X) \stackrel{?}{=} 0$ and $H^2(X, \mathbb{Z}) \cong \mathbb{Z}[K_X] \oplus H^2(X, \mathbb{Z})_{\text{prim}}$ where $H^2(X, \mathbb{Z})|_{\text{prim}} \cong \mathbb{Z}^{32}$ and the intersection form there is unimodular and even.

- ▶ \mathcal{M}_I is smooth and
 - ▶ $\dim \mathcal{M}_I = h^1(T_X) = 28$
 - ▶ the period matrix domain D_I is a homogeneous contact manifold with $\dim D_I = 57 = 2 \dim \mathcal{M}_I + 1$
- ▶ $\Phi = \mathcal{M}_I \rightarrow \Gamma_I \backslash D_I$ and Φ_* is injective (local Torelli)

⇓

$\Phi(\mathcal{M}_I) = \text{contact submanifold } \mathcal{P} \hookrightarrow \Gamma_I \backslash D_I$

- Picture of the crude stratification of $\overline{\mathcal{P}}$ ($N =$ logarithm of monodromy)



- ▶ The refined Hodge theoretic stratification of $\overline{\mathcal{P}}$ uniquely determines the stratification of $\overline{\mathcal{M}}_l^{\text{Gor}}$.[‡]
Rather than display the whole table the following is just the part for simple elliptic singularities (types I_k and III_k) — they have $N^2 = 0$ since for the semi-stable-reduction (SSR) of such a degeneration only double curves (and no triple points) occur — all of the other types occur if we include cusp singularities.

[‡]We will explain what the refined Hodge theoretic stratification means. Work in progress suggests that the refined Hodge theoretic stratification of $\overline{\mathcal{P}}$ may go a long way towards determining the full stratification off $\overline{\mathcal{M}}_l$ by singularity type.

stratum	dimension	minimal resolution \tilde{X}	$\sum_{i=1}^k (9 - d_i)$	k	codim in $\overline{\mathcal{M}}_g$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\tilde{X})=2$	8	1	9
$III_{2,2}$	12	rational surface	14	2	16
$III_{1,2}$	11	rational surface	15	2	17
$III_{1,1,R}$	10	rational surface	16	2	18
$III_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$III_{1,1,2}$	2	ruled surface with $\chi(\tilde{X})=0$	23	3	26
$III_{1,1,1}$	1	ruled surface with $\chi(\tilde{X})=0$	24	3	27

Note that the last column is the sum of the two columns preceding it (this will be explained using Hodge theory).

II. A. Moduli

- ▶ We have seen that Hodge theory, in the classical form of periods of integrals of algebraic functions together with some complex analysis, suggests what singular curves should be included to compactify the moduli space \mathcal{M}_g , leading to an essentially smooth $\overline{\mathcal{M}}_g$.
- ▶ For surfaces (and higher dimensional) varieties of general type the story thus far is both similar and different,[§] especially in the non-classical (term to be explained) case.

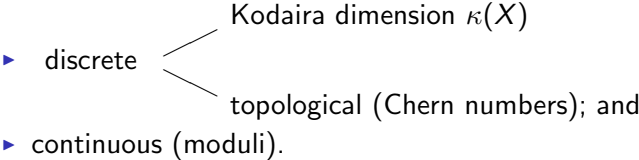
[§]Cf. [K], [Ko] and the recent Séminaire Bourbaki [B] in the references.

- ▶ Birational geometry tells us that it is possible to define a moduli space \mathcal{M} with a canonical completion $\overline{\mathcal{M}}$;
- ▶ It does not
 - (i) tell us what the singular surfaces X corresponding to the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ are;[¶]
 - (ii) tell us the stratification of $\overline{\mathcal{M}}$; and
 - (iii) in contrast to the curve case, $\overline{\mathcal{M}}$ may be highly singular along $\overline{\mathcal{M}} \setminus \mathcal{M}$, and it does not suggest how to desingularize it.

We will explain and illustrate how Hodge theory, in partnership with birational geometry, helps us understand the above points (i)–(iii).

[¶]In fact, even though they are all rational, there does not seem to be an at all practical bound on the index of the non-Gorenstein isolated singularities.

A. Moduli

- ▶ Invariants of a smooth projective variety X are basically
 - ▶ discrete Kodaira dimension $\kappa(X)$
topological (Chern numbers); and
 - ▶ continuous (moduli).
- ▶ X is a complex manifold and a basic invariant is the space $H^0(K_X^m)$ of global holomorphic forms expressed locally in holomorphic coordinates x_1, \dots, x_n as

$$\varphi = f(x)(dx_1 \wedge \cdots \wedge dx_n)^m$$

where $f(x)$ is holomorphic and transforms by the m^{th} power of the Jacobian determinant when we change coordinates.

- ▶ The Kodaira dimension $\kappa(X)$ is defined by

$$\dim H^0(mK_X) = h^0(mK_X) = Cm^{\kappa(X)} + \dots, C > 0.$$

By convention we set $\kappa(X) = -\infty$ if all $h^0(mK_X) = 0$.

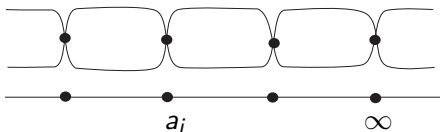
- ▶ The purpose of this part of the talk is to give an informal introduction to some aspects of moduli, to describe two simple classes of algebraic curves and surfaces, and to illustrate the semi-log-canonical (slc) singularities that arise for surfaces and to begin the discussion of how Hodge theory relates to them.

Examples when $m = 1$

- ▶ X is the smooth algebraic curve (compact Riemann surface) with affine equation

$$y^2 = \prod_{i=1}^{2g+1} (x - a_i), \quad a_i \text{ distinct.}$$

- ▶ We may picture X as a 2-sheeted branched covering of \mathbb{P}^1



and the holomorphic 1-forms on X are

$$\varphi = \frac{p(x)dx}{y}, \quad \deg p(x) \leq g - 1.$$

There are similar expressions $\frac{q(x)dx^m}{y}$ where $\deg q(x) \leq (2g - 2)m - g + 1$ for $H^0(mK_X)$, $m \geq 2$.

- ▶ The genus $g(X) = g$ and the number of parameters of X 's given as above is $2g + 1 - 2 = 2g - 1$; a general curve of genus g is represented this way for $g = 1, 2$;
- ▶ the φ 's above give the space $H^0(K_X) = H^0(\Omega_X^1)$ of holomorphic differentials on X ;

$$H_{\text{DR}}^1(X) \cong H^0(\Omega_X^1) \oplus \overline{H^0(\Omega_X^1)}$$

gives the Hodge structure on $H_{\text{DR}}^1(X, \mathbb{C}) \cong H^1(X, \mathbb{C})$;

- ▶ as a consequence for $h^0(K_X) = \dim H^0(K_X)$ we have

$$h^0(K_X) = \left(\frac{1}{2}\right) b_1(X) = g,$$

the first result in Hodge theory relating the algebro-geometric invariant $h^0(K_X)$ to the topological invariant $b_1(X)$.

- ▶ Now take X to be the smooth algebraic surface with affine equation

$$(*) \quad z^2 = f(x, y), \quad \deg f(x, y) = 2k$$

where $f(x, y) = 0$ defines a smooth algebraic curve $C \subset \mathbb{P}^2$;

- ▶ The holomorphic 2-forms on X are

$$\varphi = \frac{p(x, y)dx \wedge dy}{z}, \quad \deg p(x, y) \leq k - 3$$

and for $H^0(\Omega_X^2) = H^0(K_X)$

$$(**) \quad H_{\text{DR}}^2(X) \cong H^0(\Omega_X^2) \oplus H^1(\Omega_X^1) \oplus \overline{H^0(\Omega_X^2)}$$

gives the Hodge structure on $H^2(X, \mathbb{C})$.

*This surface is similar to but both simpler and more complicated than the I -surface.

For an initial explanation of the $H^1(\Omega_X^1)$ term, if Q is the bilinear form on $H^2(X, \mathbb{C})$ given by the cup-product, then

$$F^2 H^2(X, \mathbb{C}) = H^0(\Omega_X^2)$$

\cap

$$F^1 H^2(X, \mathbb{C}) = H^0(\Omega_X^2) \oplus H^1(\Omega_X^1)$$

where under the cup product Q in cohomology

$$F^1 = F^{2\perp}$$

and

$$F^1 \cap \bar{F}^1 = H^1(\Omega_X^1)$$

so that the Hodge decomposition (***) is determined by $H^0(\Omega_X^2)$ and Q

- ▶ in contrast to the curve case, for any $k \geq 4$ a general surface in the moduli space \mathcal{M} of surfaces of the above numerical type is equivalent to one given by (*).

- ▶ The *Kodaira number* $\kappa(X)$ for curves is

$$\kappa(X) = \begin{cases} -\infty & \text{for } g(X) = 0 \\ 0 & \text{for } g(X) = 1 \\ 1 & \text{for } g(X) \geq 2. \end{cases}$$

For the above algebraic surfaces

$$\kappa(X) = \begin{cases} -\infty & k \leq 2 & \text{(rational)} \\ 0 & k = 3 & \text{(K3)} \\ 2 & k \geq 4 & \text{(general type)} \end{cases}$$

to get $\kappa(X) = 1$ you have to allow C to be quite singular.

- ▶ General type surfaces are those with $\kappa(X) = 2$; for these the important numerical invariants are
 - ▶ $p_g = h^0(K_X) = h^0(\Omega_X^2) =$ geometric genus;
 - ▶ $q = h^0(\Omega_X^1) =$ irregularity;
 - ▶ $K_X^2 = c_1(X)^2$;

they are related by

- ▶ $p_g - q + 1 = \frac{1}{12}(K_X^2 + \chi_{\text{top}}(X))$ (Noether's formula);
- ▶ $p_g \leq \frac{K_X^2}{2} + 2$ (Noether's inequality);[†]

[†]For the above surface when $k = 4$ we have $K_X^2 = 2$ and $p_g = 3$, so that it is extremal for Noether's inequality. For the l -surface we have $K_X^2 = 1$ and $p_g = 2$ so that it is also extremal.

Theorem (Kollár-Shepherd-Barron-Alexeev) [KS-B], [A]: For general type surfaces with given numerical invariants there exists a moduli space \mathcal{M} with a canonical completion $\overline{\mathcal{M}}$.

- ▶ As noted above the proof is via birational geometry. It describes in principle what the singularities of a surface X corresponding to a boundary point in $\overline{\mathcal{M}} \setminus \mathcal{M}$ can be; * for surfaces there is no description, nor examples that I know other than the work of [FPR], of the global structure of X .

*In [K] there is a fairly short list of the singularity types that can occur. However for isolated singularities within each type there is an invariant, the *index*, and there is to my knowledge no way to bound in practice what this can be. Partial exceptions to this are in [Hoc1], [Hoc2], [H1], and [H2] where moduli of pairs that involve plane curves are considered. For non-isolated singularities there is a glueing construction whose complexity is also not effectively bounded.

- ▶ Some guiding questions are
 - ▶ how does Hodge theory limit what the X 's can be?
 - ▶ which Hodge-theoretically possible degenerations are realized algebro-geometrically?
 - ▶ does the Hodge theoretic stratification capture the algebro-geometric one?

- ▶ Given a family $\mathcal{X}^* \xrightarrow{\pi} \Delta^*$ of smooth surfaces $X_t = \pi^{-1}(t)$ for $t \neq 0$, by the theorem there is defined a *unique* limit surface $X_0 = X$ that fills in the family $\mathcal{X} \rightarrow \Delta$ where the conditions
 - ▶ \mathcal{X} has *canonical singularities* over X_{sing} ;† and
 - ▶ \mathcal{X} is of relative general type and minimal (more precisely, the relative dualizing sheaf $\omega_{\mathcal{X}/\Delta}$ is \mathbb{Q} -Cartier and relatively ample).

The first condition is local along X ; the second is global.

- ▶ In the above we are finessing some non-trivial technicalities involving base change etc. (cf. [K]).

†For normal \mathcal{X} this means that for U any open set in \mathcal{X} any holomorphic $\omega \in H^0(U \cap X_{\text{reg}}, K_{\mathcal{X}})$ satisfies

$$\int \omega \wedge \bar{\omega} < \infty$$

- ▶ As is the case for any analytic variety, $\overline{\mathcal{M}}$ has a *stratification*
 - ▶ $\overline{\mathcal{M}}$ is a union of irreducible subvarieties \mathcal{Z}_i ;
 - ▶ the incidence relation $\mathcal{Z}_j \subset \overline{\mathcal{Z}}_i$ means that singular varieties parametrized by \mathcal{Z}_i can degenerate further into those parametrized by \mathcal{Z}_j .
- ▶ The proof of the theorem does not suggest what the stratification should be; aside from [FPR] I know of no other examples where it has been analyzed.
- ▶ To give some flavor of how Hodge theory helps to organize the singularities, we note that $\mathcal{X}^* \rightarrow \Delta^*$ is topologically a fibre bundle over the circle, and thus there is a monodromy operator (here $t_0 \in \Delta^*$ is a base point)

$$T : H^2(X_{t_0}, \mathbb{Z}) \rightarrow H^2(X_{t_0}, \mathbb{Z}). \ddagger$$

\ddagger We will consider integral cohomology modulo torsion.

Denoting by

$$T = T_{\text{ss}} T_{\text{u}}$$

the Jordan decomposition of T where T_{ss} is semi-simple and T_{u} is unipotent with logarithm N , using analytic arguments [Bo], [S] arising from Hodge theory that extend the one given above in the case of elliptic curves one has the *monodromy theorem*[§]

$T_{\text{ss}}^m = 1$ (i.e., the eigenvalues of T are roots of unity)

$N^3 = 0$ (i.e., the Jordan blocks of T have length ≤ 2).

[§]Cf. [La] for a geometric proof.

A crude Hodge theoretic classification of the singularities of X is given by

- ▶ normal (a): $N = 0$
- ▶ normal (b): $N \neq 0, N^2 = 0$
- ▶ normal (c): $N \neq 0, N^2 \neq 0$
- ▶ non-normal (a): $N \neq 0$ but $N^2 = 0$ [¶]
- ▶ non-normal (b): $N^2 \neq 0$.

This may be refined by putting in the ranks of N and of N^2 .

A much finer invariant is given by including the conjugacy class of T_{ss} , usually expressed in terms of the spectrum. And if we include the extension data in the limiting mixed Hodge structure (LMHS), we obtain even more Hodge-theoretic information.*

[¶]In all examples I know, non-normal $\implies N \neq 0$.

*The above crude Hodge theoretic classification is extracted from the associated graded to the LMHS.

- ▶ *normal* (a): Then the monodromy is finite and the Hodge structures on the $H^2(X_t, \mathbb{C})$ extend across $t = 0$.[†] These include a number of quotient singularities; typically among them are those denoted

$$\frac{1}{d}(1, a), \quad \gcd(d, a) = 1$$

given by the quotient of \mathbb{C}^2 acted on by the cyclic group generated by

$$(x, y) \rightarrow (\zeta x, \zeta^a y)$$

where $\zeta = e^{2\pi i/d}$. Among these are the Wahl singularities $\frac{1}{n^2}(1, na - 1)$. For $n = 2$ and $a = 1$ this is a cone over a rational normal curve C in \mathbb{P}^4 . It is noteworthy in that for it $T = \text{Id}$.

[†]There is a Riemann extension theorem for a family of Hodge structures over Δ^* having finite monodromy.

- ▶ *normal (b): simple elliptic singularities.* (X, p) whose minimal resolution $(\tilde{X}, \tilde{C}) \rightarrow (X, p)$ is given by contracting an elliptic curve $\tilde{C} \subset \tilde{X}$ with $\tilde{C}^2 = -d$ where $d > 0$ is the *degree* of the elliptic singularity. Here the assumption that (X, p) is smoothable gives $1 \leq d \leq 9$. For $d \geq 3$, one may think of the cone over an elliptic normal curve in \mathbb{P}^{d-1} . One typically pictures such a singularity as



We are finessing the subtlety that in order to fit the desingularization \tilde{X} of X into a family $\tilde{\mathcal{X}} \rightarrow \tilde{\Delta}$ we have to do semi-stable-reduction (SSR), which involves a base change $t = \tilde{t}^m$ where $T_{\text{ss}}^m = \text{Id}$, and a normalization.

The fibre over the origin in $\tilde{\mathcal{X}} \rightarrow \tilde{\Delta}$ has \tilde{X} as one component. The other components are cyclic coverings Y_i of rational surfaces meeting \tilde{X} along the C_i , and all $p_g(Y_i) = 0$ so that $\lim_{t \rightarrow 0} H^0(\Omega_{X_t}^2)$ lives on \tilde{X} .[‡]

- ▶ For the normal (b) degeneration a similar argument applies except now for $\omega_t \in H^0(\Omega_{X_t}^2)$ we have $\lim \omega_t = \omega \in H^0(\Omega_{\tilde{X}}^2(C))$ and

$$\text{Res}_C(\omega) \in H^0(\Omega_C^1) \cong \mathbb{C}.$$

If there are e elliptic singularities, this argument leads to the bound

$$e \geq \text{rank } N.$$

[‡]An additional subtlety is that in order to have some degree of uniqueness one may want to allow $\tilde{\mathcal{X}}$ to have cyclic quotient singularities of a “simpler” type than the ones that were started with.

Another type of Hodge theoretic argument gives

$$e \leq \text{rank } N + 1.$$

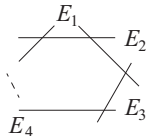
We will see that both bounds are sharp for I -surfaces.

We will also see that for I -surfaces the degrees of the elliptic singularities are determined by the eigenvalues of T_{ss} . An explicit such singular surface will be discussed in the third lecture.

- ▶ There are two types of restriction here:
 - (i) the cone is over an elliptic curve as opposed to a cone over a curve of genus $g \geq 2$;
 - (ii) the restriction $d \leq 9$ for the elliptic curve.

One can give an analytic explanation for (i); (ii) is the condition that the isolated singularity be smoothable.

- ▶ *normal (c): cusp singularity.* (X, p) where the minimal resolution $(\tilde{X}, D) \rightarrow (X, p)$ has for D a cycle of \mathbb{P}^1 's E_i with all $E_i^2 \leq -2$



and the least one $E_i^2 \leq -3$. It seems plausible, and may in fact be known, that the $-E_i^2$ are determined by the spectrum of $T_{ss,i}$:

- ▶ For the cusp, $\text{Res}_{E_i}(\omega)$ is a 1-form on \mathbb{P}^1 with log poles at the pairs of intersection points. At a point of $E_i \cap E_{i+1}$ the residues are opposite. Hence the p_g can drop by at most 1 in the limit.

- ▶ *non-normal (a)*: X has a smooth double curve C with pinch points (Whitney swallowtail given locally by $x^2y = z^2$);
- ▶ *non-normal (b)*: Informally stated, X has a nodal double curve with pinch points whose minimal resolution has a cycle of \mathbb{P}^1 's. These surfaces are frequently constructed by a glueing construction that will be illustrated below.

non-normal (a): Let $C \subset \mathbb{P}^2$ be a smooth plane quartic having an involution $\tau : C \rightarrow C$ with quotient $D = C/\tau$ an elliptic curve. Then $X = \mathbb{P}^2/\tau$ is a surface having a smooth double curve D with pinch points at the 4 branch points of $C \rightarrow D$.

The desingularization \tilde{X} of X is \mathbb{P}^2 and by pulling back 2-forms one has that

$$H^0(K_X) \cong H^0(\Omega^2(\log C))^- \cong H^0(\mathcal{O}_{\mathbb{P}^2}(1))^-$$

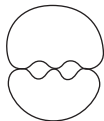
are the τ anti-invariant 2-forms on \mathbb{P}^2 having a log pole on C . Thus

$$\begin{cases} h^0(K_X) = 2 \\ K_X^2 = 1 \end{cases}$$

so that X “looks like” an I -surface. In fact X can be smoothed to such ([LR]).

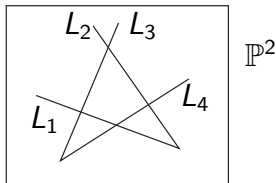
non-normal (b) ([LR]): This is a degeneration of the preceding example. Before explaining it we will give a general contextual comment.

An analogue of the “most degenerate,” meaning no equisingular deformations, $g = 2$ curve



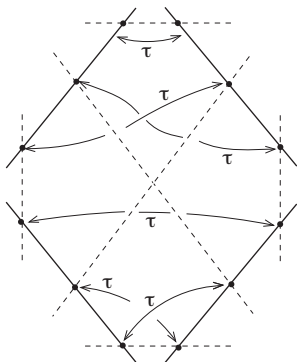
or \$ (dollar bill curve)

is conjecturally the surface



where to obtain the involution τ we identify L_1 and L_2 by

$\left\{ \begin{array}{l} 12 \longleftrightarrow 21 \\ 13 \longleftrightarrow 24 \\ 14 \longleftrightarrow 23 \end{array} \right\}$ and similarly for L_3 and L_4 . The actual construction is given by the picture



We will return to this example later.

B. Hodge theory

- ▶ Traditionally there have been two principal ways in which Hodge theory interacts with algebraic geometry:
 - ▶ topology; as previously noted many of the deeper aspects of the topology of an algebraic variety X are proved via Hodge theory;[§]
 - ▶ geometry; the Hodge structure on cohomology and its 1st order variations have been used to study the geometry of an algebraic variety X , especially the algebraic cycles that lie in X .

[§]This was true initially when X is smooth. Using mixed Hodge theory it is now the case when X is arbitrary (singular, non-complete or both), and as will be discussed below it is also the case when we have a degeneration $X_t \rightarrow X$ leading for example to a proof of the above monodromy theorem and construction of the definition of $\lim_{t \rightarrow 0} H^n(X_t)$. There is also a very rich and beautiful Hodge theory associated to isolated hypersurface singularities.

A central point of these lectures is to possibly add a third point to this list:

- ▶ it is now well understood how Hodge structures can degenerate to a *limiting mixed Hodge structure* [CKS], [CK] and [KR], [KPT], [R]; this can then be used to guide and complement the study of algebraic varieties acquiring singularities, especially as occurs in moduli.
- ▶ What is meant by a Hodge structure (HS), a mixed Hodge structure (MHS) and a limiting mixed Hodge structure (LMHS)?

- ▶ traditionally a HS or a MHS was given by a period matrix

$$\| \int_{\Gamma_\alpha} \omega_i \|$$

where the ω_i are rational (meromorphic) differential forms on an algebraic variety X and the Γ_α are cycles (including relative ones)[¶]; when X is smooth and the ω_i are regular (holomorphic) n -forms this gives the part

$$H^0(\Omega_X^n) = H^{n,0}(X) \subset H_{\text{DR}}^n(X) \cong H^n(X, \mathbb{C})$$

of the cohomology of X .

[¶]Classically (Riemann, Picard, Lefschetz, . . .) there were differentials of 1st, 2nd and 3rd kinds. It is now understood that the first kind deals with the holomorphic part of the Hodge theory of smooth varieties, the second kind with the full cohomology of smooth varieties and the third with the mixed Hodge theory of singular varieties.

As noted above, when $n = \dim X = 1, 2$ using conjugation and the cup product in cohomology $H^0(\Omega_X^n)$ determines the *Hodge decomposition*

$$\begin{cases} H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X) \\ \overline{H^{p,q}(X)} = H^{q,p}(X) \end{cases}$$

on cohomology, where

$$H^{p,q}(X) = \left\{ \begin{array}{l} \text{cohomology classes represented by} \\ C^\infty \text{ differential forms of type } (p, q) \end{array} \right\}$$

- ▶ One now defines a *Hodge structure* (V, F^\bullet) of weight n to be given by a \mathbb{Q} -vector space V and a decreasing *Hodge filtration*

$$F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}}$$

that satisfies

$$F^p \oplus \bar{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}$$

for $0 \leq p \leq n$.^{||} The relation

$$\begin{cases} F^p = \bigoplus_{p' \geq p} V^{p',q} \\ V^{p,q} = F^p \cap \bar{F}^q \end{cases}$$

gives the 1-1 correspondence between Hodge filtrations and Hodge decompositions

$$\begin{cases} V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \\ \bar{V}^{p,q} = V^{q,p}. \end{cases}$$

^{||}One may think of F^p as represented by differential forms of degree n having in holomorphic local coordinates *at least* p dz_i 's.

The reason for using Hodge filtrations is that the $F^p(X)$ vary holomorphically with X . In practice there will also be a lattice $V_{\mathbb{Z}} \subset V$ that represents integral cohomology.

- ▶ When X is of dimension n the cup products on cohomology and relations extending

$$\begin{cases} \int_X \omega \wedge \omega' = 0 & \text{(because } \omega \wedge \omega' = 0) \\ c_n \int_X \omega \wedge \bar{\omega} > 0 & \text{(because } c_n \omega \wedge \bar{\omega} > 0) \end{cases}$$

for holomorphic n -forms lead to the definition of a *polarized Hodge structure* (V, Q, F^\bullet) where

$$Q : V \otimes V \rightarrow \mathbb{Q}, \quad Q(u, v) = (-1)^n Q(v, u)$$

and the two *Hodge-Riemann bilinear relations*

- (I) $Q(F^p, F^{n-p+1}) = 0$;
- (II) $i^{p-q} Q(V^{p,q}, \bar{V}^{p,q}) > 0$

are satisfied.

- ▶ A *mixed Hodge structure* is given by $(V, W_\bullet, F^\bullet)$ where the increasing weight filtration

$$W_0 \subset W_1 \subset \cdots \subset W_m$$

is defined over \mathbb{Q} , and where the Hodge filtration F^\bullet induces on the graded quotients

$$\mathrm{Gr}_n^w V = W_n(V)/W_{n-1}(V)$$

a Hodge structure of weight n .

- ▶ The basic results connecting Hodge theory to the cohomology of algebraic varieties are
 - ▶ for X smooth and complete, $H^n(X, \mathbb{Q})$ has a Hodge structure of weight n (Hodge);

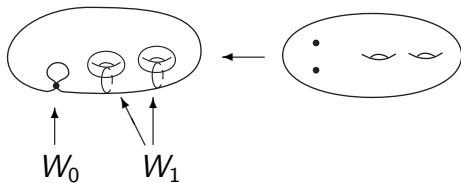
- ▶ As noted above, for $m = n = 1$ the HS is determined by the period matrix

$$\Omega = \left\| \int_{\gamma_i} \omega_\alpha \right\| \begin{cases} \omega_\alpha \in H^0(\Omega_X^1) & (\dim = g) \\ \gamma_i \in H_1(X, \mathbb{Z}) & (\cong \mathbb{Z}^{2g}) \end{cases}$$



- ▶ For $m = n = 2$ the HS on $H^2(X)$ is determined by $H^0(\Omega_X^2) = F^2$ by $F^1 = F^{2\perp}$; as in the curve case $H^0(\Omega_X^2)$ is given by the period matrix for the holomorphic 2-forms. Thus for both curves and surfaces the PHS is determined by the classical period matrix.

- ▶ For a general complete algebraic variety X , $H^m(X, \mathbb{Q})$ has a mixed Hodge structure where the weight filtration is $W_0 \subset \cdots \subset W_m$ (Deligne).

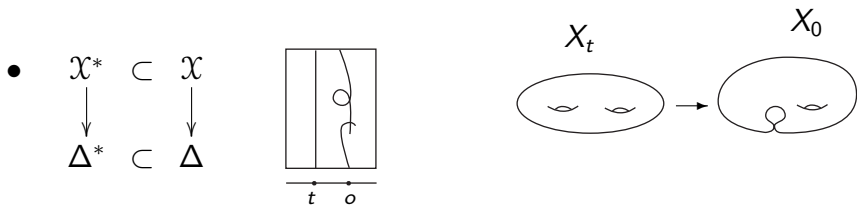


- ▶ The use of Hodge theory to study a degenerating family $X_t \rightarrow X_0 = X$ of algebraic varieties leads to the notion of a *limiting mixed Hodge structure* $(V, W(N), F^\bullet)$. Here the *monodromy weight filtration* $W(N)$ is constructed from the logarithm N of the unipotent part of monodromy and is the unique filtration

$$W_0(N) \subset W_1(N) \subset \cdots \subset W_{2n}(N)$$

satisfying

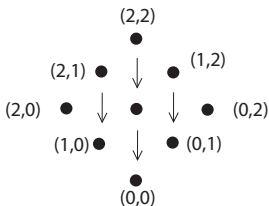
$$\begin{cases} N : W_k(N) \rightarrow W_{k-2}(N) \\ N^k : W_{n+k}(N) \xrightarrow{\sim} W_{n-k}(N). \end{cases}$$



► monodromy $T : H^n(X_t) \rightarrow H^n(X_t)$

$$\begin{cases} T = T_{ss} T_u & \text{(Jordan decomposition)} \\ T_{ss}^k = I, T_u = e^N \text{ with } N^{n+1} = 0. \end{cases}$$

- ▶ LMHS is given by a MHS $\left\{ (V, W(N), F_{\lim}^\bullet) \right.$
 $\left. \text{where } N : F_{\lim}^p \rightarrow F_{\lim}^{p-1} \right.$
- ▶ $\text{Gr}(\text{LMHS}) \cong \bigoplus_{\ell=0}^{2n} H^\ell$ where H^ℓ is a HS of weight ℓ —
 picture is a Hodge diamond. Here $n = 2$ and N is the
 vertical arrows — the dots are the $H^{p,q}$'s



- ▶ We will set $h^{p,q}$ = dimension of the (p, q) dot.
- ▶ There will also be a Q in the picture.

Theorem (Schmid)** Given $\mathcal{X} \rightarrow \Delta$ as above

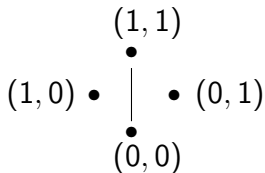
$$\lim_{t \rightarrow 0} H^m(X_t) = \text{LMHS}.$$

Proof is a combination of

- ▶ Lie theory
- ▶ complex analysis
- ▶ differential geometry

**Cf. [S], [CKS] and [CK] in the references. An algebraic approach may be found in [PS].

- ▶ For the above example the LMHS is

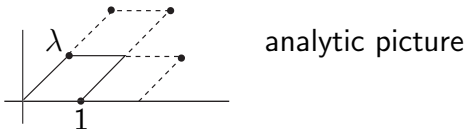


- ▶ in general the solid lines in the diagram in the introduction represent degenerations with $N \neq 0$. For these the genus of the normalizations drop by rank N .

Example:



▶ $X = \mathbb{C}/\Lambda$, $\Lambda = \{1, \lambda\}$

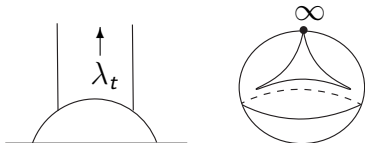


λ determined up to $\lambda \rightarrow \frac{a\lambda+b}{c\lambda+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

▶ $\mathcal{M}_1 \cong \mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$, $\mathbb{H} = \{\lambda : \mathrm{Im} \lambda > 0\}$

- ▶ the space of PHS's is $\mathbb{H} \subset \mathbb{P}^1$, $V = \begin{pmatrix} * \\ * \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,
 $F^1 = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \in \mathbb{P}^1$, HR II $\iff \text{Im } \lambda > 0$. $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and as
 $\lambda \rightarrow i\infty$ we have $F^1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = F_{\text{lim}}^1$.^{††}

$$\lambda_t = \frac{\log t}{2\pi i}$$



^{††}This picture is not indicative of what happens in the non-classical case.

How does Lie theory enter?

- ▶ *Period domain* $D = \{F^\bullet = \text{flag } \{F^n \subset \dots \subset F^0 = V_{\mathbb{C}}\} \text{ in } V_{\mathbb{C}} : (V, Q, F^\bullet) = \text{PHS}\}$
- ▶ *compact dual*
 $\check{D} = \{F^\bullet \text{ is a flag with } Q(F^p, F^{n-p+1}) = 0\}$
- ▶ $G = \text{Aut}(V, Q) = \mathbb{Q}$ -algebraic group
- ▶ $G_{\mathbb{R}}$ acts transitively on D and $G_{\mathbb{C}}$ acts transitively on \check{D} so that we have

$$D = G_{\mathbb{R}}/H \text{ with } H \text{ compact}$$

\cap

$$\check{D} = G_{\mathbb{C}}/P \text{ with } P \text{ parabolic } \begin{pmatrix} * & * & * & * \\ \circ & * & * & * \\ \circ & \circ & * & * \\ \circ & \circ & \circ & * \end{pmatrix}$$

where $D =$ open $G_{\mathbb{R}}$ -orbit in \check{D} .

Example:

$m=1$: $D = \mathrm{Sp}(2g, \mathbb{R}) / \mathcal{U}(g) = \mathcal{H}_g$ where $g = h^{1,0}$

$m=2$: $D = \mathrm{SO}(2k, \ell) / \mathcal{U}(k) \times \mathrm{SO}(\ell)$ where $k = h^{2,0}$, $\ell = h^{1,1}$

► *Classical case:*

$D =$ Hermitian symmetric domain (HSD)

\parallel

G_R / K , $K =$ maximal compact.

Two classical cases are

$m = 1$ (curves, abelian varieties)

$m = 2$ is HSD $\iff k = 1$ (K3's)

thus $h^{2,0} \geq 2$ is non-classical.

For $n \geq 3$ and X Calabi-Yau, the D corresponding to $H^n(X)$ is non-classical.

- ▶ Period domains have sub-domains corresponding to PHS's with additional structure; e.g.,

$$\begin{array}{ccc}
 D' & \subset & D \\
 \parallel & & \\
 \{ \text{reducible PHS's} \} & & \\
 \{ \text{that are } \oplus\text{'s} \} & &
 \end{array}$$

This is what the dotted lines represent in the diagram in the first lecture for $\overline{\mathcal{M}}_2$.

- ▶ In general one has Mumford-Tate sub-domains of D , defined to be those PHS's with a given algebra of Hodge tensors.

- ▶ *period mappings* arise from holomorphic mappings

$$\Phi : B \rightarrow \left\{ \begin{array}{l} \text{equivalence} \\ \text{classes of} \\ \text{PHS's} \end{array} \right\} = \Gamma \backslash D$$

where B is a complex manifold and $\Gamma \subset G_{\mathbb{Z}}$ contains the monodromy group; think of B as the parameter space for a family of smooth algebraic varieties X_b , $b \in B$, whose cohomology groups can be identified with $H^n(X_{b_0})$ for a base point $b_0 \in B$ up to the action of $\pi_1(B, b_0)$ on $H^n(X_{b_0})$.

Example: As noted above the first non-classical case is weight $n = 2$ when $h^{2,0} = 2$. In this case D has an invariant contact structure and any period mapping Φ is an integral of that structure — this means that if the contact structure is given by a 1-form θ , which is invariant up to scaling by $G_{\mathbb{R}}$, then

$$\Phi^*(\theta) = \Phi^*(d\theta) = 0.$$

In general the differential constraint satisfied by period mappings in the non-classical case is the basic new phenomenon that occurs. Thus $\Phi(\mathcal{M})$ cannot contain an open set, Γ need not be arithmetic, etc.

If we think of Φ as given locally by a holomorphically varying $2 \times k$ matrix Ω satisfying HRI in the form $\Omega Q^t \Omega = 0$, then the 2×2 matrix $d\Omega Q^t \Omega = -{}^t(d\Omega Q^t \Omega)$ is skew symmetric; writing

$$d\Omega Q^t \Omega = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

the θ gives the contact structure.

Elaborating a bit on the above, in general, in the non-classical case there is a non-trivial homogeneous sub-bundle $E \subset TD$ such that the differential of any period mapping satisfies

$$\Phi_* : T\mathcal{M} \rightarrow E \subset T(\Gamma \backslash D); \ddagger$$

as noted above the image can never be an open set in $\Gamma \backslash D$. Moreover, although it is always the case that

$$\text{vol } \Phi(\mathcal{M}) < \infty,$$

It can happen that $\Gamma \subset G_{\mathbb{Z}}$ is a thin subgroup, i.e., a subgroup with $[\Gamma : G_{\mathbb{Z}}] = \infty$.

$\ddagger E$ is defined by the differential constraint

$$\dot{F}^p \subseteq F^{p-1}.$$

- ▶ Using Lie theory the set of equivalence classes of LMHS's has been classified [KR], [KPR], [R]; they form a stratified object, and as noted above one may informally say that we know how Hodge structures degenerate; the strategy is then to use this information to help understand how algebraic varieties degenerate.

Examples:

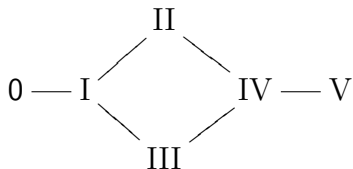
- ▶ For $n = 1$ the stratification may be pictured as

$$I_0 - I_1 - \dots - I_g .$$

reflecting (for $g = 2$)



- ▶ For $n = 2$ the picture is

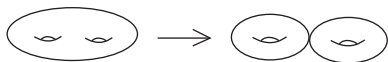


Note: The Roman numerals reflect the associated graded to the equivalence classes of LMHS's. The stratification is linear and transitive in the classical case, transitive but not linear in the non-classical $n = 2$ case, and neither transitive nor linear in the general $n \geq 3$ case.

Within each of the above strata there is a refined stratification given by PHS's with "additional" Hodge tensors (Mumford-Tate sub-domains).

Example: Curves with $N = 0$.

Using the Mumford-Tate sub-domain given by PHS's that are non-trivial direct sums $/\mathbb{Z}$ one may Hodge-theoretically detect the degeneration



which has trivial monodromy.

In general, in the stratification of $\overline{\mathcal{M}}_2$ pictured in the first lecture the solid lines refer to degenerations where $N \neq 0$ and the dotted lines to degenerations where the Jacobian of the normalization splits further into a direct sum of principally polarized abelian varieties.

Example: $n = 2$.

At least in some examples one may Hodge theoretically detect a degeneration to a $\frac{1}{d}(1, a)$ singularity where $N = 0$.

The first case is the Wahl singularity $\frac{1}{4}(1, 1)$ where $T = \text{Id}$; then for I -surfaces there are indications that the image in the period domain picks up an extra Hodge class.

III.A Generalities on Hodge theory and moduli

- ▶ The first point is that there is a moduli space

$$\mathcal{H} = \Gamma \backslash D$$

for Γ -equivalence classes of PHS's (think of $D = \mathbb{H}$ and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$).

- ▶ The second point is that there is a *period mapping*

$$(*) \quad \Phi : \mathcal{M} \rightarrow \mathcal{P} \subset \mathcal{H}$$

where Γ contains the *global monodromy group* given by the image of the monodromy representation

$$\rho : \pi_1(\mathcal{M}) \rightarrow G_{\mathbb{Z}}.$$

Note: There are several interesting technicalities here, some dealing with the singularities of \mathcal{M} and some with the presence of X 's with extra automorphisms; and of course the general X may not be smooth but rather will have canonical singularities. In the example of the I -surface to be discussed next these issues can be addressed directly.

- ▶ The following are statements that have been proved at the set-theoretic level and under various assumptions; proofs of the full results are a work in progress.

Theorem A: *The image $\mathcal{P} = \Phi(\mathcal{M}) \subset \mathcal{H}$ is a quasi-projective variety that has a canonical projective completion $\overline{\mathcal{P}}$. Set-theoretically, $\overline{\mathcal{P}}$ is obtained from \mathcal{P} by attaching the associated graded to the limiting mixed Hodge structures arising from Φ in $(*)$.**

*Cf. [BBT] and [BK] for an interesting “model-theoretic” proof of the result that \mathcal{P} is quasi-projective.

We shall call $\overline{\mathcal{P}}$ the *Satake-Baily-Borel (SBB) completion* of \mathcal{P} .

Note: For the experts, the proof of the theorem (if completed) will have the following implication: Let

$$Y \xrightarrow{f} Z$$

be a morphism of smooth, projective varieties and assume that the relative dualizing sheaf $\omega_{Y/Z}$ is a line bundle.

Then

(**) $\Lambda =: \det f_*(\omega_{Z/Y})$ is semi-ample.

It is known that Λ is nef, and if local Torelli holds for a general point of Z , then Λ is big.

One may ask: Once you know that Λ is big and nef, why don't the standard methods of birational geometry (the minimal-model-program, including the base-point-free theorem) apply to give a proof? The interesting answer is that the signs needed in the base-point-free theorem are pretty much opposite to those that may occur in the above situation. The connection of this statement with the above theorem is that if

$$\overline{\mathcal{X}} \xrightarrow{f} \overline{\mathcal{M}}$$

is a versal family of general type varieties, then

$$\overline{\mathcal{P}} = \text{Proj}(\Lambda).$$

Thus assuming (***) one may define the SBB completion of the image of the period mapping without using any Hodge theory.

- The second work-in-progress result is

Theorem B: *The period mapping Φ extends to*

$$(**) \quad \Phi_e : \overline{\mathcal{M}} \rightarrow \overline{\mathcal{P}}.$$

The above two structural statements provide a conceptual framework for the use of Hodge theory to partner with and help guide the standard algebro-geometric methods used to study the boundary structure for the KSBA moduli spaces for surfaces of general type. How this works will now be illustrated.

III.B I -surfaces and their period mappings

- ▶ Murphy's law (Vakil) — whatever nasty property a scheme can have already occurs for the moduli spaces of general type surfaces — thus unlike curves one should select “special” surfaces to study — in geometry extremal cases are frequently interesting — Noether's inequality

$$\rho_g(X) \leq \frac{K_X^2}{2} + 2$$

suggests studying surfaces close to extremal — the 1st non-classical case is

Definition: An I -surface X is a regular ($q(X) = 0$) general type surface that satisfies

$$\rho_g(X) = 2, K_X^2 = 1.$$

- ▶ One studies general type surfaces via their pluri-canonical maps

$$(h) \quad \varphi_{mK_X} : X \dashrightarrow \mathbb{P}H^0(mK_X)^* \cong \mathbb{P}^{P_m-1}$$

and pluricanonical rings $R(X) = \bigoplus H^0(mK_X)$.

- ▶ Instead of (h) it is frequently better to use weighted projective spaces corresponding to when we add new generators to $R(X)$ — from Kodaira-Kawamata-Viehweg vanishing one has for the l -surface

$$P_m(X) = m(m-1)/2 + 3, \quad m \geq 2$$

and

$$\begin{aligned} \varphi_{K_X} : X \dashrightarrow \mathbb{P}^1, \quad & \text{using adjunction } |K_X| \\ & = \text{pencil of hyperelliptic curves} \end{aligned}$$

$$\varphi_{2K_X} : X \rightarrow \mathbb{P}(1, 1, 2) \hookrightarrow \mathbb{P}^3 \text{ of degree 2;}$$

$$\varphi_{5K_X} : X \hookrightarrow \mathbb{P}(1, 1, 2, 5) \hookrightarrow \mathbb{P}^{12} \text{ an embedding.}$$

- ▶ If $C \in |K_X|$ is a general smooth fibre, then again by adjunction

$$2K_X|_C = K_C.$$

Thus the images $\varphi_{2K_X}(C) = \varphi_{K_C}(C)$ are canonical curves. The I -surface was important classically since φ_{4K_X} is not birational, while for any general type surface φ_{5K_X} always is birational.

- ▶ Equations/picture

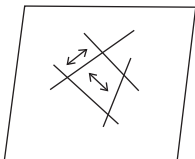
- ▶ The equation of X is $z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y)$ (weighted complete intersection in $\mathbb{P}(1, 1, 2, 5)$).

$$\left\{ \begin{array}{l} \text{▶ } \left\{ \begin{array}{l} \text{Diagram of a cone } V \text{ with vertex } P \text{ and a curve } C \text{ on its surface.} \\ \mathbb{P}(1, 1, 2) \hookrightarrow \mathbb{P}^3 \text{ given by} \\ (t_0, t_1) \hookrightarrow [t_0^2, t_0 t_1, t_1^2, y] \\ X = 2:1 \text{ map branched over } P \text{ and } V \in |\mathcal{O}_{\mathbb{P}^3}(5)| \end{array} \right. \end{array} \right.$$

- ▶ The canonical pencil $|K_X|$ is given by the two sheeted coverings of the lines L through the vertex in the quadric and branched over P and $L \cap V$.

Remark: Above we are assuming that X has at most canonical singularities. A general X is smooth. If we allow X to have a most Gorenstein singularities, then the above properties of the φ_{mK_X} for $m = 1, 2, 3$ remain valid (cf. [BHPV] and [FPR]).

Example: At the other extreme, for the surface



the equation is (cf. [LR])

$$z^2 = y(t_0^2 - y)^2(t_1^2 - y)^2.$$

Geometrically it is a double cover of a quadric cone in \mathbb{P}^3 branched over the vertex, a plane section, and two double plane sections. A general curve in $|K_X|$ is



- ▶ \mathcal{M}_I is smooth and
 - ▶ $\dim \mathcal{M}_I = h^1(T_X) = 28$
 - ▶ $\dim D_I = 57 = 2 \dim \mathcal{M}_X + 1$
- ▶ $\Phi = \mathcal{M}_I \rightarrow \Gamma_I \backslash D_I$ has Φ_* injective (local Torelli)

⇓

$$\Phi(\mathcal{M}_I) = \text{contact submanifold } \mathcal{P} \hookrightarrow \Gamma_I \backslash D_I$$

- ▶ It is suspected but not proved that global Torelli holds in the sense that $\Phi : \mathcal{M}_I \rightarrow \mathcal{P}$ is locally 1-1 and globally has degree 1.
- ▶ Γ_I is arithmetic — not known is whether $\Gamma = G_{\mathbb{Z}}$ or not.

Note: For X smooth we have $h^2(T_X) = 0$. It does not yet seem to be known if this remains true when X has canonical singularities. Here one must distinguish between the minimal model X_{\min} of X (no -1 curves) and the canonical image X_{can} of X (the -2 curves have been contracted to ordinary double points).

Before continuing with the I -surface we give a couple of global remarks about the stratification of the space of $\text{Gr}(\text{LMHS})$'s.

- ▶ For curves with $\Gamma_I = \text{Sp}(2g, \mathbb{Z})$ we have for LMHS's

$$\begin{array}{ccccccc}
 & I_0 & \text{---} & I_1 & \text{---} & I_2 & \text{---} \cdots \text{---} & I_g \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
 & \mathcal{H}_g & & \mathcal{H}_{g-1} & & \mathcal{H}_{g-2} & & \mathcal{H}_0
 \end{array}$$

- ▶ note that I_{g-m} corresponds to $[N]$ with $N^2 = 0$, rank $N = m$.

$$\begin{array}{ccccc}
 & & m & & \text{Gr}_2 \\
 & & \bullet & & \\
 & & \downarrow & & \\
 g-m & \bullet & & \bullet & \text{Gr}_1 \quad (\cong H^1(\tilde{C})) \\
 & & \downarrow & & \\
 & & \bullet & & \text{Gr}_0
 \end{array}$$

For each boundary component we have the stratification

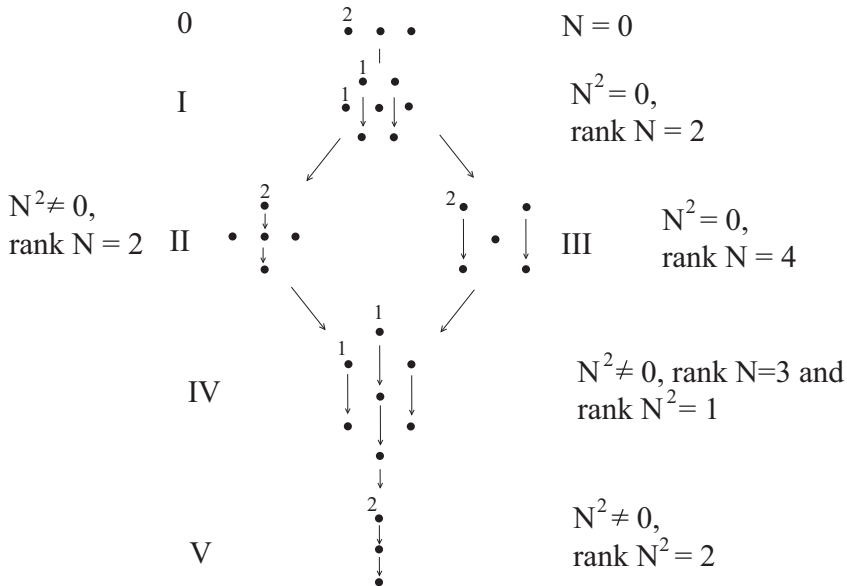
$$H^1 = \bigoplus H_i^1.$$

The composite of these induces a stratification of $\overline{\mathcal{M}}_g$ by

$$\{\# \text{ nodes}, \# \text{ components}\}.$$

Of course this is just the beginning of the story of $\overline{\mathcal{M}}_g$.

- ▶ For surfaces with $p_g = 2$ a refinement of the earlier picture by $N \neq 0$, $N^2 \neq 0$ of the classification of $\text{Gr}(\text{LMHS})'s/\mathbb{Q}$ is



- ▶ { For the refined Hodge-theoretic stratification of $\text{Gr}(\text{LHHS}/\mathbb{Z})$'s we use $T_{\text{ss}} \rightarrow \{\text{conjugacy class } [T_{\text{ss}}]$ of T_s in $\Gamma\}$. Within each of these strata we use Mumford-Tate sub-domains appearing in $\text{Gr}(\text{LMHS})$'s in $\overline{\mathcal{M}}_I$.

- ▶ Returning to the I -surface we begin by considering the Gorenstein part $\overline{\mathcal{M}}_I^{\text{Gor}} \subset \overline{\mathcal{M}}_I$ — one reason for this is the result

if $X_t \rightarrow X$ is a KSBA degeneration of a surface where all the singularities of X are isolated and non-Gorenstein, then $N = 0$.

Hence only Gorenstein singularities can non-trivially contribute to the LMHS/ \mathbb{Q} .*

*We recall that for a normal surface X , Gorenstein means that the canonical Weil divisor class K_X is a line bundle. In general the *index* is the least integer m such that mK_X is a line bundle. For example, the $\frac{1}{4}(1, 1)$ singularity has index 2.

Heuristically the reason for this is the following.[†]

- ▶ For the resolution of the singularity of a non-Gorenstein slc singularity one has a divisor $D = \sum E_i$ where the E_i are \mathbb{P}^1 's and the dual graph is a chain or perhaps a Dynkin-like diagram with forks; there are no cycles.
- ▶ For a KSBA degeneration $X_t \rightarrow X$ with $\tilde{X} \rightarrow X$ a desingularization, and $\omega_t \in H^0(\Omega_{X_t}^2)$, the limit $\lim_{t \rightarrow 0} \omega_t = \omega \in H^0(\Omega_{\tilde{X}}^2(\log D))$ and then $\text{Res}_D \omega$ gives a meromorphic 1-form on the E_i 's with log poles on $E_i \cap E_j$ and thus $\text{Res}_D \omega = 0$. It follows that $\rho_g(\tilde{X}) = \rho_g(X)$, which then implies that $N = 0$.

[†]To an analyst this might be considered to be a proof.

The following results from coupling the classification in FPR with the analysis of the LMHS's in the various cases.

Theorem B: *The Hodge theoretic stratification of $\overline{\mathcal{M}}$ given by the above diagram via the extended period mapping uniquely determines the stratification of $\overline{\mathcal{M}}_I^{\text{Gor}}$.*

- ▶ Rather than display the whole table the following is just the part for simple elliptic singularities (types I_k and III_k) — they have $N^2 = 0$ since for the semi-stable-reduction (SSR) of a degeneration only double curves (and no triple points) occur — all of the other types occur if we include cusp singularities.

- ▶ In the following
 - ▶ X is irreducible (since K_X is a line bundle with $K_X^2 = 1$ and any component of X will have positive K_X^2)[‡]
 - ▶ $d_i =$ degree of elliptic singularity
 - ▶ $k = \#$ elliptic singularities — in general, as previously noted using Hodge theory one may show that $k \leq p_g + 1$
 - ▶ $\tilde{X} =$ minimal desingularization of X — in a SSR given by $\tilde{\mathcal{X}} \rightarrow \tilde{\Delta}$ the surface \tilde{X} will appear as one component of the fibre over the origin.

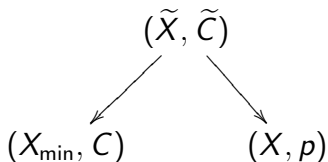
In the following table, in the 1st column subscripts denote the degrees of the elliptic singularities, which are uniquely determined by the $[T_{ss}]$'s — will explain the $\sum(9 - d_i)$ column below.

[‡]There are reducible non-Gorenstein KSBA degenerations of I -surfaces.

stratum	dimension	minimal resolution \tilde{X}	$\sum_{i=1}^k (9 - d_i)$	k	codim in $\tilde{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\tilde{X})=2$	8	1	9
$III_{2,2}$	12	rational surface	14	2	16
$III_{1,2}$	11	rational surface	15	2	17
$III_{1,1,R}$	10	rational surface	16	2	18
$III_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$III_{1,1,2}$	2	ruled surface with $\chi(\tilde{X})=0$	23	3	26
$III_{1,1,1}$	1	ruled surface with $\chi(\tilde{X})=0$	24	3	27

Note that the last column is the sum of the two columns preceding it.

Example: For I_2 the picture is



Here, $p =$ isolated normal singular point on X , $\tilde{C} =$ curve on \tilde{X} with $\tilde{C}^2 = -2$ and that contracts to p — from Hodge theory

$$2 = p_g(\tilde{X}) + g(\tilde{C}) \text{ and } p_g(\tilde{X}) = 1$$

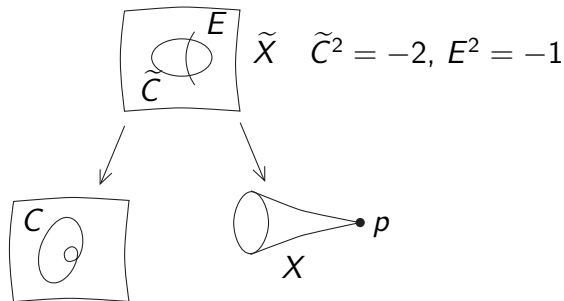
we see that $g(\tilde{C}) = 1$ (simple elliptic singularity)

- ▶ Heuristic reasoning suggests that $Hg^1(\tilde{X})$ has a \mathbb{Z}^2 summand with intersection form

$$\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix};$$

suppose we assume the basis classes are effective.

- ▶ Hodge theory then suggests the picture



$$\begin{cases} X_{\min} = K3 \\ C^2 = 2 \end{cases}$$

$$\implies X_{\min} \xrightarrow{2:1} \mathbb{P}^2 \text{ branched over } D$$



This family of K3's is studied in [L].

▶ LMHS has $\begin{cases} \text{Gr}_2 \cong H^2(X_{\min})_{\text{prim}} \\ \text{Gr}_3 \cong H^1(\tilde{C})(-1) \end{cases}$

- ▶ # of PHS's of type $\text{Gr}_3 \oplus \text{Gr}_2 = 19 + 1 = 20$ which suggests that for the boundary component of $\overline{\mathcal{M}}_l$, we have $\text{codim} = 8$.
- ▶ How to get this number? First approximation to the fibre over the origin in a SSR is blowing up p in X to have

$$\tilde{X} \cup_{\tilde{C}} (m\mathbb{P}^2)$$

where $\tilde{C} \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ and m is the multiplicity of p . Now one does base change and normalization to arrive at a SSR. Rather than proceed this way suppose we just take $\tilde{X} \cup_{\tilde{C}} \mathbb{P}^2$ and ask what we need to do to smooth this surface.

For this have to blow up $9 - (-\tilde{C}^2) = 7$ points on \tilde{C} to obtain triviality of the infinitesimal normal bundle as a necessary condition for smoothability. This suggests that

- ▶ The extension data for the LMHS contains a factor

$$\mathrm{Ext}_{\mathrm{MHS}}^1(\mathrm{Hg}^1(\tilde{\mathbb{P}}^2), H^1(\tilde{C})(-1)) \cong \oplus J(\tilde{C})$$

in which the seven points appear.

Fibre over origin in a several parameter SSR ([AK]) is given by blowing up seven points on \tilde{C} — this is a del Pezzo surface.

- ▶ Hodge theory suggests where to look — the seven parameters arise from the possible extension data for $\text{GR}(\text{LMHS})$ — and following FPR one may go back and prove things algebraically.
- ▶ Finally, *what about the non-Gorenstein singularities?* This is work in progress. To begin with from the list of normal slc singularities of surfaces these are mostly quotient singularities. Moreover their contribution to monodromy is $T = T_{\text{ss}}$ which is of finite order. For those for which T_{ss} is non-trivial, one might say that at least they are detected Hodge-theoretically.

However as previously mentioned there is at least one notable exception to this, namely the *Wahl singularity* $\frac{1}{4}(1, 1)$ where $T = \text{Id}$.^{*} For I -surfaces it may be the case that

- ▶ the period mapping gives $\Phi : \Delta \rightarrow D$ (there is no need to quotient by a Γ);[†]
- ▶ the point $\tilde{\Phi}(o) \in D$ is a PHS with an extra Hodge class arising from $\text{Hg}^1(\tilde{X})$, where $\tilde{X} \rightarrow X$ is the minimal desingularization of X .

^{*}This is the quotient of \mathbb{C}^2 by $(x, y) \rightarrow (x, \zeta y)$, $\zeta = e^{2\pi i/4}$.

[†]This is OK; for a family $\mathcal{X}^* \rightarrow \Delta^*$ with $T = \text{Id}$ the period mapping extends across $t = 0$ to give a point $\Phi(0)$ in D (i.e., an honest PHS).

The second point raises an interesting issue. There is a standard cohomological formalism for computing the differential of the period mapping at a smooth X . Here we have a singular X where there is no monodromy and one needs to compute the map $T \text{Def } X \rightarrow T_{\Phi(0)} D$. This has yet to be done.

Conclusion

The SBB completion $\overline{\mathcal{P}}$ of the image of moduli under the period mapping gives an invariant that has a rich structure and that provides an important guide to the boundary structure of the moduli space. For a desingularization $\widetilde{\mathcal{M}}$ of \mathcal{M} the fibre of $\widetilde{\mathcal{M}} \rightarrow \overline{\mathcal{P}}$ maps to the extension data associated to a LMHS with fixed associated graded. The geometrically interpreted extension data suggests how one may carry out the SSR and the desingularization of $\overline{\mathcal{M}}$.

Thank you

References

- [FPR] {
- ▶ M. Franciosi, R. Pardini, and S. Rollenske, *Computing invariants of semi-log-canonical surfaces*, Math. Z. 280 no. 3-4 (2015), 1107–1123.
 - ▶ M. Franciosi, R. Pardini and S. Rollenske, *Gorenstein stable surfaces with $K_X^2 = 1$ and $p_g > 0$* , Math. Nachr. 290 (2017), no. 5-6, 794–814.
 - ▶ M. Franciosi, R. Pardini and S. Rollenske, *Log-canonical pairs and Gorenstein stable surfaces with $K_X^2 = 1$* , Compos. Math. 151 (2015), no. 8, 1529–1542.

- [AK] D. Abramovich and K. Karu, *Weak semistable reduction in characteristic 0*, Invent. Math. **139** (2000), 241–273.
- [A] V. Alexeev, *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810.
- [BBT] B. Bakker, Y. Brunebarbe, and J. Tsimerman, *0-minimal GAGA and a conjecture of Griffiths*, 2018, arXiv:1811.12230.
- [BKT] B. Bakker, B. Klingler, and J. Tsimerman, *Tame topology of arithmetic quotients and algebraicity of Hodge loci*, 2018. arXiv:1810.04801.
- [BHPV] W. Barth, K. Hulek, C. Peters and A. van de Ven, *Compact Complex Surfaces*, Ergeb. Math. Grezgeb. **4**, 436pp., Springer-Verlag, New York, 2004.

- [B] O. Benoist, *Réduction stable en dimension supérieure [d'après Kollár, Hacon–Xu...]*, Séminaire BOURBAKI 71^e année 2018–2019, no. 1155, 2019, arXiv:1904.03465.
- [C] F. Catanese, *A superficial working guide to deformations and moduli*, in *Handbook of Moduli, Vol. 1*, Adv. Lectures in Math. **25**, pp. 161–216, (G. Farkas and I. Morrison, eds.), Int. Press, Somerville, MA, 2013.
- [CK] E. Cattani and A. Kaplan, *Polarized mixed Hodge structure and the local monodromy of a variation of Hodge structure*, Invent. Math. **67** (1982), no. 1, 101–115.
- [CKS] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** no. 3 (1986), 457–535. MR 0840721. Zbl 0617.14005. Available at <https://doi.org/10.2307/1971333>.

- [Dem] J.-P. Demailly, *Analytic methods in algebraic geometry*, *Surveys of Modern Mathematics* **1**, International Press, Somerville, MA; Higher Education Press, Beijing, 2012. MR 2978333. Zbl 1271.14001.
- [GG] M. Green and P. Griffiths, *Positivity of vector bundles and Hodge theory*, 2018, arXiv 1803.07405.
- [GGLR] M. Green, P. Griffiths, R. Laza and C. Robles, *Completion of period mappings and ampleness of the Hodge bundle*, 2017, arXiv 1708.09524.
- [Hac1] P. Hacking, *Compact moduli of plane curves*, *Duke Math. J* **124**, no. 2 (2004), 213.
- [Hac2] P. Hacking, *Compact spaces of surfaces of general type*, in *Compact Moduli Spaces and Vector Bundles*, *Contemp. Math.* **564**, 1–18, Amer. Math. Soc., Providence, RI, 2012.
- [Hac3] P. Hacking, *Exceptional bundles associated to degenerations of surfaces*, 2011, arXiv:1107.2644.

- [H1] B. Hassett, *Stable log surfaces and limits of quartic plane curves*, Manuscr. Math. **100**, no. 4 (1999), 469–487.
- [H2] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), no. 2, 316–352.
- [Ho] E. Horikawa, *Algebraic surfaces of general type with small c_1^2 . IV*, Invent. Math. **50** (1979), 103–128.
- [KR] M. Kerr and C. Robles, *Classification of smooth horizontal Schubert varieties*, 2016, arXiv:1605.09003.
- [KPR] M. Kerr, G. Pearlstein, and C. Robles, *Polarized relations on horizontal $SL(2)$'s*, 2017, arXiv:1705.03117.

- [K] J. Kollár, *Moduli of varieties of general type*, in *Handbook of moduli, Vol. 2*, Adv. Lect. Math. 25, Int. Press, Somerville, MA, 2013, pp. 131–157.
- [KS-B] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), 299–338.
- [Ko] S. J. Kovács, *Singularities of stable varieties*, in *Handbook of Moduli, Vol. II*, Adv. Lectures in Math. **25**, pp. 159–204, (G. Farkas and I. Morrison, eds.), Int. Press, Somerville, MA, 2013.
- [La] A. Landman, *On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities*, Trans. Amer. Math. Soc. **181** (1973), 89–126.

- [L] R. Laza, *The KSBA compactification for the moduli space of degree two K3 pairs*, 2014.
- [LR] W. Liu and S. Rollenske, *Geography of Gorenstein stable log surfaces*, *Trans. Amer. Math. Soc.* **368** (2016), no. 4, 2563–2588.
- [P] V. P. Palamodov, *Deformation of complex spaces*, *Russian Math. Surveys* **31** (1976), no. 3, 129–197.
- [PS] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*, *Eggeb. Math. Grenzegeb.* **52**, Springer-Verlag, Berlin, 2008. MR 2393625. Zbl 1138.14002.
- [R] C. Robles, *Classification of horizontal $SL(2)$ s*, 2014, arXiv:1405.3163.
- [S] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, *Invent. Math.* **22** (1973), 211–319.