Hodge Theory and Moduli

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Outline

I. Introduction
II. Moduli
III. Hodge theory
IV. The extended period map — Theorems A and B
V. Examples of Theorem C

I. Introduction

- talk will be at the interface of the two topics
  - moduli and singularities
  - Hodge theory and degenerations of Hodge structures
- KSBA moduli space $\mathcal{M}$ for varieties X of general type — in this talk restrict to surfaces
  - local structure of X_{sing} is understood
  - global structure less so — one exception is work of Fransciosi-Pardini-Rollenske [FPR][†]
- moduli space of polarized Hodge structures (PHS's) and their degenerations is better understood
  - classification of limiting mixed Hodge structures/$\mathbb{Q}$ (LMHS's) and the incidence relations among them (cf. work of Brosnan, Kerr, Pearlstein, Robles)[‡]

[†]References at the end of section

[‡]References also at the end of this section.

- classical work on several variable degenerations of PHS's (Cattani-Kaplan-Schmid, also Kashiwara, ...)
- needed: LMHS's $/\mathbb{Z}$ — this will be discussed below

- goal of this talk is to relate the two topics — specifically to discuss and illustrate how to use Hodge theory to study the boundary structure of $\overline{\mathcal{M}}$
- three main results — definitions and notations to be explained
- we consider a period mapping

$$(*) \qquad \Phi : B \longrightarrow \Gamma\backslash D$$

B: smooth quasi-projective variety

$\Gamma\backslash D$: quotient of a period domain by a discrete group containing the monodromy group

Theorem A

Canonically constructed from the $(*)$ *there is an extension*

$$(**) \qquad \begin{array}{ccc} \Phi_e : \overline{B} & \longrightarrow & (\Gamma\backslash D)_\Phi \\ \cup & & \cup \\ B & \longrightarrow & \Gamma\backslash D \end{array}$$

where $\overline{B}$ = *smooth completion of* B *such that* $\overline{B}\backslash B = Z$ *is a normal crossing divisor. The image* $\Phi_e(\overline{B})$ *is a compact complex analytic variety over which the augmented Hodge bundle* Λ_e *is ample.*

For weights $n = 1$ and 2, including the cases of algebraic curves and surfaces, Λ_e is the usual Hodge line bundle.

Remarks: (i) Set-theoretically $\Phi_e(\overline{B})$ is obtained by adding to $\Phi(B)$ the associated graded to the variations of polarized mixed Hodge structures associated to monodromy cones along the strata of Z. For this reason we will call $(**)$ the *Satake-Baily-Borel* (SBB) completion of $(*)$.

(ii) The construction is necessarily a *relative* one — it depends on the Φ

(iii) In the non-classical case when D is not a Hermitian-symmetric domain (HSD), the construction of $\Phi_e(\overline{B})$ is by constructing local quasi-charts and using the full CKS theory to glue them together.[§]

[§]Even in the classical case this construction is quite different from the usual one.

The projectivity of $\Phi_e(\overline{B})$ is proved by extending the classical Kodaira theorem to the case where the variety and, especially, the metric and curvature have singularities. Bigness and nefness are relatively easier — ampleness is more subtle.

(iii) To further illustrate the non-classical nature of things, $\mathbb{C}(\Gamma\backslash D) = \mathbb{C}$ and Γ may be a *thin* matrix group — then $\mathrm{vol}(\Gamma\backslash D) = \infty$ although $\mathrm{vol}(\Phi(B)) < \infty$.

———— < ∘ > ————

For the next result we restrict to the case of general type algebraic surfaces and weight $n = 2$ PHS's.

- $\mathcal{M}$ =KSBA moduli space with canonical completion $\overline{\mathcal{M}}$
- to apply the above result to period mappings

$$\Phi : \mathcal{M} \to \Gamma\backslash D$$

and their completions, one traditionally uses blowing up/branched coverings (base change) to arrive at

$$\begin{array}{ccccc} B & \subset & \overline{B} & \xrightarrow{\Phi_e} & (\Gamma\backslash D)_\Phi \\ \downarrow & & \downarrow & \nearrow & \\ \mathcal{M} & \subset & \overline{\mathcal{M}} & & \end{array}$$

For the study of moduli it is desirable to descend Φ_e to the dotted arrow above.

“Theorem B”

The mapping Φ_e factors by the dotted arrow above.

The “ ” is because there are issues of finite monodromy groups and relatedly LMHS's$/\mathbb{Z}$ that have yet to be properly formulated and understood. What is established is that at the set-theoretic level and for $\overline{\mathcal{M}}^{\mathrm{Gor}}$ the dotted arrow mapping is defined. The proof is by a detailed analysis of Gorenstein KSBA degenerations. In the case at hand it contains an extension and refinement of classical results of Shah-Steenbrink and more recent work of Kollár-Kovács and others on the relation between semi-log-canonical (slc) and Du Bois singularities. With notations to be explained below, the result has as a consequence the isomorphism

$$I^{2,p}_{\mathrm{lim}} \cong I^{2-p,0}(X)(-p)$$

of Hermitian vector spaces.

Note: This is sometimes written as a non-canonical isomorphism $I^{p,0}_{\lim} \cong I^{p,0}(X)$.

Remark: In case Γ is arithmetic and assuming the existence of a fan, Kato-Usui have constructed toroidal-type completions $(\Gamma\backslash D)_{\mathrm{KU}}$, and in this situation it seems feasible that there will be a diagram

$$\begin{array}{ccc} \overline{B} & \xrightarrow{\Phi_{\mathrm{KU}}} & (\Gamma\backslash D)_{\mathrm{KU}} \\ & \searrow^{\Phi_e} & \downarrow \\ & & (\Gamma\backslash D)_{\Phi}. \end{array}$$

The dotted arrow means that the map is only defined on the image of Φ_{KU}. The K-U construction is absolute, not relative, and depends on Γ being arithmetic.

Remark: In the examples to be discussed, we will see that for normal Gorenstein degenerations we have

$$\begin{array}{ccc} & & (\Gamma\backslash D)_{\mathrm{KU}} \\ & \nearrow & \downarrow \\ \overline{\mathcal{N}} & \xrightarrow{\Phi_e} & (\Gamma\backslash D)_{\Phi} \end{array}$$

where here the dotted arrow means that the map Φ_e lifts up to finitely many choices — i.e., the extension data in the LMHS's is *discrete*; this is in contrast to the case of algebraic curves, and other classical cases.

- *Examples* — Interested in non-classical phenomena so consider algebraic surfaces X with
  - $q(X) = 0$ (regular)
  - $p_g(X) = 2$
  - small K_X^2; specifically equal to 1,2.

  Also will consider Noether extremal surfaces.
- Those with $K_X^2 = 1$ are classical, and the boundary structure of their moduli space $\mathcal{M}_I$ has been studied in the nice series of papers by [FPR].
- In both cases the period mapping

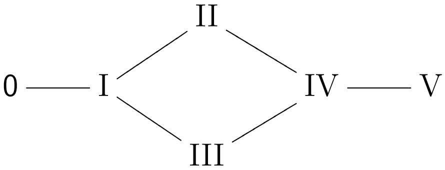
$$\Phi : \mathcal{M} \to \Gamma \backslash D$$

  satisfies generic local Torelli (for I-surfaces also done independently by Carlson-Toledo and by Pearlstein-Zhang).

- For $p_g = 2$ the infinitesimal period relation is a contact system — for I surfaces $\Phi(\mathcal{M})$ is a contact submanifold — for H-surfaces it is of codimension 1 in a contact submanifold.
- In both cases the canonical model is a complete intersection in a weighted projective space and one may suspect that the result is general for such surfaces.

For Noether extremal surfaces local Torelli holds but the canonical model is far from being a complete intersection.

- The boundary structure for $p_g = 2$ has the picture



  which represents different equivalence classes of LMHS's$/\mathbb{Q}$ — not a linear ordering as in the classical case, but it is transitive (not always the case in non-classical case).
- There is also a classification of diagrams as above where the full monodromy cone σ is used (cf. references below) — however, don't know of examples where dim $\geqq 2$.
- By LMHS$/\mathbb{Z}$ we will mean the data of LMHS$/\mathbb{Q}$ and semi-simple part T^s of monodromy. Consideration of these gives refinements of the above diagram.

Theorem C

The extended period mapping

$$\Phi_e : \partial\mathcal{M}^{\mathrm{Gor}}_{I,\mathrm{ref}} \to \partial(\Gamma\backslash D)_{\Phi,\mathbb{Z}}$$

is a map of stratified varieties that is

(i) *1-1 mapping components to components*

(ii) *surjective to $\mathbb{Q}$-components of $\partial(\Gamma\backslash D)_\Phi$.*

Thus in the Gorensetin I surface case as will be further explained below the extended period mapping may be said to capture the structure of the boundary moduli. Much of the result extends to H-surfaces but the complete story is work in progress.

- In the following tables all the singular X's are normal and Gorenstein.

Stratum	HT bdry component	minimal resolution	$K(\widetilde{X})$	$p_g(\widetilde{X})$	$q(\widetilde{X})$
$\mathcal{N}_1 \supset \mathcal{N}_1^0$	I, II	blow up K3	0	1	0
$\mathcal{N}_2 \supset \mathcal{N}_2^0$	I, II	minimal elliptic	1	1	0
$\mathcal{N}_{2,2} \supset \cdots$	III, IV, V	rational	$-\infty$	0	0
$\mathcal{N}_{1,2} \supset \cdots$	III, IV, V	rational	$-\infty$	0	0
$\mathcal{N}_{1,1}^R \supset \cdots$	III, IV, V	rational	$-\infty$	0	0
$\mathcal{N}_{1,1}^E \supset \cdots$	III, IV, V	{blow up Enriques}	0	0	0
$\mathcal{N}_{1,1,2}$	V	ruled	$-\infty$	0	1
$\mathcal{N}_{1,1,1}$	V	ruled	$-\infty$	0	1

- elliptic singularities are hypersurface ones — by inspection of tables in Arnold the degree is determined by Coxeter element which is in T^s. In this sense the LMHS$/\mathbb{Z}$ captures the fine stratification of $\partial\mathcal{M}_I^{\mathrm{Gor}}$.

- H-surface cases have additionally

Stratum	HT	type	κ	$p_g(\widetilde{X})$	$q(\widetilde{X})$
$\mathcal{N}_{1,1,2}$	I	$T_1 \times T_2$	0	1	2
$\mathcal{N}_4$	I	$K^2_{\widetilde{X}} = 1$	2	1	0

- most of the rest of the normal H-surface classification follows I-surface pattern
- non-normal case
  - $K(\widetilde{X}) = -\infty, 0$
  - $\widetilde{X}$ —I→ $\begin{cases} \mathbb{P}^2, \text{ dP of degree 1} \\ \text{minimal ruled with } q(\widetilde{X}) = 1 \end{cases}$

    $\widetilde{X}$ —H→ $\mathbb{P}^1 \times \mathbb{P}^1$, dP of degree 2??

References

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II. Moduli

- Consider surfaces X such that
  - X is of general type
  - X is either smooth or has canonical singularities
  - X has given numerical characters

$$K_X^2,\ q(X),\ p_g(X).$$

  Usually one says given $\chi(\mathcal{O}_X)$, but as we are interested in Hodge theory we use $q(X)$, $p_g(X)$.
- Then it is known that there is a good moduli theory.

(General reference: Kollár, Moduli of varieties of general type, Handbook of Moduli, Vol. II)

- For such a moduli space $\mathcal{M}$ there is a canonical completion $\overline{\mathcal{M}}^{\mathrm{KSBA}}$ due to Kollár, Shepherd-Barron, Alexeev.
- For a family

$$\begin{array}{ccc} \mathcal{X}^* & & \\ \downarrow & & \\ \Delta^* & \longrightarrow & \mathcal{M} \end{array}$$

one wants to uniquely fill in the fibre X_0 over the origin — informally one does this by requiring

(a) X_0 has semi-log-canonical (slc) singularities (local)

(b) K_{X_0} is ample (global).

For both of these K_{X_0} is assumed $\mathbb{Q}$-Cartier.

Equivalently for $\mathcal{X} \xrightarrow{\pi} \Delta$

(a′) $\mathcal{X}$ has canonical singularities

(b′) $\omega_{\mathcal{X}/\Delta}$ is relatively ample.

- The resulting $\overline{\mathcal{M}}$ exists, is unique and is projective.
- The following is a partial (4 exceptions) list of the possible singularities taken from Kollár (loc. cit.) — the ones marked with * are Gorenstein, which for Hodge-theoretic purposes to be explained below are particularly important.
- For the Gorenstein ones there are also natural and relatively simple semi-stable-reductions (SSR).
- The innocent looking class (3.3.4) contains a wealth of examples that may be constructed combinatorially by gluing — we will give one such below.

Kollár's list

3.2 (List of log canonical surface singularities).

∗(3.2.1) Terminal = smooth.

∗(3.2.2) Canonical = Du Val (= rational double point).

(3.2.3) Log terminal = quotient of $\mathbb{C}^2$ by a finite group of $\mathrm{GL}(2, \mathbb{C})$ that acts freely outside the origin. A more detailed list is the following:

(a) (Cyclic quotient)

$$c_1 - \cdots - c_n.$$

(b) (Dihedral quotient) Here $n \geqq 2$ with dual graph

$$\begin{array}{ccc} & & 2 \\ & \diagup & \\ c_1 - \cdots - c_n & & \\ & \diagdown & \\ & & 2 \end{array}$$

(c) (Other quotients) The dual graph has one fork (with Γ_i as in (a))

$$\begin{array}{ccccc} \Gamma_1 & — & c_0 & — & \Gamma_2 \\ & & | & & \\ & & \Gamma_3 & & \end{array}$$

with three cases for $(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3))$:

(Tetrahedral) $(2, 3, 3)$
(Octahedral) $(2, 3, 4)$
(Icosahedral) $(2, 3, 5)$.

$*$(3.2.4) Log canonical

(a) (Simple elliptic) $\Gamma = \{E\}$ has a single vertex which is a smooth elliptic curve with self intersection $\leqq -1$.

(b)* (Cusp) Γ is a circle of smooth rational curves, at least one of them with $c_i \geqq 3$. (The cases $n = 1, 2$ are somewhat special.)

$$\begin{array}{ccccc} & / & c_n - \cdots - c_{n+1} & \backslash & \\ c_1 & & & & c_r \\ & \backslash & c_2 - \cdots - c_{r-1} & / & \end{array}$$

If X is a non-normal semi-log-canonical surface singularity, then we describe its normalization $\widetilde{X}$ together with the preimage of the double curve $\widetilde{B} \subset \widetilde{X}$.

3.3 (List of semi-log-canonical surface singularities).
There are three irreducible cases.

(3.3.1) (Cyclic quotient, one branch of $\widetilde{B}$)

$$\bullet \xrightarrow{1-\frac{1}{\det \Gamma}} c_1 - \cdots - c_n$$

$*$(3.3.4) (Possibly reducible cases) We can take several components as above and glue them together along two local branches of $\widetilde{B}$.

It is this last class that allows one to do combinatorial (or tropical) type constructions.

As mentioned one may describe a natural semi-stable (SSR) reduction prescription for the $*$-surfaces. They are of the general form

$$\widetilde{X} \cup Y \cup Z$$

where $Y = \mathbb{P}^1$-bundle over $\widetilde{B}$ and $Z =$ rational surface arising from $\widetilde{B}_{\mathrm{sing}}$.

III. Hodge theory

- *Polarized Hodge structure* (PHS)
  $(V, Q, F^\bullet)$ with $V = \mathbb{Q}$-vector space
  - $Q : V \otimes V \to \mathbb{Q}$ non-degenerate
  - $F^p V_{\mathbb{C}}$ with $F^p \oplus \overline{F^{n-p+1}} \xrightarrow{\sim} V_{\mathbb{C}}$
  - Hodge-Riemann I and II
- $V^{p,q} = F^p \cap \overline{F}^q$ and
  - $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$, $V^{p,q} = \overline{V}^{q,p}$ (Hodge decomposition)
  - $F^p = \bigoplus_{p' \geq p} V^{p',q}$
  - $\dim V^{p,q} = h^{p,q}$ (Hodge numbers)
- *Example:* $H^n(X, \mathbb{Q})_{\mathrm{prim}}$ where X is a smooth projective variety $/\mathbb{C}$

- *Mixed Hodge Structure* (MHS)
  $(V, W_\bullet, F^\bullet)$ where $F^\bullet$ induces a Hodge structure of weight m on
  $$\mathrm{Gr}_m^W V = W_m / W_{m-1}.$$
- $V_{\mathbb{C}} \cong \bigoplus_{p,q} I^{p,q}$ where $I^{p,q} \equiv \bar{I}^{q,p} \bmod W_{p+q-2}$
  (Deligne decomposition). Then
  $$I^{p,q} \cong \left(\mathrm{Gr}_m^W V\right)^{p,q}.$$

Example: $V = H^n(X, \mathbb{Q})$ where X = complete variety $/\mathbb{C}$ and the weights satisfy $0 \leqq m \leqq n$

- $N \in \mathrm{End}_Q(V)$ nilpotent with $N^n \neq 0$, $N^{n+1} = 0$ gives $W_k(N)$ with
  - $0 \leqq k \leqq 2n$
  - $N : W_k(N) \to W_{k-2}(N)$
  - $N^k : \mathrm{Gr}_{n+k}^{W(N)} V \xrightarrow{\sim} \mathrm{Gr}_{n-k}^{W(N)} V$

- *limiting mixed Hodge structure* (LMHS) $(V, W_\bullet(N), F^\bullet_{\lim})$
  - will always have a "Q" in the background but will omit reference to it
- *Example:* smooth projective family

$$\mathcal{X}^* \to \Delta^*$$

with unipotent monodromy $T = e^N$
$\rightsquigarrow H^n_{\lim} = (V, W_\bullet(N), F^\bullet_{\lim})$

- D = *period domain* of PHS's with given $h^{p,q} = \dim V^{p,q}$
  - $D = G_{\mathbb{R}}/H$, H compact
  - $D = \text{HSD} \iff \begin{cases} n = 1 \\ n = 2 \text{ and } h^{2,0} = 1 \end{cases}$ (classical case)
  - for $n = 2$

$$D = \mathrm{SO}(2a, b)/\mathcal{U}(a) \times \mathrm{SO}(b).$$

- in non-classical case with all $h^{p,q} \neq 0$ there is a unique minimal, $G_{\mathbb{R}}$-invariant and bracket generating $I \subset TD$ (*infinitesimal period relation* (IPR))

Example: $n = 2$ and $h^{2,0} = 2 \implies I =$ contact system

- *compact dual* $\check{D} = \{F^\bullet : Q(F^p, F^{n-p+1}) = 0\}$
    - $\check{D} = G_{\mathbb{C}}/P =$ homogeneous rational projective variety
    - $D =$ open $G_{\mathbb{R}}$-orbit in $\check{D}$
    - $G_{\mathbb{R}}$-orbit structure of ∂D is very rich (Matsuki duality, etc.)
- *period mapping* is

$$\Phi : B \to \Gamma \backslash D$$

- locally liftable ($\implies$ monodromy representation $\Phi_* : \pi_1(B) \to \Gamma$ is defined)
- holomorphic
- $\Phi_* : TB \to I$.

- *Example*: $\Phi : \Delta^* \to \Gamma_{\mathrm{loc}} \backslash D$ where $\Gamma_{\mathrm{loc}} = \{T^m\}$
  $\rightsquigarrow$ LMHS $(V, W_\bullet(N), F^\bullet_{\lim})$ where $F_{\lim} \in \check{D}$.
- *Example*: *nilpotent* orbit is $F \to \exp(zN) \cdot F_0$, $z \in \mathcal{H}$ and $F_0 \in \check{D}$
  - $N \cdot F_0^P \subseteq F_0^{p-1}$
  - $\exp(zN)F_0 \in D$ for $\mathrm{Im}\, z \gg 0$

  $\rightsquigarrow$ $\Phi_\nu : \Delta^* \to \Gamma_{\mathrm{loc}} \backslash D$.

Theorem (Schmid)

Any Φ strongly approximated by a nilpotent orbit Φ_ν.

- Classification of (equivalence classes of) nilpotent orbits (Brosnan, Kerr, Pearlstein, Robles)
- *Conclusion:* Given $\mathcal{X}^* \to \Delta^*$ we know what the possible $\mathrm{Gr}^{W(N)}$ (LMHS's)$/\mathbb{Q}$ are.

- Now consider the several parameter situation — localizing $\Phi : B \to \Gamma \backslash D$ around a point of $Z = \overline{B} \backslash B$ (= NCV) leads to the
- $\Phi : \Delta^{*k} \times \Delta^{\ell} \to \Gamma_{\mathrm{loc}} \backslash D$ where Γ_{loc} arises from a *monodromy cone* $\sigma = \mathrm{span}_{\mathbb{R}^+}\{N_1, \ldots, N_k\}$ with $[N_i, N_j] = 0$
  - $W(N)$ independent of $N \in \sigma$
  - relative weight filtration property (RWFP)
  - for $t = (t_1, \ldots, t_k) \in \Delta^{*k}$ and $w = (w_1, \ldots, w_\ell) \in \Delta^{\ell}$, setting $\ell(t_j) = \log t_j / 2\pi i$.

Theorem

$\exp\left(\sum_j \ell(t_j) N_j\right) \cdot F_0(w)$ *strongly approximates* Φ.

- Asymptotics of several variable families of PHS's are quite subtle (Cattani-Kaplan-Schmid) (for $f(x_1, \cdots, x_k)$ defined for $x_j > 0$, $\lim f(\lambda_1 t, \cdots, \lambda_k t)$ may exist for all λ but $\lim_{x \to 0} f(x)$ may not exist — need to analyze sectoral behavior).

IV. The extended period map: Theorems A and B

- Given a smooth quasi-projective variety B and a period mapping
$$\Phi : B \to \Gamma\backslash D$$
one seeks to define an extension $(\Gamma\backslash D)_\Phi$, *depending on* Φ, such that for any smooth completion $\overline{B}$ with $\overline{B}\backslash B = Z = \cup Z_i$ a NCV we have
$$\begin{array}{ccc} B & \xrightarrow{\Phi} & \Gamma\backslash D \\ \downarrow & & \cap \\ \overline{B} & \xrightarrow{\Phi_e} & (\Gamma\backslash D)_\Phi \end{array}$$

- This is work in progress — here we shall only deal with an extension $\overline{M}$ of the image $M = \Phi(B)$ to have

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & M \subset \Gamma\backslash D \\ \cap & & \cap \\ \overline{B} & \xrightarrow{\Phi_e} & \overline{M}. \end{array}$$

- The main point here is that the global Lie theoretic methods used by Baily-Borel are not applicable; a different approach — which is even interesting in the classical case — is necessary.¶

¶For $D \neq$ HSD, the quotient $\Gamma\backslash D$ has no non-constant meromorphic functions and we may have $\mathrm{vol}(\Gamma\backslash D) = \infty$.

- The steps are
  (i) analyze the structure of nilpotent orbits to define monomial maps
  $$\mu : \Delta^k \longrightarrow \mathbb{C}^N$$
  whose fibres are those of the set-theoretic map given on strata by
  $$\Phi_I : \Delta_I^* \to \mathrm{Gr}(\mathrm{LMHS_I});$$
  (ii) extend (i) to arbitrary local period maps to give quasi-charts
  $$\Delta^k \times \Delta^\ell \to \Gamma_{\mathrm{loc}} \backslash D;$$
  (iii) show that the local quasi-charts patch together to give a global mapping
  $$\overline{B} \xrightarrow{\Phi_e} \overline{M}$$
  whose image is a compact, complex analytic variety;‖

‖We may think of $\overline{M}$ as the quotient of $\overline{B}$ by the relation to have equivalent $\mathrm{Gr}(\mathrm{LMHS})$'s — operative word here is "equivalent."

(iv) show that the augmented Hodge line bundle is defined as a holomorphic line bundle $\Lambda_e \to \overline{M}$ and that it is ample.

- *Key local questions:*
  (a) What are the fibres of a nilpotent orbit?
  (b) How do the closures of the fibres of nilpotent orbits on the strata Δ_I^* meet the faces of Δ^k?

(a) When is a monomial $t^B = t_1^{b_1} \cdots t_k^{b_k}$, $b_j \in \mathbb{Z}^{\geqq 0}$ constant on the fibres of a nilpotent orbit? Set

$$R = \left\{ A = (a_1, \dots, a_k) : \sum_j a_j N_j = 0 \right\}.$$

Since the vector field given by a non-zero $\sum_j a_j N_j$ doesn't vanish on D, using

$$\Phi_*(t_j \partial/\partial t_j) = N_j$$

the condition is

$$0 = \left(\sum_j a_j t_j \partial/\partial t_j \right) t^B = (A, B) t^B$$
$$\implies B \in R^\perp.$$

- $R = \mathbb{Q}$-vector space — from Farkas' lemma in linear programming

$$R^\perp \textit{ spanned by vectors in first quadrant } \mathbb{Q}^{k+}$$
$$\implies \left\{ \begin{matrix} \text{monomial mapping } \mu : \Delta^k \to \mathbb{C}^N \text{ has} \\ \text{same connected fibres as nilpotent orbit} \end{matrix} \right\}$$

- use coordinate change

$$t'_j = e^{t_j(t,w)} t_j$$

  adapted to Φ to define

$$\mu : \Delta^k \times \Delta^\ell \to \mathbb{C}^N$$

  with same connected fibres as Φ in $\Delta^{*k} \times \Delta^\ell$.
- $\mu(\Delta^k \times \Delta^\ell)$ fibres over the parameter space with toroidal-type fibres which are open sets in $W \backslash V^*$ where $W \subset \mathbb{C}^N$ is an algebraic variety, $V^* \subset V$ is a Zariski open in a proper subvariety $V \subset W$.
- relative weight filtration property (RWFP) now leads to

$$\textit{for } I \subset J \textit{ closure of fibres of } \mu_J \subseteq \textit{fibres of } \mu_I$$

  (compatibility across strata)

- the relation $\sum a_i N_i = 0$ above is replaced by

$$\sum_{j \in I^c} a_j N_j \in W_{-1}^{(W(N_I))}(V);$$

- construction does not fall in standard analytic or algebraic geometry frameworks; standard methods of quotienting by an equivalence relation don't apply — need to use special circumstances plus global results from VHS

(b) from CKS with refinements by Kawamata, Kollár and others
  - Chern form of $\Lambda_e \to \overline{B}$ is represented by a closed (1,1) current (= differential form with distribution coefficients) — not sufficient for what is needed here —

- traditional problems with distributions are
  - cannot be multiplied
  - cannot be restricted to submanifolds

Theorem

Chern forms of Hodge bundles can be multiplied and restricted to Z_I^* to give Chern forms of Hodge bundles associated to $\mathrm{Gr}(\mathrm{LMHS_I})$

Requires analysis of singularities of Chern forms in $T^*\overline{B}$ (refined wave-front-set analysis in sectors in $N^*_{Z_I^*/\overline{B}}$)

- for $\xi \in T\overline{B}$ the condition

$$\omega_e(\xi) = 0$$

  can be defined (although in general the "value" $\omega_e(\xi)$ cannot be). Then

  equation $\omega_e = 0$ *in* $T\overline{B}$ *defines the fibres of* $\overline{B} \xrightarrow{\Phi_e} \overline{M}$

  $\rightsquigarrow$ ampleness of $\Lambda_e \to \overline{M}$.

V. Examples of Theorem C

- Will illustrate the normal cases and one non-normal case of I-surfaces X — recall
  - $K_X^2 = 1$, $h^2(\mathcal{O}_X) = 2$
  - K_X ample
  - We shall restrict to the Gorenstein case since only these singularities can contribute non-trivially to the LMHS$/\mathbb{Q}$.
  - The non-Gorenstein singularities contribute finite (including trivial) monodromy, and bounding these is interesting but we have nothing much to say here.

- $\mathcal{N}_{d_1,\dots,d_k} = \begin{cases} \text{normal } I\text{-surfaces with simple} \\ \text{elliptic singularities } p_1, \dots, p_k \\ \text{of degrees } d_1, \dots, d_k. \end{cases}$
- $k \leqq 3$ — in general Hodge theoretic argument gives $k \leqq h^2(\mathcal{O}_X) + 1$.
- For I-surfaces $d_i \leqq 3$ (for H-surfaces $d_i \leqq 4$ — don't yet know a general result other than $d_i \leqq 9$).
- General philosophy: *One can frequently use Hodge theory to bound the complexity of* X_{sing}.

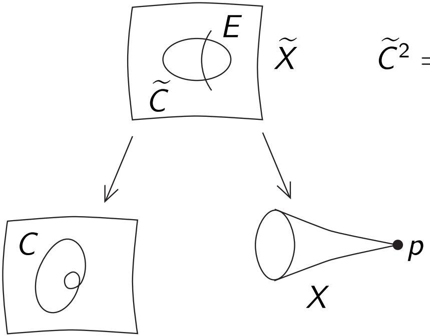
- Warm up: Picture for $\mathcal{N}_2$ suggested by $\mathrm{LMHS}/\mathbb{Z}$
- General X has $\mathrm{Hg}^1(\widetilde{X}, \mathbb{Z})$ containing $\mathbb{Z}^2$ with intersection form $\left(\begin{smallmatrix} -2 & 2 \\ 2 & -1 \end{smallmatrix}\right)$ — basis classes are effective
- LMHS has

$$\text{LMHS has} \begin{cases} \mathrm{Gr}_2 \cong H^2(X_{\mathsf{min}})_{\mathrm{prim}} \\ \mathrm{Gr}_3 \cong H^1(\widetilde{C})(-1) \quad (\implies \mathrm{Gr}_1 \cong H^1(\widetilde{C})) \end{cases}$$

- MHS $H^2(X)$ computed from

$$(\widetilde{X}, \widetilde{C}) \to (X, p)$$

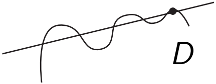
- # of PHS with parameters $\mathrm{Gr}_3 \oplus \mathrm{Gr}_2 = 19 + 1 = 20$
- Hodge theory suggests picture



$\widetilde{C}^2 = -2$

$$\begin{cases} X_{\min} = K3 \\ C^2 = 2 \end{cases} \implies X_{\min} \xrightarrow{2:1} \mathbb{P}^2$$

branched over D



- $\dim \mathcal{M}_I = 28$
- $\dim \widetilde{\mathcal{N}}_2 = 27 = 20 + (9-2)$
- from the picture obvious that
  - $I^{2,0}_{\text{lim}} \cong I^{2,0}(X) \cong H^{2,0}(X_{\text{min}})$
  - $I^{2,1}_{\text{lim}} \cong I^{1,0}(X)(-1) \cong H^{1,0}(\widetilde{C})(-1)$
  - $I^{2,2}_{\text{lim}} = 0$ since $N^2 = 0$.
- in general for $\mathcal{X} \to \Delta$ on $\widetilde{\mathcal{X}} = \mathrm{Bl}_{p_1 \cdots p_k} \mathcal{X}$ we have exceptional surfaces $Y_i \cong \mathbb{P}^2$ with $\widetilde{C}_i \in |\mathcal{O}_{Y_i}(3)|$ — Blow up $9 - d_i$ points on Y_i to get Del Pezzo of degree d_i — $\#$ moduli $= 1 + (9 - d_i) - 1 = 9 - d_i$
  $\implies$ SSR's for X's has $\sum_i 9 - d_i$ moduli

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
$\mathcal{N}_\emptyset = \mathcal{M}_{1,3}$	28	general type			
$\mathcal{N}_2$	20	blow up of a K3-surface	7	1	8
$\mathcal{N}_1$	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathcal{N}_{2,2}$	12	rational surface	14	2	16
$\mathcal{N}_{1,2}$	11	rational surface	15	2	17
$\mathcal{N}_{1,1}^R$	10	rational surface	16	2	18
$\mathcal{N}_{1,1}^E$	10	blow up of an Enriques surface	16	2	18
$\mathcal{N}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathcal{N}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

- case $k = 3$ and rank $N = 2$ occurs when the p_i fail to impose independent conditions on $\lim_{t\to 0} H^0(\Omega^2_{X_t})$.
- $\mathcal{N}^0_{d_1,\ldots,d_k}$ refinements obtained by degenerating the elliptic curves to cusps; then $N^2 \neq 0$ and all the possibilities with rank $N^2 = 1, 2$ can be achieved.

Conclusion:

$$\mathrm{codim} = \sum_{i=1}^{k}(9 - d_i) + k;$$

i.e.,

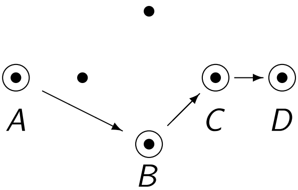
$$\left\{ \begin{array}{c} \text{SSR's for } X\text{'s with} \\ k \text{ elliptic singularities} \end{array} \right\} \text{ has codim } k$$

- Non-normal case have $(\widetilde{X}, \widetilde{C}, E_i, \tau)$ and (X, C) where $C = \widetilde{C}/\tau$ and E_i give cycles that are contracted to singular points on C.
- Will give two examples due to Liu-Rollenske

  Example 1: $\widetilde{X} = \mathbb{P}^2$, $\widetilde{C} =$ smooth plane quartic and $\tau : \widetilde{C} \to \widetilde{C}$ elliptic involution, no E_i

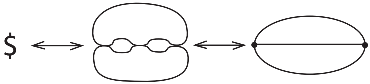
$$\rightsquigarrow \begin{cases} I^{2,0}_{\text{lim}} = (0) \\ I^{2,1}_{\text{lim}} \cong H^0(\Omega^1_{\widetilde{C}})^- \\ I^{2,2}_{\text{lim}} = (0). \end{cases}$$

- degeneration picture in non-normal case



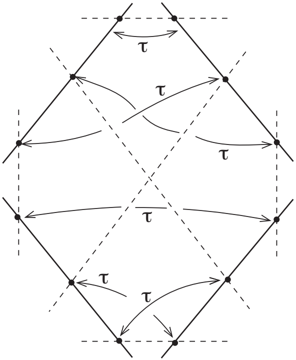
  - $A \longleftrightarrow$ smooth
  - $B \longleftrightarrow \widetilde{C}$ smooth
  - $C \longleftrightarrow \left\{\begin{matrix} \widetilde{C} + E \\ \widetilde{C}_1 + \widetilde{C}_2 + E_1 + E_2 \end{matrix}\right\}$
  - $D \longleftrightarrow \left\{\begin{matrix} \text{LMHS is Hodge-Tate; will} \\ \text{need to analyze monodromy} \\ \text{cone to classify — yet to be done} \end{matrix}\right\}$

- conjectural case of most degenerate I-surface
  - for $g = 2$ curves have



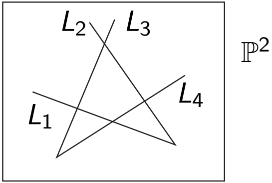
rigid, monodromy cone σ maximal

  - replace ($\mathbb{P}^1$, three points) by ($\mathbb{P}^2$, four lines)



One choice of τ is drawn in. Dotted lines are exceptional divisors E_{ij}

- for I surface



- identify L_1 and L_2 by $\left\{\begin{matrix} 12 \longleftrightarrow 21 \\ 13 \longleftrightarrow 24 \\ 14 \longleftrightarrow 23 \end{matrix}\right\}$

  similarly for L_3 and L_4

Question: *Is this X uniquely determined by its $(LMHS/\mathbb{Z}, \sigma)$?*

- Noether extremal surfaces
  - $p_g \leqq \frac{1}{2}K_X^2 + 2$
  - case of equality can be analyzed and canonical image is 2:1 branched covering

$$X \to S \subset \mathbb{P}^{p_g - 1}$$

    where S has minimal degree
  - local Torelli holds
  - pluricanonical ring is complicated but geometry is relatively simple

Question: Can we use GIT for the analysis of branch curve $B \subset S$ (non-reductive group)?

- Conclusions
  - Given $\mathcal{M}$ there is a canonical minimal completion of the image of the period mapping.
  - The Hodge-theoretic boundary structure is understood, and in early examples this provides a guide to the structure of $\partial\mathcal{M}$.

$$\text{———} < \circ > \text{———}$$

Thank you