Curvature properties of Hodge bundles

Hong Kong University

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Phillip Griffiths

Outline

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I. Introduction

- In algebraic geometry the use of complex analytic methods — differential geometry and PDEs — is long standing and far reaching — for rather deep reasons the results these methods give (Hodge theory, vanishing theorems) have not been replaced by purely algebraic techniques.
- Two central areas in the subject are moduli and classification of varieties — today we will discuss two results in which the use of analytic methods through the curvature properties of the Hodge bundles plays a central role.
- For the first result* we let
  - $\mathcal{M} = \mathrm{KSBA}$ moduli space for general type algebraic varieties with fixed Chern numbers;
  - $\Phi : \mathcal{M} \to \Gamma\backslash D$ the period mapping.

*Based on joint work with Mark Green, Radu Laza and Colleen Robles

Theorem

There exists a canonical extension $(\Gamma\backslash D)_\Phi \supset \Gamma\backslash D$ *and Hodge line bundle* $\Lambda_e \to (\Gamma\backslash D)_\Phi$ *with*

- $\Phi_e : \overline{\mathcal{M}} \to (\Gamma\backslash D)_\Phi$ *with image a compact, complex analytic variety;*
- $\Lambda_e \to \Phi_e(\overline{\mathcal{M}})$ *is ample.*

Application: The boundary of $\Phi(\mathcal{M})$ is constructed using Lie theory and complex analysis. There is a fairly well understood story of how Hodge structures degenerate. In some early examples this provides a very effective tool to analyze the algebro-geometric boundary structure of $\mathcal{M}$ — gives the first non-classical examples.†

† Joint with GLR and Marco Franciosi, Rita Pardini and Sönke Rollenske.

In the classical case $(\Gamma\backslash D)_\Phi = (\overline{\Gamma\backslash D})^{\mathrm{SBB}}$ is the Satake-Baily-Borel compactification — does not depend on Φ — in the non-classical case the construction is a relative one. It is rather subtle and even in the classical case gives new information.

$$———<\circ>———$$

For the second result recall the Kodaira dimension of a smooth variety W

$$\kappa(W) = k \text{ if } h^0(mK_W) \sim m^k.$$

W is general type if $\kappa(W) = \dim W$ (means $g \geqq 2$ for curves).

Theorem (Itaka conjecture proved by Viehweg with refinements by a number of people)

Given

$$X \xrightarrow{f} Y$$

with

- $\mathrm{var} f = \dim Y$;
- X_y *general type for general* $y \in Y$

Then

$$\kappa(X) \geqq \kappa(X_y) + \kappa(Y).$$

There is a vast literature, much of it recent, concerning extensions and refinements of this result (cf. Păun, arXiv1606.0017 (2016))

II. Preliminaries

- $E \to X$ is a holomorphic vector bundle with Hermitian metric h.
- *Chern connection* is the unique connection

$$D : A^0(E) \to A^1(E)$$

  satisfying
  - $D^{0,1} = \overline{\partial}$;
  - $d(e, e') = (De, e') + (e, De')$.
- *Curvature* is

$$D^2 = \Theta_E = \left\| \Theta^{\alpha}_{\bar{\beta} i \bar{j}} dz^i \wedge d\bar{z}^j \right\| \in A^{1,1}(\mathrm{End}\, E)$$

  and it satisfies

$$(\Theta_E(e), e') + (e, \Theta_E(e')) = 0.$$

- *Curvature form* is

$$\begin{aligned}\Theta_E(e,\xi) &= \left\langle (\Theta_E(e), e), \xi \wedge \overline{\xi} \right\rangle \\ &= \Theta^{\alpha}_{\bar{\beta} i \bar{j}} e_\alpha \bar{e}_\beta \xi^i \overline{\xi}^j\end{aligned}$$

  where $e \in E_x$, $\xi \in T_x X$.
- *Example:* $E = TX$ — then

$$\Theta_{TX}(\eta,\xi) = R(\eta,\xi) = \left\{ \begin{array}{c}\text{holomorphic}\\ \text{bisectional}\\ \text{curvature}\end{array} \right\}.$$

- $\det \left\| I + \left(\frac{i}{2\pi}\right) \Theta_E \right\| = \sum_q c_q(\Theta_E)$ where $c_q(\Theta_E)$ are the Chern forms.
- $E \to X$ is *positive*, written $E > 0$, if there exists h such that $\Theta_E(e,\xi) > 0$ — semi-positive if $\Theta_E(e,\xi) \geqq 0$.

- $(\mathbb{P}E)_x = \mathbb{P}E_x^*$ and $\mathcal{O}_{\mathbb{P}E}(1)_{x,[e^*]} = E_x/e^{*\perp}$ $H^0(S^m E) \cong H^0(\mathcal{O}_{\mathbb{P}E}(m))$ — metric on E induces one on $\mathcal{O}_{\mathbb{P}E}(1)$ — $E > 0 \implies \mathcal{O}_{\mathbb{P}E}(1) > 0$ (converse is a long standing open problem).
- *Example:* Universal quotient bundle $Q \to G(k,n)$ is positive $\iff k = n-1$; dual of universal sub-bundle $E \to G(k,n)$ positive $\iff k = 1$.
- a line bundle $L \to Y$ is
  - (i) *nef* if $\langle c_1(L), [C]\rangle \geqq 0$ for all curves $C \subset Y$ $(\iff \deg L\big|_C \geqq 0)$
  - (ii) *big* if $h^0(L^m) \sim m^d$, $d = \dim Y$;
  - (iii) *free* if $H^0(L^m) \to L_y^m \to 0$, $m \gg 0$ and all $y \in Y$; then we have $\varphi_L : Y \to \mathbb{P}H^0(Y, L^m)^* = \mathbb{P}$ with $\varphi_L^*(\mathcal{O}_{\mathbb{P}}(1)) = L^m$.
  - (iv) *ample* if there exists $Y \hookrightarrow \mathbb{P}^N$ with $L^m = \mathcal{O}_{\mathbb{P}^N}(1)\big|_Y$ for some $m > 0$;
- same properties for $E \to X$ if they hold for $\mathcal{O}_{\mathbb{P}E}(1) \to \mathbb{P}E$.

Basic results

- $\Theta_E > 0 \implies E$ is ample (Kodaira);
- $\Theta_E \geqq 0 \implies E$ is nef;
- $\Theta_E > 0$ on an open set $\implies E$ is big (Siu, also Demailly).

Basic question: *Give metric conditions that imply E is free*

- suppose we have $L \to X$ with h and Chern form $\omega = \left(\frac{i}{2\pi}\right) \Theta_E$ such that
  - $\omega \geqq 0$ ($\implies L$ is nef),
  - normal crossing divisor Z on X with $\omega^d > 0$ on $X^* := X \backslash Z$ ($\implies L$ is big),
  - $\omega(\xi) = 0 \iff \xi \in TZ$ — then ω defines a Kähler metric ω^* on X^*.

Question: *Are there properties of* $\lim_{p \to Z} R_{\omega^*}$ *that imply $L \to X$ is free?*

III. Strong semi-positivity of vector bundles

- In general positivity suggests sections — given that $\Theta_E > 0$ is rare but $\Theta_E \geqq 0$ is common, especially when $\operatorname{rank} E \geqq 2$, one has the

Question

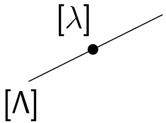
What are natural conditions that imply some $S^m E$ is big?

- **Definition:** *$E \to X$ is strongly semi-positive if there exists an h such that*
  - $\Theta_E \geqq 0$
  - $\operatorname{Tr} \Theta_E > 0$ *on an open set.*
- Although rather simple I am not aware of the following in the literature

Theorem

$E \to X$ strongly semi-positive $\Longrightarrow$ $S^m E$ is big for some $m \leqq \operatorname{rank} E$.

- *Example:* $E =$ dual of the universal sub-bundle over $G(2,4) = \mathbb{G}(1,3)$. We have the picture

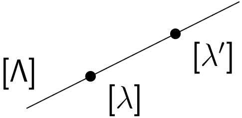


$\Lambda = \mathbb{C}^2 \subset \mathbb{C}^4,\ [\Lambda] = \mathbb{P}\Lambda$ in a line in $\mathbb{P}^3$
and $[\lambda] =$ point on $[\Lambda]$
$E_\Lambda = \Lambda^* =$ and $\mathbb{C}^{4*} \xrightarrow{\sim} H^0(E)$
$[\mathbb{P}E]_\Lambda \cong [\Lambda]$

$\rightsquigarrow \varphi : \mathbb{P}E \to \mathbb{P}^3$ given by

$$\varphi([\lambda]) = [\lambda]$$

$\implies$ fibres of φ are the $\mathbb{P}^2$ of lines through $[\lambda]$.
Now look at S^2E — points of $\mathbb{P}S^2E$ are



and λ, λ' uniquely determine $\Lambda \implies S^2E$ is big.

- *Example:* $X =$ compact Kähler with

$$R(\eta,\xi) \leqq 0 \text{ and } \mathrm{Ric}(\xi) < 0 \text{ on an open set}$$

$$\implies S^m T^* M \text{ is big for some } m \leqq \dim X.$$

Example: $\mathcal{A}_g = S_p(2g,\mathbb{Z})\backslash\mathcal{H}_g$ with Hodge vector bundle $F \to \mathcal{A}_g$ — then $F_e \to \overline{\mathcal{A}}_g^{\mathrm{Tor}}$ is not big but $S^2 F_e$ is (and more generally for the Hodge vector bundle over toroidal compactifications of Shimura varieties) — there will be similar bundles arising in general from moduli and Hodge theory.

IV. Singularities of metrics, curvatures and Chern forms

- In many questions, such as those concerning the boundaries of moduli spaces and in birational geometry, the use of curvature necessitates considering singular metrics and their curvatures — in fact the use of singular metrics gives considerable flexibility to the use of analytic methods in complex algebraic geometry.
- Roughly speaking there are three types of singularities that have arisen:
  (i) analytic singularities,
  (ii) logarithmic singularities,
  (iii) singularities that arise from solving non-linear PDEs (frequently of Monge-Ampere type).

- Will discuss (i) and (ii) — especially *mild* logarithmic singularities.
- Will restrict to singularities of metrics in line bundles $L \to X$ — these will be of the local form

$$h = e^{-\varphi} h_0$$

where $h_0 =$ smooth metric and φ is a pluri-subharmonic (psh) function — thus

$$i\partial\overline{\partial}\varphi \geqq 0 \text{ as a (1,1) current}$$
$$\Updownarrow$$
$$\frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\xi^i\overline{\xi}^j \geqq 0 \text{ as a measure.}$$

Example (i): $\varphi = c\log(|f_1|^2 + \cdots + |f_m|^2) + C^\infty$ term, $c > 0$ (or $\log(|f_1|^{\alpha_1} + \cdots |f_m|^{\alpha_m}), \alpha_j > 0$).

Example (ii):

$\varphi = \log(-\log|f_1|)^2 + \cdots + \log(-\log|f_m|)^2 + C^\infty$ term, and where we are near $f_1 = 0, \ldots, f_m = 0$.

- In (i)
  - for $z \in \Delta \subset \mathbb{C}$ we have $\left(\frac{i}{\pi}\right)\partial\overline{\partial}\log|z| = \delta_0\left(\frac{i}{2}\right)dz \wedge d\bar{z}$
  - more generally $\left(\frac{i}{\pi}\right)\partial\overline{\partial}\log|f| = \left\{\begin{smallmatrix}\text{integration}\\ \text{over } f=0\end{smallmatrix}\right\}$
  - $\mathcal{U} \xrightarrow{f} \mathbb{P}^{m-1}$ and $\left(\frac{i}{\pi}\right)\partial\overline{\partial}\varphi = f^*\begin{pmatrix}\text{Fubini study}\\ \text{metric on } \mathbb{P}^{m-1}\end{pmatrix}$

  In (ii)

$$\partial\overline{\partial}\log(-\log|z|)^2 = \frac{dz \wedge d\bar{z}}{|z|^2(-\log|z|)^2} = PM$$

is in L^1 — in computing $\partial, \overline{\partial}$ in the sense of currents no "residue terms" appear — only L^1 terms appear and one just computes formally using rules of calculus.

- $I(\varphi) = \{f \in \mathcal{O}_X : \int |f|^2 h < \infty\}$
  $= \text{coherent sheaf of ideals.}$
  - (i) for $f_i = z_i$ in $\mathbb{C}^m$ and $c \geqq 1$ has $I(\varphi) = \mathfrak{m}_0$.
  - for (ii), $I(\varphi) = \mathcal{O}_X$ (just barely)
- Given $(E \to X, h)$ with h singular one may ask about the singularities of the Chern forms $c_q(\Theta_E)$ and the polynomials $P(c_1(\Theta_E), \ldots, c_r(\Theta_E))$ in the Chern forms — e.g., for line bundles

$$c_1(\Theta_E) = \left(\frac{i}{2\pi}\right)\left[\partial\overline{\partial}\log\varphi + \partial\overline{\partial}\log h_0\right].$$

where 1^{st} term is a current — i.e., a differential form with distribution coefficients
  - analytic singularities — e.g., given $s \in H^0(L)$ define $\|s\| = 1$ — any other $s' = fs$ and $\|s'\| = |f|$ and

$$c_1(\Theta_L) = [Z], \quad Z = (s);$$

  - logarithmic singularities; to be discussed.

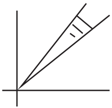
- traditional problems with distributions
  - cannot be multiplied
  - cannot be restricted to submanifolds Z (for the restriction of a current to Z defined by $f_i = 0$ — first set $df_i = 0$ and then try to restrict the distribution coefficients of what remains to Z)
- *Definition:* $E \to X$ has *mild singularities* if there exists $h, Z = \sum Z_i$ a normal crossing divisor such that
  - h has logarithmic singularities along Z
  - Θ_E is computed formally and its entries are L^1_{loc} currents
  - $P(\Theta_E)$ exists as a closed current and the cohomology class
$$[P(\Theta_E)] = P(c_i(E))$$
  - $P(\Theta_E)\big|_{Z_I}$ exists.

- **Main theorem:** (a) *Let $F_e \to X$ be a Hodge bundle given by the canonical extension to X of a variation of Hodge structure over $X \backslash Z$. Then $F_e \to X$ has mild singularities.*

  (b) *Over each stratum Z_I we have a variation of mixed Hodge structures with associated graded being a direct sum of Hodge structures having total Hodge bundle $F_{e,I}$, and*
$$P(\Theta_{F_e})\big|_{Z_I} = P\big(\Theta_{F_{e,I}}\big).$$
- Will not define all the terms here — will illustrate and give an application in the next section — the proof is by extending the deep asymptotic analysis of variations of Hodge structure due to Cattani-Kaplan-Schmid — basically one has to interpret analysis of the singularities of $P(\Theta_{F_e})$ along Z up in the co-normal bundle $N^*_{Z/X}$ —

this is where the wave front sets of distributions live and is the natural place for dealing with the above mentioned two classical problems about distributions.

- Ordinarily the wave front set is a conical neighborhood in $N^*_{Z/Y}$ — we mention two subtleties



(i) $WF =$ whole quadrant

(ii) no uniform bound by PM_i's but

(bound) $\times$ (width of sector) is bounded by PM_i's — this keeps $\int P(\Theta) < \infty$.

Other computational subtleties arise and require explicit combinatorial analysis of the Chern forms.

V. Application to algebraic geometry

- $$\begin{tikzcd} Y^* \arrow[d, "\pi"'] & \subset & Y \arrow[d, "\pi_e"] \\ X^* & \subset & X \end{tikzcd}$$
- $Y_x = \pi^{-1}(x)$ smooth for $x \in X^*$
- $Y_x = \pi_e^{-1}(x)$ generally singular for $x \in Z$.
- **Example:** $Z = Z_1 + Z_2$

$x \in X^*$ $\quad$ $x \in Z_1$ $\quad$ $x \in Z_1 \cap Z_2$

- $F^p \to X^*$ has fibres $H^q(\Omega^p_{Y_x})$
  $F = F^n$, $q = 0$ and $p = n = \dim Y_x$ is the Hodge vector bundle $F = \pi_* \Omega^n_{Y^*/X^*}$
- $F^p_e \to X$ is the canonical extension

- $F_e = \pi_* \omega_{Y/X}$
- Hodge metrics in $F^p \to X^*$

$$(\psi, \eta) \in F_x \implies (\psi, \eta) = c_n \int_{Y_x} \psi \wedge \overline{\eta}.$$

- *Hodge metrics have mild logarithmic singularities.* Logarithmic singularities follows from the Gauss-Manin connection having regular singular points
- as discussed above more subtle is that with the Hodge metrics the Hodge bundles have mild singularities
- now restrict to the case $n = 1$ (curves) and $n = 2$ (surfaces) — then

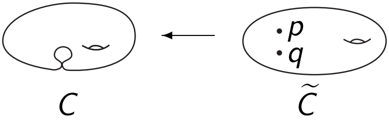
Definition

$\Lambda_e = \det F_e$ is the *Hodge line bundle*.

- Denote by ω, ω_e the Chern forms of Λ, Λ_e — to interpret ω we have
  - period mapping $\Phi : X^* \to \Gamma \backslash D$
  - for $\xi \in T_x X^*$
$$\omega(\xi) = \|\Phi_*(\xi)\|^2$$
  - $\omega > 0 \iff$ local Torelli.
- We note that even though the curvature in a bundle is usually a 2^{nd} order invariant, for the Hodge bundles it is a 1^{st} order invariant — this may at least partly explain the central role played by the curvature of the Hodge bundle.
- as $x \to x_0 \in Z_I$
  - $Y_x \to Y_{x_0} =$ singular variety
  - $H^n(Y_{x_0})$ has a mixed Hodge structure
  - $\mathrm{Gr}\, H^n(Y_{x_0}) = \oplus$ (pure Hodge structures) giving period mappings
$$\Phi_I : Z_I^* \to \Gamma_I \backslash D_I.$$

- **Theorem:** $\omega_e\big|_{Z_I^*} =$ *Chern form* ω_I *of* Λ_I.
- Geometric interpretation: $\omega_I = 0$ defines a fibration of Z_I whose fibres are the variation of the extension data in the MHS.

Example:



extension data is $\mathrm{AJ}_{\widetilde{C}}(p-q)$

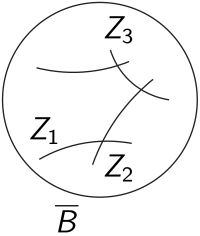
- **Theorem:** $\Lambda_e \to X$ *is*
  (i) *nef*
  (ii) *big if* Φ_* *generically injective*
  (iii) *free.*

  Here (iii) is the hard part.
- **Definition:** The *Satake-Baily-Borel (SBB) completion* of $\Phi(X^*) \subset \Gamma\backslash D$ is $\mathrm{Proj}(\Lambda_e)$.
- We note that ω_e captures $\mathrm{Gr}(\mathrm{LMHS's})$ but does not see the extension data — $\mathrm{Proj}(\Lambda_e \to \overline{\mathcal{M}})$ is canonical and is the minimal completion of the image of the period mapping.

- *Application:* Fine structure of the compactified moduli space of general type algebraic surfaces S with $q(S) = 0$, $p_g(S) = 2$, $K_S^2 = 1$ (first non-classical case to be analyzed) — in this case the structure of $\Phi_e(\overline{\mathcal{M}})$ as a stratified projective variety completely captures that of $\overline{\mathcal{M}}$ — in general $\Phi_e(\overline{\mathcal{M}})$ serves to "organize" the boundary structure of moduli.

Appendix

- Construction of $\mathrm{Proj}(\Lambda_e)$ when $\dim B = 2$



$Z' = \sum_{i\in I} Z_i$

where $\Lambda_e\big|_{Z_i} \cong \mathcal{O}_{Z_i}$

  - $\omega_e \geqq 0$ and $\omega_e^2 > 0$ on an open set
  - $\omega_e\big|_{Z_i} = 0 \iff i \in I$.
- Then $\|Z_i \cdot Z_j\| < 0$ by the Hodge index theorem;
- By Grauert Z' can be contracted to give a singular surface B';
- $\Lambda_e' \to B'$ is ample (Kodaira theorem for singular varieties).

Thank you