

# COMPARISONS BETWEEN THE HODGE-THEORETIC AND ALGEBRO-GEOMETRIC APPROACHES TO LIMITING MIXED HODGE STRUCTURES\*

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## I. INTRODUCTION

In the study of degenerating families of Hodge structures in the literature there are two complementary approaches. These are

**Hodge theoretic:** This is given by a period mapping

$$(I.1)_{\text{HT}} \quad \Phi : \Delta^{*\ell} \rightarrow \Gamma \backslash D$$

where  $\Gamma$  is generated by unipotent local monodromy transformations  $T_i = \exp N_i$ . We denote by  $\sigma = \text{span}_{\mathbb{R}^+} \{N_1, \dots, N_\ell\}$  the monodromy cone associated to (I.1)<sub>HT</sub>.

Here, we are given a  $\mathbb{Q}$ -vector space  $V$  and a non-degenerate form  $Q : V \otimes V \rightarrow \mathbb{Q}$ , and  $D$  is a period domain consisting of polarized

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\*These notes are a work in progress and are not for circulation.

Hodge structures of weight  $n$  given by a filtration  $F^\bullet = \{F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}}\}$  that satisfies the first and second Hodge-Riemann bilinear relations (cf. [CM-SP] and [Ca]). We set  $G = \text{Aut}(V, Q)$  and then the monodromy transformation around the  $i^{\text{th}}$  coordinate axis in  $\Delta^\ell$  is given by  $T_i \in G$ .

**Complex algebro-geometric:** This is given by the Hodge-theoretic properties associated to a projective family

$$(I.1)_{\text{AG}} \quad \pi : \mathcal{X} \rightarrow \Delta^\ell$$

which has the local form (I.7) below of semi-stable reduction (cf. [AK]). We assume that the restriction

$$\pi : \mathcal{X}^* \rightarrow \Delta^{*\ell}$$

of (I.1)<sub>AG</sub> to  $\Delta^{*\ell}$  has smooth fibres and that the local monodromies around the coordinate axes in  $\Delta^\ell$  are unipotent.

In what follows we will frequently use the notation  $S^*$  both for the domain of a period mapping  $\Phi : S^* \rightarrow \Gamma \backslash D$  and the parameter space for a family of smooth varieties  $\mathcal{X}^* \rightarrow S^*$ , and by  $S$  a smooth variety containing  $S^*$  as an open set where  $S^* = S \setminus T$  with  $T$  being a normal crossing divisor. Then in (I.1)<sub>HT</sub> and (I.1)<sub>AG</sub> we have  $S = \Delta^\ell$ ,  $S^* = \Delta^{*\ell}$  and  $T =$  union of the coordinate hyperplanes in  $\Delta^\ell$ . We will denote by  $X_s$  the fibre  $\pi^{-1}(s)$  for  $s \in S$  and by  $X = \pi^{-1}(s_0)$  when  $S = \Delta^\ell$  and  $s_0$  is the origin. It will be assumed that  $X$  is connected of dimension  $d$ .

There are three main purposes of these notes. One is to compare the approaches (I.1)<sub>HT</sub> and (I.1)<sub>AG</sub>. For 1-parameter families ( $\ell = 1$ ) the asymptotic analysis of (I.1)<sub>HT</sub> is due to Schmid [Sc], and the algebro-geometric analysis of (I.1)<sub>AG</sub> and tight connection to [Sc] is carried out in the basic work [St1].

For general families the asymptotic analysis of (I.1)<sub>HT</sub> was initiated in [CKS1], [CKS2], [Kas1] and [KK]. Concerning (I.1)<sub>AG</sub> the direct extension of some of [St1] is done in a number of papers, frequently in the setting of logarithmic geometry; cf. [Fu1] and the references cited there. There is an extensive literature centering around (I.1)<sub>HT</sub>,

and there is also one centering around  $(I.1)_{AG}$  in both the local case above and the global case of  $\mathcal{X} \rightarrow S$  where  $\mathcal{X}$  and  $S$  are complete algebraic varieties.<sup>1</sup> We shall not attempt to give here an account of the literature, but rather refer to the references in the articles [C1] and [C2] by Cattani for  $(I.1)_{HT}$  and by Brosnan-El Zein [BE-Z] for  $(I.1)_{AG}$  which appeared in the recently published volume [C<sup>†</sup>] on Hodge theory arising from the 2010 summer school held at ICTP in Trieste.

In order to carry out the comparison between the approaches to  $(I.1)_{HT}$  and  $(I.1)_{AG}$  we shall need to review, and in some cases amplify, what is known about  $(I.1)_{AG}$  for general  $\ell$ . As will be explained more fully in Section II below, we shall do this in two steps, labelled (A) and (B) there. Step (A) consists of summarizing some of what is known about  $(I.1)_{AG}$  in the 1-parameter case  $\ell = 1$ . Here we closely follow the original work [St1] and the nice presentation in Chapter 11 of [PS]. We shall verbally describe the main conceptual ideas, the point being to prepare the way for step (B), which is to explain how the analogous formulation of the conceptual ideas enables one to know how to take the technical steps that are needed to extend the theory from the 1-parameter case to the several parameter case in  $(I.1)_{AG}$ .

As second purpose of these notes is the following: In the setting of  $(I.1)_{HT}$  Deligne predicted from the results that hold in the  $\ell$ -adic case using the Weil conjectures three remarkable properties of monodromy cones (cf. [CK] and [CKS1]):

- (I.2)(i) the weight filtration  $W(N)$  is independent of  $N \in \sigma$ ;
- (I.2)(ii) the compatibility of the weight filtrations among the faces of  $\sigma$ ;
- (I.2)(iii) the vanishing of the Koszul homology groups  $H_p(N_1, \dots, N_\ell; V)$  in positive weight (purity).

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<sup>1</sup>The penultimate story here is the Hodge theory of maps and the decomposition theorem (cf. [M], [dC] and [dCM]).

Regarding (I.2)(i), given a finite dimensional vector space  $E$  and a nilpotent endomorphism  $A \in \text{End}(E)$ , there is a unique weight filtration  $W(A)$ , central at zero and characterized by

- $A : W_k(A) \rightarrow W_{k-2}(A)$
- $A^k : \text{Gr}_k^W E \xrightarrow{\sim} \text{Gr}_{-k}^W E$  is an isomorphism for  $k \geq 0$ .

Each  $N \in \sigma$  then has a weight filtration, and (I.2)(i) says that these are the same for all  $N$ ; thus, one may set  $W(\sigma) = W(N)$  for any  $N \in \sigma$ .

Regarding (I.2)(ii), given a vector space  $E$ , a filtration  $W_0$  on  $E$ , and a nilpotent endomorphism  $A$  that preserves  $W_0$ , we denote by  $\text{Gr}_k^{W_0} A \in \text{End}(\text{Gr}_k^{W_0} E)$  the induced endomorphism. Then a *weight filtration of  $A$  relative to  $W_0$*  is a filtration  $W$  of  $E$  such that

- $AW_k \subset W_{k-2}$ ,
- $W \text{Gr}_k^{W_0} E = W(\text{Gr}_k^{W_0} A)$ .

The notation  $W \text{Gr}_k^{W_0} E$  means the filtration on  $\text{Gr}_k^{W_0} E$  induced by  $W$ . There is at most one such  $W$ , and we will be interested in the situation when  $W = W(A)$ . Taking the case  $\ell = 2$  in (I.2)(ii), compatibility means that  $W(N)$  is a weight filtration of  $N_1$ , relative to  $W(N_2)$ . In general, there is an extension of this to a general  $\sigma$  and its faces (cf. [CK], (3.3) on page 113).

Further Hodge-theoretic properties of (I.1)<sub>HT</sub> that are proved in [CKS1] and are here stated in the case  $\ell = 2$  are

- (I.3)(i) for each  $s_2 \neq 0$ , the limit of  $\Phi(s_1, s_2)$  as  $s_2 \rightarrow 0$  gives a limiting mixed Hodge structure  $\Phi(s_1, 0)$ , and when  $s_1$  varies this gives a variation of limiting mixed Hodge structure in the sense of [SZ] and [E-Z];<sup>2</sup>
- (I.3)(ii) for  $\Delta_\lambda = \{(s, \lambda s) \text{ where } \lambda \neq 0, \infty\}$ , the restriction  $\Phi_\lambda$  of  $\Phi$  to  $\Delta_\lambda^*$  gives a 1-parameter case of (I.1)<sub>HT</sub>, and the equivalence class of limiting mixed Hodge structures given by the limit of  $\Phi(s, \lambda s)$  as  $s \rightarrow 0$  is independent of  $\lambda$ ;

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<sup>2</sup>In these references the notion of variation of mixed Hodge structures is discussed. Here we are adding the additional conditions that we have *limiting* mixed Hodge structures; these may readily be included in the theory developed in loc. cit.

(I.3)(iii) the limit as  $s_2 \rightarrow 0$  in the sense of [SZ] of the mixed Hodge structures  $\Phi(s_1, 0)$  exists as a limiting mixed Hodge structure and its equivalence class is equal to  $\Phi_\lambda(0)$ .

In the picture below the limit in (I.3)(i) corresponds to  $\Delta_{s_2}$ , that in (I.3)(ii) to  $\Delta_\lambda$ , and that in (I.3)(iii) to  $\Delta_{s_2}$  followed by  $\Delta_{s_1}$ .

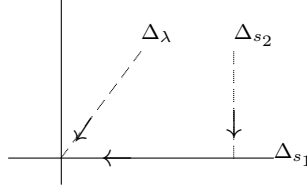


FIGURE 1

One may informally say that the best results that can be defined are in fact true.

To establish some notations and terminology we recall that a mixed Hodge structure is given by  $(V, W, F^\bullet)$  where  $W$  is an increasing weight filtration on  $V$ ,  $F^\bullet$  is a decreasing Hodge filtration on  $V_{\mathbb{C}}$ , and where  $F^\bullet$  induces on each  $\text{Gr}_k^W V$  a Hodge structure of weight  $k$ . A *limiting mixed Hodge structure* is given by a mixed Hodge structure  $(V, W, F^\bullet)$  where  $W = W(N)$  is the weight filtration associated to a nilpotent endomorphism  $N \in \text{End}(V)$ . A *polarized limiting mixed Hodge structure* has the additional data of a bilinear form  $Q$  with  $N \in \text{End}_Q(V)$ , and where the polarization conditions (2.2) in [CK] are satisfied. Finally, two limiting mixed Hodge structures  $(V, W(N), F^\bullet)$  and  $(V, W(N), F'^\bullet)$  are said to be *equivalent* if

$$F'^\bullet = \exp(\lambda N) \cdot F^\bullet$$

for some  $\lambda \in \mathbb{C}$ . We note that equivalent mixed Hodge structures induce the same Hodge structures on  $\text{Gr}_k^{W(N)} V$ , and also on the 2-step mixed Hodge structures on  $W_k(N)/W_{k-2}(N)$ . The equivalence class of a limiting mixed Hodge structure will be denoted  $(V, W, [F^\bullet])$ .

Finally there is the issue of identifying the limiting mixed Hodge structures in (I.3)(i)–(iii) in the algebro-geometric case (I.1)<sub>AG</sub>. In [Fu1], the methods in [St1] are extended to show that

(I.4) the groups  $\mathbb{H}^n \left( \Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_{X_s} \right)$  give mixed Hodge structures for each  $s \in S$ .

As will be discussed below, these are limiting mixed Hodge structures that will be naturally identified with those in (I.3)(i)–(ii).

More precisely, there is a mixed Hodge structure  $(V, W, F^\bullet)$  where  $V_{\mathbb{C}} = \mathbb{H}^n(\Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$  and where the Hodge filtration  $F^\bullet$  is induced from the bétè filtration on  $\Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X$ . The  $\mathbb{Q}$ -structure on  $V_{\mathbb{C}}$ , i.e., the  $\mathbb{Q}$ -vector space  $V$  with  $V \otimes \mathbb{C} \cong V_{\mathbb{C}}$ , is somewhat subtle to define. A very nice approach to this, using logarithmic structures, is in [St2] and is used in [PS], [Fu1]. These methods may be adapted to the several parameter case as presented below and, since this is not directly relevant to the central points we are seeking to make, will not be taken up here.

To complete the second purpose of these notes we need to first show that these mixed Hodge structures are limiting mixed Hodge structures and that the properties (I.2)(i)–(ii) and (I.3)(i)–(iii) hold for them. The essential point in (I.2)(i) is to compare two filtrations  $W$  and  $W(N)$  on the space  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$ . Here  $W$  is the filtration induced on this space by a filtration (VIII.9) on the complex of sheaves  $A^\bullet$  given by (VIII.3) and that is quasi-isomorphic to  $\Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X$ . This filtration in  $A^\bullet$  then induces the filtration  $W$  on  $\mathbb{H}^n(A^\bullet) \cong \mathbb{H}^n(\Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$ . The other filtration is the monodromy filtration  $W(N)$ . Under the assumption that (I.1)<sub>AG</sub> is a projective family, an argument using the full strength of the Hodge-Riemann bilinear relations gives that  $W = W(N)$ . This proves that  $(V, W, F^\bullet)$  is a limiting mixed Hodge structure, and examination of the argument then gives (I.2)(i) in the case (I.1)<sub>AG</sub>.

As for (I.2)(ii) in the algebro-geometric case, we suppose that  $\ell = 2$  and set  $X_{(s_1, s_2)} = \pi^{-1}(s_1, s_2)$ . Referring to Figure 1 above, the family over  $\Delta_{s_2}$  gives a limiting mixed Hodge structure with weight filtration  $W(N_2)$ . Call this family  $(V, W(N_2), F_{(s_1, 0)}^\bullet) =: V_{(s_1, 0)}$ . Then  $\mathrm{Gr}^{W(N_2)} V_{(s_1, 0)}$  is a variation of Hodge structure over  $\Delta_{s_1}^*$  with monodromy

logarithm  $N_1$ . Thus on  $\mathrm{Gr}^{W(N_2)} V_{(s_1,0)}$  there is an induced action  $\mathrm{Gr}^{W(N_2)} N_1$ , and it has a monodromy weight filtration  $W(\mathrm{Gr}^{W(N_2)} N_1)$ . On the other, on  $\Delta_\lambda$  there is the monodromy logarithm  $N = N_1 + N_2$ , and the result is that  $W(N)$  is weight filtration of  $N_1$  relative to  $W(N_2)$ .

A third purpose of these notes is this: Letting  $s = (s_1, \dots, s_\ell)$  denote a point in  $\Delta^{*\ell}$  and  $s_0 = (0, \dots, 0)$  the origin, in each of the cases (I.1)<sub>HT</sub> and (I.1)<sub>AG</sub>, there is an equivalence class of limiting mixed Hodge structures associated to the limit of the Hodge structures as  $s \rightarrow s_0$ . In the case (I.1)<sub>HT</sub> this equivalence class of limiting mixed Hodge structures is polarized by the nilpotent cone  $\sigma$ .<sup>3</sup> Denoting by  $X = \pi^{-1}(s_0)$  the fibre over the origin, in the case (I.1)<sub>AG</sub> there is defined a subspace

$$(I.4) \quad T_{s_0} \subset T_X \mathrm{Def}(X)$$

where  $\mathrm{Def}(X)$  is the Kuranishi space parametrizing the versal germ of deformations of  $X$  and  $T_X \mathrm{Def}(X)$  is its Zariski tangent space ([Pa]). Here, for simplicity of exposition we are assuming that the natural map  $T_{s_0} \Delta^\ell \rightarrow T_X \mathrm{Def}(X)$  is injective with image  $T_{s_0}$ . We will see that

$$(I.3)_{\mathrm{HT}} \quad T_{s_0} \cong \sigma_{\mathbb{C}} =: \sigma \otimes \mathbb{C}$$

and that, setting  $\mathcal{Y} = \pi^{-1}$  (union of the coordinate hyperplanes in  $\Delta^\ell$ ),

$$(I.3)_{\mathrm{AG}} \quad T_{s_0} \subset \mathbb{E}xt_{\mathcal{O}_X}^1 \left( \Omega_{\mathcal{X}_\epsilon / \Delta(\epsilon)^\ell}^1(\log \mathcal{Y}) \otimes \mathcal{O}_X, \mathcal{O}_X \right).$$

Here,  $\mathcal{X}_\epsilon$  is a 1<sup>st</sup> order neighborhood of  $X$  in  $\mathcal{X}$  and  $\Delta(\epsilon) = \mathrm{Spec} \mathbb{C}[\epsilon]$  where  $\epsilon^2 = 0$ . Using the identification (I.3)<sub>HT</sub>, in the open set

$$T_{s_0}^\circ = \{\xi = \sum z_i N_i, z_i \neq 0\} \subset T_{s_0}$$

each point  $\xi$  determines a mixed Hodge structure  $(V, W(\sigma), F_\xi^\bullet)$  where

$$F_\xi^p = F^p \mathbb{H}^n \left( \Omega_{\mathcal{X}_\epsilon / \Delta(\epsilon)^\ell}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X \right).$$

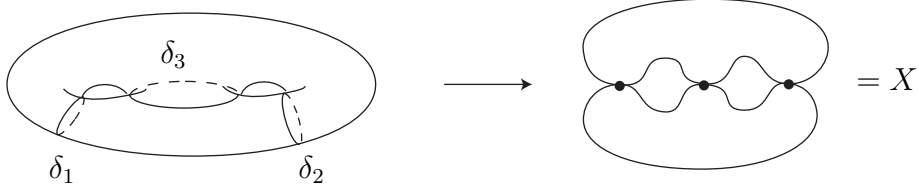
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<sup>3</sup>This means that the bilinear forms  $\tilde{Q}_k$  defined in [CK] by  $Q$  and each  $N$  on the primitive spaces  $(\mathrm{Gr}_k^{W(N)} V)_{\mathrm{prim}}$  satisfy the first and second Hodge-Riemann bilinear relations. We note that the primitive spaces  $(\mathrm{Gr}_k^{W(N)} V)_{\mathrm{prim}}$  and bilinear forms  $\tilde{Q}_k$  depend on  $N$ , even though  $W(N)$  is independent of  $N$ .

Implicit here is that given  $\xi \in T_{s_0}^o$ , there is a 1<sup>st</sup> order deformation  $\mathcal{X}_\epsilon \rightarrow \Delta(\epsilon)^\ell$  and the  $\Omega_{\mathcal{X}_\epsilon/\Delta(\epsilon)^\ell}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X$  may be defined in terms of  $\xi$ , a point that in the  $\ell = 1$  case appears implicitly in [Fr1] and explicitly in [St2]. When  $\xi \in \sigma$  this is a limiting mixed Hodge structure that is polarized by the cone  $\sigma$ . With these notations, the 1<sup>st</sup> order variation of the limiting mixed Hodge structure is induced from the natural pairing

$$(I.4)_{AG} \quad \mathbb{E}xt_{\mathcal{O}_X}^1 \left( \Omega_{\mathcal{X}_\epsilon/\Delta(\epsilon)^\ell}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X, \mathcal{O}_X \right) \\ \rightarrow \text{End}_{\text{LMHS}} \mathbb{H}^n \left( \Omega_{\mathcal{X}_\epsilon/\Delta(\epsilon)^\ell}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X \right).$$

Whereas the usual differential of the period mapping at infinity loses some of the information in the variation of the extension data in the limiting mixed Hodge structure, the refinement  $(I.4)_{AG}$  captures the variation in all of the extension data. For example, in the curve degeneration



with the three vanishing cycles  $\delta_1, \delta_2, \delta_3$  above, all of the 3-parameters in the extension data are lost in the traditional differential of the period mapping at  $X \in \overline{\mathcal{M}}_2$  the Deligne-Mumford compactification of genus 2 curves. It was in seeking to understand this phenomenon, and put it in a form amenable to the computation of examples, that led to the paper [GG] and to these explanatory notes as an accompaniment to that paper.

We will now explain some of the terms and notations that appear above.

- The local normal form for the map  $(I.1)_{AG}$  is given, following [AK], as follows: in a neighborhood  $U$  around each  $x \in X$ , there is an embedding

$$U \subset \mathbb{C}^{d+k} \times \mathbb{C}^k$$



and index sets  $I_1 = \{1, \dots, i_1\}$ ,  $I_2 = \{i_1 + 1, \dots, i_2\}, \dots, I_k$  in  $\{1, \dots, n + k\}$  and  $\{j_1, \dots, j_k\} \subset \{1, \dots, \ell\}$  such that (I.1)<sub>AG</sub> is given by

$$(I.7) \quad \begin{cases} x_{I_1} = s_{j_1} \\ \vdots \\ x_{I_k} = s_{j_k}. \end{cases}$$

Thus the fibres  $X_s \cap U$  over  $s_{j_1} = \dots = s_{j_k} = 0$  are

$$(I.8) \quad U_s = U_1 \times \dots \times U_k \times U_0$$

where  $U_j$  is a normal crossing variety in  $\mathbb{C}^{i_j - i_{j-1}}$ , and the parameters  $U_0$  are an open set in  $\mathbb{C}^{d - i_k}$ . The number  $k$  of local factors is described by

$$k = \left\{ \begin{array}{l} \text{number of factors in } \pi(x) \in \Delta^\ell \text{ that are} \\ \text{over the origin in the corresponding disc} \end{array} \right\}.$$

In order to simplify the notations, we shall usually take  $j_1 = 1, \dots, j_k = k$  in (I.7). In fact, one of the problems in expositions of the subject is that the multi-index notations may obscure the basically simple local geometry. This issue is at least partially alleviated by the use of divided power Koszul complexes as in [St2].

In the appendix we will describe the various log complexes that appear, including  $\Omega_{x/S}^\bullet(\log \mathcal{Y})$ , locally in terms of the normal form (I.7) above.

- As noted above,  $\text{Def}(X)$  refers to the Kuranishi space for the deformations of the compact analytic variety  $X$  [Pa], and we shall use the identification

$$(I.9) \quad T_X \text{Def}(X) = \mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)$$

of its Zariski tangent space.

We remark that the formalism of logarithmic geometry provides an alternative setting for the theory, especially in the study of (I.1)<sub>AG</sub> (cf. [A<sup>†</sup>] and the references cited there). As noted above, this setting is especially convenient for defining the  $\mathbb{Q}$ -structure on the mixed Hodge structures constructed from de Rham type complexes (cf. [St2], [Fu2] and [FN]).

Regarding (I.1)<sub>HT</sub>, there is also a formalism of logarithmic Hodge theory (cf. [KU]) which leads to extensions of period maps

$$(I.10) \quad \begin{cases} \Phi : \Delta^{*\ell} \rightarrow \Gamma \backslash D \\ \quad \cap \quad \quad \cap \\ \Phi_e : \Delta^\ell \rightarrow \Gamma \backslash D_\sigma \end{cases}$$

where  $\Phi_e$  is a morphism of log-analytic varieties. Here, setting

$$\begin{cases} T_j = \Delta_j \times \cdots \times \{0\} \times \cdots \times \Delta_\ell \\ T = \sum_{j=1}^{\ell} T_j = \text{coordinate hyperplanes in } \Delta^\ell, \end{cases}$$

we have that  $\mathcal{Y} = \pi^{-1}T$  is a normal crossing divisor in  $\mathcal{X}$ . Then with the canonical logarithmic structures associated to a normal crossing divisor in a smooth variety,  $(\mathcal{X}, \mathcal{Y}) \rightarrow (S, T)$  is a morphism of log-analytic varieties. Moreover, setting

$$B_\sigma = D_\sigma \backslash D$$

from [KU] one has that  $\Gamma \backslash D$  has the structure of a log-analytic variety with slits and

$$\Phi_e : (S, T) \rightarrow (\Gamma \backslash D_\sigma, \Gamma \backslash B_\sigma)$$

is a morphism of log analytic varieties.

In part because some of the audience for these notes are people who work on (I.1)<sub>HT</sub> from a complex analytic/geometric perspective, we have decided to give the exposition in a more traditional framework.

## II. OUTLINE OF THE ALGEBRO-GEOMETRIC APPROACH

As was mentioned above, for  $\ell = 1$  the analysis of the situation (I.1)<sub>AG</sub> is originally due to [St1]. Since then there have been a number of works building on [St1]; we mention especially the presentation in Chapter 11 of [PS] (where part of the discussion is adapted from [GN]). In recent years some, but not all, aspects of the theory have been extended to the case of (I.1)<sub>AG</sub> for general  $\ell$ ; cf. [Fu1]. In presenting the theory it may sometimes be the case that the notations may not make transparent the central geometric ideas. This is especially true in the case of several parameter families where multi-multi-index notations

appear. Of course, something like this is necessary for the detailed proofs.

What we shall attempt to do in these notes is the following:

- (A) Give a summary presentation of the case  $\ell = 1$ , explaining verbally the key steps;
- (B) Explain how each of the key steps in (A) may be extended to the case of general  $\ell$ , and give an outline of how these steps may be carried out in the case  $\ell = 2$ . This case captures the essential features and where the notations are less complex, but even here we shall not give the detailed calculations. Here in greatly simplified form the main points are
  - (i) the results of the  $\ell = 1$  case extend to the case of a global product

$$\mathcal{X}_1 \times \cdots \times \mathcal{X}_\ell \rightarrow \Delta_1 \times \cdots \times \Delta_\ell;$$

- (ii) the theory is based on standard natural “local to global” techniques, and locally the case of general  $\ell$  is a product of the  $\ell = 1$  case.

Although there are a number of subtleties along the way, we emphasize that there is nothing particularly new or original in what follows. The purpose is to give an exposition of (I.1)<sub>AG</sub> in a way that enables us to connect as directly as possible with (I.1)<sub>HT</sub>.

We now list the key steps in (A) that will be carried out below.

**Step one:** *Localization to  $X$ .* We are seeking to describe the limiting mixed Hodge structure  $(V, W, F^\bullet)$  associated to  $\mathcal{X} \xrightarrow{\pi} \Delta$ . For this one needs the  $\mathbb{Q}$ -vector space  $V$ , weight filtration  $W$ , and a Hodge filtration  $F^\bullet$ . One thinks of

$$V = H^n(X_\mathfrak{s}, \mathbb{Q})$$

where  $\mathfrak{s}$  is a “general point” of  $\Delta^*$ ,

$$W = W(N),$$

where  $N \in \text{End}(V)$  is the nilpotent transformation given by the logarithm of monodromy and  $W(N)$  is the filtration associated to this

nilpotent transformation, and

$$F^\bullet = \lim_{s \rightarrow 0} F^\bullet H^n(X_s)_{\text{prim}}$$

is a to-be-defined limit. As previously noted, the issue of the  $\mathbb{Q}$ -structure is an interesting one, one that however is not directly relevant to these notes. Here we shall mainly deal with the  $\mathbb{C}$ -structure, referring to [St2], [PS] and [Fu2], [FN] for the  $\mathbb{Q}$ -structure.

Following [St1] one *defines*

$$(II.1) \quad V_{\mathbb{C}} = H^n(\tilde{\mathcal{X}}^*, \mathbb{C})_{\text{prim}}$$

where

$$\begin{array}{ccc} \tilde{\mathcal{X}}^* & \longrightarrow & \mathcal{X}^* \\ \downarrow & & \downarrow \\ \mathbb{H} & \longrightarrow & \Delta^* \end{array}$$

is obtained by passing to the pullback of  $\mathcal{X}^* \rightarrow \Delta^*$  under the universal covering map  $\mathbb{H} \rightarrow \Delta^*$  given by

$$s = \exp 2\pi iz, \quad \text{Im } z > 0.$$

With this definition it is made precise what is meant by the cohomology of a general fibre and the action of monodromy, as induced by the deck transformations in the above diagram, on that cohomology.

We introduce the notation, used in [Fr1],

$$(II.2) \quad \Lambda_X^\bullet = \Omega_{\mathcal{X}/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X.$$

We will see that  $\Lambda_X^\bullet$  depends only on the first order neighborhood of  $X$  in  $\mathcal{X}$ ; i.e., on a vector  $\xi \in T_X \text{Def}(X)$ . The first step consists in establishing the identification

$$(II.3) \quad V_{\mathbb{C}} = \mathbb{H}^n(\Lambda_X^\bullet).$$

The Hodge filtration will then be *defined* by

$$(II.4) \quad F^\bullet V = F^\bullet \mathbb{H}^n(\Lambda_X^\bullet),$$

where the right-hand side is the filtration on hypercohomology induced by the bétè filtration

$$F^p \Lambda_X^\bullet = \{0 \rightarrow \Lambda_X^p \rightarrow \Lambda_X^{p+1} \rightarrow \dots\}$$

on  $\Lambda_X^\bullet$ . We will see that scaling  $\xi$  by  $\xi \rightarrow \lambda\xi$  induces the change

$$F^\bullet \rightarrow \exp(\lambda N) \cdot F^\bullet.$$

The eventual conclusion will be

- the standard picture  $\mathcal{X} \rightarrow \Delta$  in (I.1)<sub>AG</sub> in the case  $\ell = 1$  defines an equivalence class of limiting mixed Hodge structures;
- the data  $(X, \xi)$  defines a limiting mixed Hodge structure (no equivalence class).

Having defined the vector space and the Hodge filtration, the next step is

**Step two:** *The weight filtration and  $E_1$ -term of the weight spectral sequence.* We will first note that the naïve filtration induced on  $\Lambda_X^\bullet$  by the standard weight filtration  $W_r\Omega_X^\bullet(\log X)$  given by “ $\leq r$   $dx_i/x_i$  terms” does not lead to a good answer. One reason for this is presented in [PS]; below we shall give another heuristic reason. The key insight that arises in [St1] is to consider the first sequence

(II.5)

$$0 \rightarrow \Omega_{X/\Delta}^p(\log X) \otimes \mathcal{O}_X \xrightarrow{ds/s} \Omega_X^{p+1}(\log X) \otimes \mathcal{O}_X \rightarrow \Omega_{X/\Delta}^{p+1}(\log X) \otimes \mathcal{O}_X \rightarrow 0$$

that arises naturally when one interprets the regularity theorem for the Gauss-Manin connection (cf. [De1] and [K]). Then replacing the first map by the inclusion

$$(II.6) \quad 0 \rightarrow \Omega_{X/\Delta}^p(\log X) \otimes \mathcal{O}_X \rightarrow \Omega_X^{p+1}(\log X)/W_0\Omega_X^{p+1}(\log X)$$

leads to Steenbrink’s double complex

$$A^{p,q} = \Omega_X^{p+q+1}(\log X)/W_q\Omega_X^{p+q+1}(\log X)$$

with commuting differentials  $d' = \text{usual } d$  and  $d'' = \wedge(ds/s)$ . Continuing the sequence (II.5) leads to a quasi-isomorphism

$$\Lambda_X^\bullet \rightarrow A^\bullet$$

between  $\Lambda_X^\bullet$  and the total complex  $(A^\bullet, d' + d'')$  associated to  $(A^{\bullet, \bullet}; d', d'')$ . Simple numerology, given below, then suggests the definition

$$W_r A^{p,q} = W_{r+2q+1} \Omega_X^{p+q+1}(\log X) / W_q \Omega_X^{p+q+1}(\log X)$$

for the weight filtration  $W$  for the mixed Hodge structure  $(V, W, F^\bullet)$ .

Via Poincaré residues the associated graded to the weight filtration is a direct sum of Tate twists of the complexes

$$((a_i)_* \Omega_{X^{[i]}}^\bullet, d)$$

(cf. [De1]), and consequently the  ${}_W E_1^{p,q}$ -term of the weight spectral sequence for  $(A^\bullet, d)$  is a direct sum of the groups

$$(II.7) \quad {}_W E_1^{p,q} = \bigoplus H^i(X^{[j]})(-k)$$

where  $i, j, k$  run over an index set determined by  $p, q, d = \dim X$ , and  $n$  where the spectral sequence abuts to  $\mathbb{H}^n(\Omega_{X/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X)$ .<sup>4</sup>

**Step three:** *The  $\mathfrak{sl}_2 \times \mathfrak{sl}_2$  action and construction of the limiting mixed Hodge structure.* Given a nilpotent endomorphism  $M \in \text{End}(U)$  of a vector space  $U$ , as noted above there is an associated weight filtration  $W(M)$ , by convention centered at zero and with the defining properties

$$M : W_k(M) \rightarrow W_{k-2}(M),$$

and for  $k \geq 0$

$$M^k : \text{Gr}_k^W \xrightarrow{\sim} \text{Gr}_{-k}^W.$$

A *grading element*  $Y$  for  $W(M)$  is given by a semi-simple  $Y \in \text{End}(U)$  with eigenvalues in  $\mathbb{Z}$  and with

$$W_k(M) = \bigoplus_{\ell \leq k} U_\ell$$

where the  $U_\ell$  are the eigenspaces of  $M$ . It follows that

$$[Y, M] = -2M.$$

---

<sup>4</sup>One of the main difficulties in expositions of the theory is in the book-keeping of the indices that appear. Here we shall try to explain the conceptual ideas which then indicate how the correct indexing may be determined, referring to [St1], [PS] and [Fu1] for the details.

By the Jacobson-Morosov theorem such a grading element always exists and we may uniquely complete  $M, Y$  to an  $\mathfrak{sl}_2$ -triple  $(M, Y, M^+)$ . Conversely, such an  $\mathfrak{sl}_2$ -triple gives a grading element  $Y$  for  $W(M)$ .

The use of such  $\mathfrak{sl}_2$  triples is very common in Hodge theory. Examples include (i) the structure of the cohomology ring of a smooth projective variety  $Z$  where  $U = H^*(Z)$  and  $L = c_1(\mathcal{O}_Z(1))$  is the Lefschetz operator and the grading element is determined by the direct sum decomposition  $H^*(Z)$ , and (ii) the analysis of  $(I.1)_{\text{HT}}$  by Schmid [Sc] and Cattani-Kaplan-Schmid [CKS1] using the  $\mathfrak{sl}_2$ -orbit theorems. Especially in the presence of an invariant bilinear form, the decomposition of  $U$  into irreducible summands yields very rich structures.

In the  $\ell = 1$  situation  $(I.1)_{\text{AG}}$  where  $\mathcal{X} \subset \mathbb{P}^A$  is a family of projective varieties, there are two commuting nilpotent endomorphisms

$$N, L \in \text{End}({}_W E_1).$$

The direct sum decomposition (II.7) gives canonical grading elements, and it was noted in [GN] and used effectively in [PS] that this gives commuting  $\mathfrak{sl}_2$ 's acting on  ${}_W E_1$  as morphisms of Hodge structures and commuting with the differential

$$d_1 : {}_W E_1 \rightarrow {}_W E_1$$

of the weight spectral sequence. The spectral sequence degenerates at  ${}_W E_2$ , and on  ${}_W E_\infty = \text{Gr}^W V$  we have an action of commuting  $\mathfrak{sl}_2$ 's. Exploiting the very rich consequences of this structure leads to the results in [St1] that  $(V, W(N), F^\bullet)$  is a *polarized limiting mixed Hodge structure*.

We now turn to (B):

**Step four:** Steps 1–3 above for the case  $\ell = 1$  are given by passing from local to global using hypercohomology of the complex  $\Lambda_X^\bullet = \Omega_{X/\Delta}(\log X) \otimes \mathcal{O}_X$ . The Hodge filtration is the usual bête one. The weight filtration is described in terms of Steenbrink's quasi-isomorphism

$$\Lambda_X^\bullet \rightarrow A^\bullet,$$

and as we shall see below the monodromy operator is induced by the projection

$$\nu : A^{p,q} \rightarrow A^{p-1,q+1}$$

in the bi-complex  $A^{\bullet,\bullet}$ . In case  $X$  is locally a product of normal crossing divisors,  $\Lambda_X^\bullet$  is locally the tensor product of the analogous sheaves on the factors of the tensor product of the factors. The point now is to check that steps two and three above together with the description of the monodromy operator extend to the situation where we have only locally a product of normal crossing divisors. Not unexpectedly, the most involved part is to show that the filtration on  $V_{\mathbb{C}} = \mathbb{H}^n(\Lambda_X^\bullet)$  induced by the tensor product of the weight filtrations on the  $A_i^\bullet$ 's is well defined and agrees with the filtration  $W(N)$  on  $V_{\mathbb{C}}$ . For example, (II.5) is replaced by a filtration on  $\Omega_X^p(\log X) \otimes \mathcal{O}_X$ , monodromy comes out of the  $d_1$  in the associated spectral sequence and (II.7) becomes

$$0 \rightarrow \Omega_{X/S}^p(\log X) \otimes \mathcal{O}_X \rightarrow \Omega_X^{p+\ell}(\log X) / \sum_{i=1}^{\ell} W_{i,1} \Omega_X^{p+\ell}(\log X)$$

where the notations will be explained below.

More subtle is the argument for (I.2)(i). Using the notations from the introduction, because of the elementary result that if we have nilpotent endomorphisms  $A_i \in \text{End}(E_i)$  where  $E_1, E_2$  are *finite dimensional* vector spaces, the weight filtration on  $E_1 \otimes E_2$  given by  $\lambda_1 A_1 \otimes \text{Id}_{E_2} + \text{Id}_{E_1} \otimes \lambda_2 A_2$  is independent of  $\lambda_1, \lambda_2 \in \mathbb{C}^*$ , at first one might suspect that (I.2)(i) should hold in this generality. But the result is a Hodge-theoretic one whose proof is the algebro-geometric case (I.1)<sub>AG</sub>, as in the Hodge-theoretic case (I.1)<sub>HT</sub>, requires the full use of the Hodge-Riemann bilinear relations. It is here that the conditions  $\lambda_1, \lambda_2 \in \mathbb{R}^{>0}$  come in. Similar considerations hold for the proof of (I.2)(ii).

As a closing aside to this section, there are three algebro-geometric phenomena that may arise in the situation (I.1)<sub>AG</sub> when  $\ell \geq 2$ :

- (a) the  $N_i$  are the nil-negative elements of commuting  $\mathfrak{sl}_2$ 's action on  $V$ ;



- (b) the Koszul cohomology groups  $H_p(N_1, \dots, N_\ell; V)$  vanish for  $p \neq 0$ ;  
(c) the weight filtration  $W(N_\lambda) = W(N)$  for all  $N_\lambda = \sum_{i=1}^\ell \lambda_i N_i$ ,  
 $\lambda_i \in \mathbb{C}^*$  (and not just  $\lambda_i \in \mathbb{R}^{>0}$ ).

These phenomena are related; e.g.,

$$(a) \Rightarrow (b), (c),$$

but the exact relationship among them is as yet not understood. For example, there are obstructions that commuting  $\mathfrak{sl}_2$  action on  $\mathrm{Gr}^W V = {}_W E_\infty$  lift to commuting  $\mathfrak{sl}_2$  actions on  $V$ . It seems that these have not yet been computed.

### III. LOCALIZATION TO $X$

We consider the standard situation (I.1)<sub>AG</sub>

$$(III.1) \quad \mathcal{X} \xrightarrow{\pi} \Delta$$

when  $\ell = 1$ , and we want to construct from this data a limiting mixed Hodge structure  $(V, W, F^\bullet)$  that reflects the limit of the family of Hodge structures

$$\lim_{s \rightarrow 0} H^n(X_s, \mathbb{Q}).$$

With the definition

$$(III.2) \quad V_{\mathbb{C}} = H^n(\tilde{\mathcal{X}}^*, \mathbb{C}),$$

localization to  $X$  means the natural identification in [St1]

$$(III.3) \quad V_{\mathbb{C}} \cong \mathbb{H}^n(\Omega_{\mathcal{X}/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X).$$

We will not repeat the argument from loc. cit., but will make some comments with an eye towards extending this identification to the general case.

- (i) One way of viewing the underlying geometry is via the *Clemens retraction map* [C]

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{r} & X \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \{0\} \end{array}$$

which gives that the inclusion  $X \hookrightarrow \mathfrak{X}$  is a homotopy equivalence and leads to the specialization or collapsing map

$$r_s : X_s \rightarrow X.$$

It is the analysis of this map, together with the monodromy transformation

$$T_s : X_s \rightarrow X_s$$

with which it homotopy commutes, that from a classical Lefschetz-style approach leads to the limiting mixed Hodge structure (loc. cit.). Although we will not follow this approach here, we mention it because the above picture extends to the general situation (I.1)<sub>AG</sub> (personal communication from Herb Clemens). Thus with the same definition (III.2) for  $V$  it is plausible that the identification (III.3) will extend to give

$$(III.4) \quad V_{\mathbb{C}} \cong \mathbb{H}^n(\Omega_{\mathfrak{X}/\Delta^\ell}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X).$$

(ii) If one thinks of the period mapping (I.1)<sub>HT</sub> in the case  $\ell = 1$  as given by a variation of Hodge structure over  $\Delta^*$  consisting of a vector bundle  $\mathbb{V}^* \rightarrow \Delta^*$  with connection  $\nabla$ , then because the monodromy is unipotent there is the canonical *Deligne extension* [De1] to a vector bundle  $\mathbb{V} \rightarrow \Delta$  where

$$\nabla : \mathbb{V} \rightarrow \mathbb{V} \otimes \Omega_\Delta^1(\log 0)$$

has regular singular points, and where the logarithm  $N$  of monodromy is given by  $1/2\pi i$  times the residue  $\text{Res}_{\{0\}} \nabla$  of the connection. In the case (I.1)<sub>AG</sub> one has the identifications

$$\mathbb{V}^* = \mathcal{O}_{\Delta^*}(R_\pi^n \mathbb{C}) \cong \mathbb{R}_\pi^n \Omega_{X^*/\Delta^*}^\bullet$$

and

$$\mathbb{V} = \mathbb{R}_\pi^n \Omega_{\mathfrak{X}/\Delta}^\bullet(\log X).$$

For the Deligne extension there are two types of frames defined in a neighborhood of the origin:

- (a) single-valued holomorphic frames  $f_\alpha(s)$ ;
- (b) multi-valued horizontal frames  $e_\alpha(s)$ .

The latter are defined for  $s \neq 0$  and satisfy  $\nabla e(s) = 0$ .

Setting  $\ell(s) = \log s/2\pi i$ , these frames are related by

$$(III.5) \quad e_\alpha(s) = \sum_{\lambda=0}^n A_{\alpha\lambda}^\beta \ell(s)^\lambda f_\beta(s)$$

where  $A_{\alpha\lambda}^\beta(s)$  is holomorphic. Given the horizontal frame  $e_\alpha(s)$  and coordinates in  $\Delta$ , there is a distinguished choice of holomorphic frame defined as follows. Analytic continuation of  $e_\alpha(s)$  around  $s = 0$  transforms  $e_\alpha(s)$  by the monodromy matrix  $T = \exp N$ . Then

$$f_\alpha(s) = \sum_{\beta} \exp(-\ell(s)N)_\alpha^\beta e_\beta(s)$$

gives a holomorphic frame relative to which

$$(III.6) \quad \nabla f_\alpha(s) = \sum_{\beta} N_\alpha^\beta f_\beta(s) \frac{ds}{s};$$

i.e., the connection matrix is  $N ds/s$ .

The reason for recalling this classical material is two-fold. One is the introduction of the complex of sheaves  $\mathcal{L}^\bullet$ , introduced in [St1] and perhaps suggested by (III.5), whose sections are of the form

$$\omega = \sum_{\alpha} \omega_\alpha \ell(s)^\alpha$$

with  $\omega_\alpha$  being a section of  $i^* \Omega_{\mathcal{X}}^\bullet(\log X)$  where  $i \cdot X \hookrightarrow \mathcal{X}$  is the inclusion. In loc. cit. it is shown that

$$(III.7) \quad H^n(\tilde{X}^*, \mathbb{C}) \cong \mathbb{H}^n(\mathcal{L}^\bullet),$$

$$(III.8) \quad \mathbb{H}^n(\mathcal{L}^\bullet) \cong \mathbb{H}^n(\Omega_{\mathcal{X}/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X).$$

The isomorphism (III.7) is first obtained by using the standard isomorphism

$$H^n(\tilde{\mathcal{X}}^*, \mathbb{C}) \cong \mathbb{H}^*(\Omega_{\tilde{\mathcal{X}}^*}^\bullet),$$

and then localizing the right-hand side along  $X \subset \mathcal{X}$  by restricting to discs  $\Delta(\epsilon)$  of radius  $\epsilon$  and letting  $\epsilon \rightarrow 0$ . The right-hand side then becomes  $\mathbb{H}^*(k_* i^* \Omega_{\tilde{\mathcal{X}}^*}^\bullet)$  where  $k \cdot \tilde{X}^* \rightarrow X$  is the composition of  $\tilde{X}^* \rightarrow \mathcal{X}^*$

and the inclusion  $X \hookrightarrow \mathcal{X}$ . Then (III.7) follows by establishing a quasi-isomorphism

$$\mathcal{L}^\bullet \rightarrow k_* i^* \Omega_{\mathcal{X}^*}^\bullet.$$

The isomorphism (III.8) is obtained by showing that the map of complexes

$$(III.9) \quad \begin{array}{ccc} \mathcal{L}^* & \longrightarrow & \Omega_{\mathcal{X}/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X \\ \Psi & & \Psi \\ \omega & \longrightarrow & \omega_0|_X \end{array}$$

is also a quasi-isomorphism; here  $\omega_i|_X$  is the image of the natural map  $\Omega_{\mathcal{X}}^\bullet(\log X) \rightarrow \Omega_{\mathcal{X}/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X$ .<sup>5</sup> The composition of the isomorphisms (III.7) and (III.8) may be thought of as expressing the Clemens retraction mapping in terms of holomorphic de Rham-type complexes.

The second reason is that the discussion in (ii) extends to the case of general  $\mathcal{X} \rightarrow \Delta^\ell$  in (I.1)<sub>AG</sub>, where now (III.5) is replaced by

$$e_\alpha(s) = \sum_{\lambda=(\lambda_1, \dots, \lambda_\ell)} A_{\alpha\lambda}^\beta(s) \ell(s_1)^{\lambda_1} \cdots \ell(s_\ell)^{\lambda_\ell} f_\beta(s)$$

and (III.6) by

$$\nabla f_\alpha(s) = \sum_{\beta, i} N_{\alpha i}^\beta f_\beta(s) ds_i/s_i.$$

Combining (III.7) and (III.8) will give the desired localization (III.3). The necessary local considerations needed to establish the extensions of (III.7) and (III.8) will be taken up in subsequent sections.

**Historical remark:** Let  $Y$  be a compact, complex manifold,  $Z \subset Y$  a normal crossing divisor and  $U = Y \setminus Z$  with the inclusion  $j : U \hookrightarrow Y$ . As Grothendieck pointed out in [Gr], in their paper [HA] Hodge and Atiyah proved what essentially amounts to the natural isomorphism

$$H^*(U, \mathbb{C}) \cong \mathbb{H}^*(\Omega_Y^\bullet(\log Z)).$$

<sup>5</sup>As explained in [St2], this quasi-isomorphism depends on the choice of an isomorphism  $T_0^* \Delta \cong \mathbb{C}$ . Such an isomorphism is given by a coordinate  $s$  on  $\Delta$ , and a change of coordinates  $s' = f(s)$  scales the quasi-isomorphism by  $f'(0)$ .

This was based on the quasi-isomorphism

$$(III.10) \quad \Omega_Y^\bullet(\log Z) \hookrightarrow j_*\Omega_U^\bullet.$$

The main points here are

- (i) the complex  $\Omega_Y^\bullet(\log Z)$  is intrinsically defined;
- (ii) if  $Z$  is given locally by  $z_1 = \cdots = z_k = 0$  in  $\mathbb{C}^n$ , then for the Stein manifold  $W = \Delta^{*k} \times \Delta^{\ell-k}$  we have

$$(III.11) \quad H^*(W, \mathbb{C}) \cong H_{\text{DR}}^*(W) = H_{\text{DR}}^*(\Gamma(W, \Omega_W^\bullet));$$

- (iii) the left-hand side of (III.11) is  $H^*(\Delta^{*k}, \mathbb{C})$ , and the right-hand side contains the subspace  $\Lambda^*\{dz_1/z_1, \dots, dz_k/z_k\}$ , which is isomorphic to the left-hand side.

Basically, using

$$(III.12) \quad dz_i^k = kz_i^{k-1} dz_i$$

all but the terms having only  $dz_i/z_i$ 's for  $1 \leq i \leq k$  may be eliminated in  $H_{\text{DR}}^*(\Gamma(W, \Omega_W^\bullet))$ .<sup>6</sup>

When instead of  $U \subset Y$  we have the situation

$$\begin{array}{c} \tilde{\mathcal{X}}^* \\ \downarrow \\ j : \mathcal{X}^* \subset \mathcal{X} \end{array}$$

then instead of differential forms with coefficients in  $j_*(\mathcal{O}_{\mathcal{X}^*})$ , i.e., function having poles along  $X$ , we have to adjoin power series in  $\log s_1, \dots, \log s_k$  to obtain the complex  $\mathcal{L}^\bullet$ , and then the top row in (III.9) is an analogue of (III.10). Then the relation

$$d(\log s_i)^k = k(\log s_i)^{k-1} dz_i/s_i$$

will be used to eliminate all the  $\log s_i^k$ 's for  $k \neq 0$ , thus leading leading to (III.8).

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<sup>6</sup>The Grothendieck algebraic de Rham theorem [Gr] is treated in detail in [E-ZT].

## IV. FILTRATIONS ON THE COMPLEXES

Each of the complexes

$$\Omega_X^\bullet(\log X), \Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X$$

introduced above has two filtrations, a *Hodge filtration* and a *weight filtration*. We will now describe these.

The first of these, the Hodge filtration, is the same in both cases. It is the “bétè” filtration, given for any complex  $(C^\bullet, d)$  by

$$F^p C^\bullet = \{0 \rightarrow C^p \rightarrow C^{p+1} \rightarrow \dots\};$$

i.e., we truncate  $\{C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \dots\}$  by simply setting equal to zero the  $C^q$  for  $q < p$ . It is motivated by Hodge theory, where for a smooth complete variety  $Y$  the Hodge filtration  $F^p H_{\text{DR}}^n(Y)$  means “ $\geq p$   $dz_i$ ’s in differential forms representing cohomology classes.”

For  $\Omega_X^\bullet(\log X)$  the weight filtration is informally described by

$$\varphi \in W_k \Omega_X^\bullet(\log X) \iff \varphi \text{ has } \leq k \text{ } dx_i/x_i \text{’s.}$$

More precisely,

$$W_k \Omega_X^\bullet(\log X) \text{ is the image of } \Omega_X^k(\log X) \otimes \Omega_X^\bullet \rightarrow \Omega_X^{\bullet+k}(\log X).$$

We recall the description of the associated graded. If  $X = X_1 + \dots + X_N$  then for each index set  $I = \{i_1, \dots, i_k\}$  where  $0 \leq i_1 < \dots < i_k \leq N$  we set  $X_I = X_{i_1} \cap \dots \cap X_{i_k}$ . This is a complex manifold of dimension  $d - k$  together with a map  $a_k : X_I \rightarrow X$ . Setting

$$X^{[k]} = \coprod_{|I|=k} X_I$$

we then have

$$a_k : X^{[k]} \rightarrow X.$$

The iterated Poincaré residue map gives an isomorphism

$$(IV.1) \quad \text{Gr}_k^W \Omega_X^\bullet(\log X) \xrightarrow{\sim} \Omega_{X^{[k]}}^{\bullet-k}[k].$$

In coordinates, if

$$\varphi = dx_{i_1}/x_{i_1} \wedge \dots \wedge dx_{i_k}/x_{i_k} \wedge \psi$$

then

$$\text{Res } \varphi = (2\pi i)^k \psi|_{X_I}.$$

For the complex  $\Omega_{X/S}^\bullet(\log X)$  and its restriction to  $X$  there is what might be called the naïve weight filtration induced from the  $W_k$ 's just described. In contrast to the situation for  $\Omega_X^\bullet(\log X)$ , when combined with the Hodge filtration the naïve weight filtration does not lead to a mixed Hodge structure<sup>7</sup> (cf. the footnote below and the discussion in Chapter 11 of [PS]). As may be expected from the above discussion, the weight filtration on  $\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X$  will be induced locally from the weight filtrations on the local normal crossing divisor case using (IV.1). Thus the crucial case is the weight filtration for  $\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X$ .

Here the basic insight is due to Steenbrink [St1]. It originates from the observation that there is a natural identification

(IV.2)

$$\Omega_{X/S}^p(\log X) \otimes \mathcal{O}_X \cong \text{coker} \left\{ \Omega_X^{p-1}(\log X) \xrightarrow{\theta} \Omega_X^p(\log X) / W_0 \Omega_X^p(\log X) \right\}$$

where  $\theta$  is the map given by  $\wedge(ds/s)$  (cf. (II.5) and (II.6) above). An intrinsic proof of (IV.2) is given in [St1], and also in Chapter 11 of [PS]. We will give a coordinate argument that illustrates the mechanism. This coordinate argument will then be used in Section VIII below to suggest what the extension of (IV.2) to the general case of  $\mathcal{X} \rightarrow \Delta^\ell$  should be.

For  $\mathcal{X} \rightarrow S$  locally given by  $x_1 \cdots x_k = s$ , we first lift a local section  $\varphi$  of  $\Omega_{X/S}^p(\log X) \otimes \mathcal{O}_X$  to a section  $\tilde{\varphi}$  of  $\Omega_X^p(\log X) \otimes \mathcal{O}_X$ , and then we extend  $\tilde{\varphi}$  to a section of  $\tilde{\tilde{\varphi}}$  of  $\Omega_X^p(\log X)$ . The composite map

$$\varphi \rightarrow \tilde{\varphi} \rightarrow \tilde{\tilde{\varphi}} \wedge \frac{ds}{s}$$

---

<sup>7</sup>One reason why the naïve weight filtration on  $\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X$  does not work may be explained as follows: Assume that the limit  $\lim_{s \rightarrow 0} H^n(X_s)$  may be defined as a mixed Hodge structure. For simplicity of explanation take  $n = d = \dim X_s$ . One may expect the mixed Hodge structure on  $H^n(X)$  to appear in the limit. For this the weights are  $0 \leq w \leq n$ . On the other hand, the cup-product

$$H^n(X_s) \otimes H^n(X_s) \rightarrow H^{2n}(X_s) \cong \mathbb{Q}(-n)$$

should exist in the limit to pair  $\lim_{s \rightarrow 0} H^n(X_s)$  to a Hodge structure of weight  $2n$ . Thus in  $\lim_{s \rightarrow 0} H^n(X_s)$  the weights should run from 0 to  $2n$ .

is the one from the left-hand side of (IV.2) to the right-hand side. For any other lifting  $\tilde{\varphi}'$  we have  $\tilde{\varphi} - \tilde{\varphi}' = ds \wedge \psi$ , and when we extend from  $\Omega_{\mathcal{X}}^p(\log X) \otimes \mathcal{O}_X$  to  $\Omega_{\mathcal{X}}^p(\log X)$  and take  $\wedge ds/s$  we will have  $\tilde{\varphi} \wedge ds = \tilde{\varphi}' \wedge ds$ .

Somewhat more subtly, if we have a local section  $\psi$  of  $\Omega_{\mathcal{X}}^p(\log X) \otimes \mathcal{O}_X$  and take two extensions  $\tilde{\psi}$  and  $\tilde{\psi}'$  to a local section of  $\Omega_{\mathcal{X}}^p(\log X)$ , then we have  $\tilde{\psi} - \tilde{\psi}' = s\gamma$  where  $\gamma$  is a local section of  $\Omega_{\mathcal{X}}^p(\log X)$ . We thus have to show that

$$\gamma \wedge ds \in W_0 \Omega_{\mathcal{X}}^{p+1}(\log X) = \Omega_{\mathcal{X}}^{p+1}.$$

We recall that for  $I = \{i_1, \dots, i_\ell\}$  we write  $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_\ell}$ . Then  $\gamma$  is a sum of terms

$$\frac{dx_I}{x_I} \wedge \lambda$$

where  $\lambda$  is holomorphic. Now

$$ds = \sum_j x_1 \cdots \hat{x}_j \cdots x_\ell dx_j$$

which gives

$$ds \wedge x_I dx_I = \sum_{j \neq I} x_1 \cdots \hat{x}_j \cdots x_k dx_j \wedge \frac{dx_I}{x_I}.$$

For each term in the sum we have  $I \subset \{1, \dots, \hat{j}, \dots, k\}$ , so that the numerator cancels the denominator and the whole expression is holomorphic.

Once we have (IV.2), and using the fact that, for a vector space  $E$  and non-zero  $e \in E$ , the complex  $(\wedge^\bullet E, \wedge e)$  is acyclic, what is suggested is to define a resolution

$$0 \rightarrow \Omega_{\mathcal{X}/S}^p(\log X) \otimes \mathcal{O}_X \rightarrow \frac{\Omega_{\mathcal{X}}^{p+1}(\log \mathcal{X})}{W_0 \Omega_{\mathcal{X}}^{p+1}(\log \mathcal{X})} \xrightarrow{\theta} \frac{\Omega_{\mathcal{X}}^{p+2}(\log \mathcal{X})}{W_1 \Omega_{\mathcal{X}}^{p+2}(\log \mathcal{X})} \rightarrow \dots$$

This in turn suggests defining a double complex  $(A^{\bullet, \bullet}, d', d'')$  whose associated single complex gives a resolution of the complex  $(\Omega_{\mathcal{X}/S}^\bullet(\log X) \otimes$



$\mathcal{O}_X, d$ ). For this, following the notation in [St1], we define

$$(IV.3) \quad \begin{cases} A^{p,q} = \Omega_X^{p+q+1}(\log X) / W_q \Omega_X^{p+q+1}(\log X) \\ d' = \text{usual } d \\ d'' = \theta = \wedge ds/s. \end{cases}$$

Then (IV.2) has the following consequence:

(IV.4) *The map  $\tilde{\theta} : \Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X \rightarrow A^\bullet$  given by*

$$\tilde{\theta}(\varphi) = (-1)^p \theta \wedge \tilde{\varphi} \Big|_X$$

*is a well-defined quasi-isomorphism.*

The Hodge and weight filtrations are then defined by

$$(IV.5) \quad \begin{aligned} & \bullet \quad F^p A^\bullet = \bigoplus_{\substack{p' \geq p \\ q \geq 0}} A^{p',q}; \\ & \bullet \quad W_r A^{p,q} = W_{2q+r+1} \Omega_X^{p+q+1}(\log X) / W_q \Omega_X^{p+q+1}(\log X). \end{aligned}$$

To give an heuristic explanation of how we may arrive at the weight filtration in (IV.5), we first note that in the vector space  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$  it is difficult to see how any direct local residue-theoretic construction on  $\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X$  will lead to weights in the range  $0 \leq w < n$ . The only geometry here resides in the normal crossing variety  $X$  together with the class  $\xi \in T_X \text{Def}(X)$  that gives a smoothing of  $X$  to 1<sup>st</sup> order and provides the data to be able to define  $\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X$ . As observed above the inclusion

$$0 \rightarrow \Omega_{X/S}^p(\log X) \otimes \mathcal{O}_X \xrightarrow{\tilde{\theta}} \Omega_X^{p+1}(\log X) / W_0 \Omega_X^{p+1}(\log X)$$

suggests completing this sequence to a resolution of  $\Omega_{X/S}^p(\log X) \otimes \mathcal{O}_X$  whose terms reflect geometric information beyond that given by simply completing (II.5) to a resolution of  $\Omega_{X/\Delta}^p(\log X) \otimes \mathcal{O}_X$ . This leads to the  $A^\bullet = \bigoplus A^{p,q}$  above. The issue then is what weight filtration should be put on the  $A^{p,q}$  so that in the abutment of the resulting spectral

sequence the higher cohomology groups of the normalizations of the strata of  $X$  will appear?<sup>8</sup>

To answer this for any  $\ell$  one may define

$$(IV.6) \quad W_r A^{p,q} = W_{r+\ell} \Omega_X^{p+q+1}(\log X) / W_q \Omega_X^{p+q+1}(\log X)$$

and then compute the resulting weight spectral sequence. One knows a priori that

- $\mathrm{Gr}_r^W A^\bullet \cong \bigoplus (a_k)_* \Omega_{X^{[k]}}^\bullet[-j]$  where  $j, k$  run over an index set determined by the choice of  $\ell$  in (IV.6);
- by the general results in [De3] the spectral sequence will degenerate at  $E_2$  and will abut to  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$ .

If we then require that the weight filtration on the abutment should be (at most) length  $2n$  and be centered at zero, then essentially by inverting the calculations in [St1] one may solve for  $\ell$  in (IV.6) to obtain  $\ell = 2q + 1$ .

## V. MONODROMY WEIGHT FILTRATION IN THE 1-PARAMETER CASE (I)

Having determined the filtration (IV.6)

$$W_r A^\bullet = W_{r+2q+1} = \Omega_X^{p+q+1}(\log X) / W_q \Omega_X^{p+q+1}(\log X)$$

that may be expected to induce one of length  $2n$  on  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$ , we now ask for a map

$$(V.1) \quad \nu : A^{\bullet,\bullet} \rightarrow A^{\bullet,\bullet}$$

that induces the action by monodromy  $N$  on  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$ . We have noted earlier that  $N$  is induced by the connecting map in the long exact hypercohomology sequence associated to

$$(V.2) \quad 0 \rightarrow \mathcal{O}_{X/S}^{\bullet-1}(\log X) \otimes \mathcal{O}_X \rightarrow \Omega_X^\bullet(\log X) \otimes \mathcal{O}_X \rightarrow \Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X \rightarrow 0.$$

---

<sup>8</sup>The subtle point here is that although  $\xi \in T_X^0 \mathrm{Def}(X)$  is sufficient information to define  $\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X$ , it is only in the quasi-isomorphic complex  $A^\bullet$  that includes higher order information of  $X$  in  $\mathcal{X}$  that one defines the weight filtration.

At this point there is the standard mapping cone construction that will give the mapping  $\nu$  in (V.1). Before explaining this we will give an heuristic argument to show that, up to a constant,

$$(V.3) \quad \nu \text{ is the natural projection } A^{p,q} \rightarrow A^{p-1,q+1}.$$

The reasons are

- Since  $\nu$  preserves the total degree we should have that

$$\nu : A^{p,q} \rightarrow A^{p-k,q+k}$$

for some  $k$ ;

- Since  $\nu$  is supposed to be of type  $(-1, -1)$  acting on  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$ , we should have  $k = 1$  so that

$$(V.4) \quad \nu \in F^{-1} \text{End}_{\text{MHS}}(\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X));$$

- Finally, as a check we should also have

$$(V.5) \quad \nu : W_r \mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X) \rightarrow W_{r-2} \mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X);$$

this is indeed satisfied by  $\nu$  given by (V.3).

Informally we may describe  $\nu$  as follows:

- locally in coordinates, recalling that  $A^{p,q}$  means we mod out by terms having  $\leq q$  of the  $dx_i/x_i$ 's,  $\nu$  kills the terms with exactly  $q+1$  of the  $dx_i/x_i$ 's and is the identity on the rest;
- when we pass to  $\text{Gr}_r^W A^{\bullet,\bullet}$ , in any local coordinate presentation of  $X \subset \mathcal{X}$  this has the effect on  $A^{p,q}$  of picking out the non-zero terms with exactly  $r+2q+1$  of the  $dx_i/x_i$ 's (note that this non-zero condition restricts  $r$  to  $-n \leq r \leq n$ );
- from this we infer that on the individual pieces in  $\text{Gr}^W A^\bullet$ ,  $\nu$  is either zero or the identity;
- thus when we pass to  $\mathbb{H}^n(\text{Gr}_r^W A^\bullet)$ , on the individual summands  $\nu$  will either be zero or the identity, depending on the particular indices;
- a consequence of this will be that, when the indices are worked out we will have almost tautologically that for  $r \geq 0$

$$(V.6) \quad \nu^r : \text{Gr}_r^W A^\bullet \xrightarrow{\sim} \text{Gr}_{-r}^W A^\bullet.$$

As noted in [St1], this relatively easy result is far from sufficient to show that  $W(\nu)$  induces the monodromy weight filtration on  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$ .

- another consequence is that there is a natural grading element for the nilpotent operator  $\nu$  on  ${}_W E_1$ , and consequently  $\nu$  is canonically part of an  $\mathfrak{sl}_2$  action on  ${}_W E_1$  (the terms will be explained below).

We now review the mapping cone construction to check that  $\nu$  given by (V.3) will, up to a constant, induce the cohomology map in the exact hypercohomology sequence of (V.2). The cone construction is expressed by the diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \Omega_{X/S}^{\bullet-1}(\log X) \otimes \mathcal{O}_X & \xrightarrow{\theta} & A^{\bullet-1} \\
 \downarrow & & \downarrow \\
 \Omega_X^\bullet(\log X) \otimes \mathcal{O}_X & \xrightarrow{\eta} & B^\bullet \\
 \downarrow & & \downarrow \\
 \Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X & \xrightarrow{\theta} & A^\bullet \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

where  $B^\bullet$  is the total complex associated to the double complex  $B^{\bullet,\bullet}$  with

$$B^{p,q} = A^{p-1,q} \oplus A^{p,q}$$

and with differentials

$$\begin{aligned}
 d'(\omega_1, \omega_2) &= (d\omega_1, d\omega_2) \\
 d''(\omega_1, \omega_2) &= (\theta(\omega_1) + (-1)^{p+q+1}\nu(\omega_2), \theta(\omega_2)).
 \end{aligned}$$

The maps

$$\eta \cdot \Omega_X^p(\log X) \otimes \mathcal{O}_X \rightarrow A^{p-1,0} \oplus A^{p,0}$$

are defined by

$$\eta(\omega) = (\omega \bmod W_0, (-1)^p \theta \wedge \omega).$$

The above diagram is commutative, and because the top and bottom horizontal arrows are quasi-isomorphisms the five lemma shows that  $\eta$  must also be a quasi-isomorphism. After passing to the exact hypercohomology sequence, the connecting map  $N$  from the left-hand column corresponds, up to a constant, to the map induced by  $\nu$  in the right column.

In summary, the conditions that  $\nu$  should induce  $N$  and the cone construction force, up to a constant, the definition (V.3) of  $\nu$ .

It follows that the map induced by  $\nu$  on  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$  shifts the weight filtrations  $W_r =: W_r(\nu)$  down by 2, i.e.,

$$\nu : W_r \mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X) \rightarrow W_{r-2} \mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X).$$

On the other hand, as will now be explained and identifying  $\nu$  with  $N$ , there is a canonical filtration  $W(N)$  on  $\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X)$  defined by the nilpotent endomorphism  $N$ , and the main result is that

$$(V.7) \quad \boxed{\text{the filtrations } W \text{ and } W(N) \text{ coincide.}}$$

This means that not only is  $(\mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X, W, F^\bullet))$  a mixed Hodge structure, it is of the *very* special type of a limiting mixed Hodge structure. Here we recall that as mentioned above a limiting mixed Hodge structure is given by a mixed Hodge structure  $(V, W, F^\bullet)$  together with a nilpotent endomorphism  $N \in F^{-1} \text{End}(V)$  such that  $W = W(N)$  is the monodromy weight filtration associated to  $N$ .

In addition to the property that  $N(W_r) \subset W_{r-2}$ , the crucial so-called *Hard-Lefschetz property* for defining the monodromy weight filtration is that for  $r \geq 0$

$$(V.8) \quad N^r : \text{Gr}_r^W \mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X) \xrightarrow{\sim} \text{Gr}_{-r}^W \mathbb{H}^n(\Omega_{X/S}^\bullet(\log X) \otimes \mathcal{O}_X).$$

We have noted the elementary fact that this result is true at the  ${}^w E_1$  level. Much more subtle is that for projective families it holds for  ${}^w E_\infty$ .

Digressing for a moment, recall that associated to any nilpotent endomorphism  $N \in \text{End}(V)$  is its monodromy weight filtration  $W(N)$ . The usual definition is that, if  $N^{n+1} = 0$  but  $N^n \neq 0$ ,  $W(N)$  is the

unique increasing filtration  $W_k(N)$  centered at zero and of length  $2n+1$  that satisfies

$$(V.9) \quad \begin{cases} N : W_r(N) \rightarrow W_{r-2}(N) \\ N^r : \mathrm{Gr}_r^{W(N)} V \xrightarrow{\sim} \mathrm{Gr}_{-r}^{W(N)} V. \end{cases}$$

For the purposes of these notes, we will use the Jacobson-Morosov theorem to complete  $N$  to an  $\mathfrak{sl}_2$ -triple  $(N, Y, N^+)$  in  $\mathrm{End}(V)$  where

$$(V.10) \quad \begin{aligned} [Y, N] &= -2N \\ [Y, N^+] &= 2N^+ \\ [N^+, N] &= Y. \end{aligned}$$

The  $Y$  is then a grading element for the filtration  $W(N)$ . Recall that this means that  $Y$  is semi-simple with eigenvalues  $\{-n, 1-n, \dots, n-1, n\}$  and corresponding eigenspaces  $V_k \subset V$  and where

$$W_k(N) = \bigoplus_{p \leq k} V_p.$$

The choice of  $N$  and  $Y$  satisfying the first equation in (V.10) uniquely determines the  $N^+$  in the  $\mathfrak{sl}_2$ -triple. Moreover, for any other such choice  $Y'$  we have

$$Y' \equiv Y \pmod{\ker(\mathrm{mod} N) \cap \mathrm{im}(\mathrm{mod} N)},$$

so that the grading defined by  $Y'$  leads to the same filtration as that defined by  $Y$ .

We have noted above, and it will be further discussed below, that for the weight spectral sequence there is a canonical grading element associated to the nilpotent endomorphism  $N \in \mathrm{End}(E_1)$ . Thus *the monodromy weight filtration  $W(N)$  on  $E_1$  is canonically split*, and we then have a canonical  $\mathfrak{sl}_2$ -action on  $E_1$ , one that will be seen to commute with the  $d_1$  of the spectral sequence.

## VI. MONODROMY WEIGHT FILTRATION IN THE 1-PARAMETER CASE (II)

For the standard picture of  $\mathcal{X} \rightarrow S$  of a 1-parameter family where  $S = \Delta$ , in the previous section we have defined

- a mixed Hodge structure  $\mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log X) \otimes \mathcal{O}_X, W, F^\bullet)$ , and

- a nilpotent monodromy operator  $N \in \text{End}(\mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log X) \otimes \mathcal{O}_X))$  with the properties

$$\begin{aligned} N(W_r) &\subset W_{r-2}, \\ N(F^p) &\subset F^{p-1}. \end{aligned}$$

We want to show that

$$W(N) = W.$$

For this it is necessary to establish the second of the defining properties (V.9) of  $W(N)$ ; that is, we must prove the Hard Lefschetz property (V.8).

This property will not hold without further assumptions on the standard situation (I.1)<sub>AG</sub> when  $\ell = 1$ . Perhaps the most natural of these is that the family  $\mathcal{X} \rightarrow S$  is projective; i.e., we have  $\mathcal{X} \subset \mathbb{P}^A$  with the Lefschetz class  $L_s = c_1(\mathcal{O}_{X_s}(1))$ . We set  $L = L_0$  and note that  $L$  induces the class of an ample line bundle on the normalizations  $X^{[k]}$  of the strata of  $X$ . Noting that

- (i) the  $E_1^{p,q}$  term of the weight spectral sequence is given by a direct sum of the groups

$$(VI.1) \quad H^i(X^{[j]})(-k),$$

where the indices  $i, j, k$  that appear depend on  $p, q, n$ , and  $d$ ;

- (ii)  $L$  operates on the  $E_1$  term and this action commutes with  $d_1$ ;
- (iii) the action of  $L$  also commutes with the action of the monodromy operator  $N$  (which we have seen is either zero or the identity on each factor (VI.1) in  $E_1$ )

what is suggested is that we decompose the  $E_1$ -term under the action of  $L$  and see what this gives.

More precisely, we have already observed that  $N$  is part of an  $\mathfrak{sl}_2$  acting on  $E_1$ , and the same is true of the action of  $L$  on each  $H^*(X^{[k]})$ , with suitable Tate twists on the individual groups. Thus one may hope that  $L$  is also part of an  $\mathfrak{sl}_2$  acting on  $E_1$ , so that we may have a *pair of commuting  $\mathfrak{sl}_2$ 's acting on  $E_1$  and commuting with  $d_1$* . This is the

approach taken in [GN] and with an exposition in Chapter 11 of [PS]. We will say more about this below.

One additional point. From the earliest days one has known that many of the deepest results in Hodge theory require the 2<sup>nd</sup> Hodge-Riemann bilinear relation, especially the positivity of the Hermitian forms. So it will not be just the irreducible  $\mathfrak{sl}_2 \times \mathfrak{sl}_2$  factors, or bi-primitive decompositions of the  $E_1$  terms, but also the non-degenerate bilinear forms on those pieces, that should come into play. In fact, this is the case in the original proof that  $W = W(N)$  in [St1].

The main difficulty in implementing the above general strategy is in the book-keeping of the complicated indices that enter. This is very nicely organized in [GN] and [PS], as follows. Using now real coefficients so as to have the operation of complex conjugation, we set

$$K^{ijk} = H^{i+j-2k+n}(X^{[2k-i+1]})(i-k).$$

Then, recalling that  $\dim X = d$  and that we are considering a mixed Hodge structure  $(V, W, F^\bullet)$ , the  $E_1$ -term of the monodromy weight spectral sequence is ([PS], page 273)

$$E_1^{-r, n+1} = \bigoplus_k K^{-r, n-d, k} \implies V_{\mathbb{R}}.$$

The differential  $d_1$  in the weight spectral sequence is given by  $d_1 = d'_1 + d''_1$  where the differentials are maps

- $d'_1 : K^{i, j, k} \rightarrow K^{i+1, j+1, k+1}$
- $d''_1 : K^{i, j, k} \rightarrow K^{i+1, j+1, k}$

and are described by

- $d'_1$  are the signed restriction maps on cohomology induced from the maps  $X^{[\ell+1]} \rightarrow X^{[\ell]}$ ;
- $d''_1$  are the signed Gysin maps on cohomology induced from these same maps.

We note that

- $d_1 : K^{i, j, k} \rightarrow K^{i+1, j+1, k+1} \oplus K^{i+1, j+1, k}$

and that  $d'_1$ ,  $d''_1$  and  $d_1$  are all morphisms of real Hodge structures.



There are additional maps on the  $K^{i,j,k}$  induced from the monodromy operator  $N$  and Lefschetz operator  $L$ . Both are also morphisms of real Hodge structures. In terms of the  $K^{i,j,k}$  we have

$$N : K^{i,j,k} \rightarrow K^{i+2,j,k+1}$$

$$L : K^{i,j,k} \rightarrow K^{i,j+2,k}(1).$$

This implies that on the bi-graded complex

$$K^{i,j} = \bigoplus_k K^{i,j,k},$$

for which the cohomology of the associated single complex gives  $E_2$ , that  $N$  has bi-degree  $(2, 0)$  and  $L$  has bi-degree  $(0, 2)$ .

All together one has a very rich representation theoretic structure on  $E_1$ , from which using the standard representation theory of  $\mathfrak{sl}_2 \times \mathfrak{sl}_2$  one infers a double primitive decomposition

$$K^{r,s} = \sum N^i L^j K_0^{r-2i,s-2j}$$

with symmetric positive definite bilinear forms

$$Q : K_0^{-i,-j} \otimes K_0^{-i,-j} \rightarrow \mathbb{R}$$

derived from the usual ones giving polarized Hodge structures on the primitive cohomologies on the normalization of the strata of  $X$ . Using the diagonal action of the  $\mathfrak{sl}_2 \times \mathfrak{sl}_2$  on  $K^{\bullet,\bullet}$ 's, these in turn lead to symmetric and positive definite bilinear forms

$$\varphi : K^{\bullet,\bullet} \otimes K^{\bullet,\bullet} \rightarrow \mathbb{R}.$$

The final step is to use  $\varphi$  to define the adjoint  $d^*$  of  $d$  and Laplacian  $\Delta = dd^* + d^*d$  and use the representation theory of  $\mathfrak{sl}_2 \times \mathfrak{sl}_2$  on  $\text{End}(K^{\bullet,\bullet})$  to show that

$$\ker \Delta \text{ is an invariant subspace of } K^{\bullet,\bullet}.$$

This leads to the basic result (V.7).

**Remark:** One cannot help but be struck by the several uses of “finite dimensional harmonic theory” to prove deep results, such as the one above. One such is Kostant’s theorem [Ko] on the  $\mathfrak{n}$ -cohomology of irreducible  $\mathfrak{g}$ -modules, where  $\mathfrak{g}$  is a semi-simple complex Lie algebra

and  $\mathfrak{n}$  is the unipotent radical of a Borel subalgebra (cf. [Ko]). The other is the purity result in [CKS2].

## VII. LOCALIZATION TO $X$ IN THE SEVERAL PARAMETER CASE

Conceptually the steps are analogous to those from [St1] in the 1-parameter case: Starting with the situation (I.1)<sub>AG</sub>

$$\pi \cdot \mathcal{X} \rightarrow S$$

where  $S = \Delta^\ell$ , and where around points  $x \in X = \pi^{-1}(s_0)$  where  $X$  is locally a product of normal crossing divisors and a space of parameters, the mapping has the standard local normal form

$$(VII.1) \quad \begin{cases} x_{I_1} = s_1 \\ \vdots \\ x_{I_k} = s_k. \end{cases}$$

Then there is a diagram

$$\begin{array}{ccccc} \tilde{\mathcal{X}}^* & \xrightarrow{k} & \mathcal{X} & \xleftarrow{i} & X \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{S}^* & \xrightarrow{j} & S & \xleftarrow{i_0} & \{s_0\} \end{array}$$

where  $\tilde{\mathcal{X}}^* \xrightarrow{k} \mathcal{X}$  is the composition of  $\tilde{\mathcal{X}}^* \rightarrow \mathcal{X}^*$  and the inclusion of  $\mathcal{X}^*$  in  $\mathcal{X}$ . Similarly for  $j$ , while  $i$  and  $i_0$  are both inclusions. The group  $\mathbb{Z}^\ell$  of deck transformations acts equivalently on the left-hand side of the diagram and  $e_i = (0, \dots, 1, \dots, 0)$  induces the monodromy transformation  $\exp N_i$  on  $\mathbb{H}^n(\tilde{X}^*, \mathbb{C})$ .

We recall the steps in the localization to  $X$ :

- (i)  $H^n(\tilde{\mathcal{X}}^*, \mathbb{C}) \cong \mathbb{H}^n(\Omega_{\tilde{\mathcal{X}}^*}^\bullet)$ ;
- (ii)  $\mathbb{H}^n(\Omega_{\mathcal{X}^*}^\bullet) \cong \mathbb{H}^n(\mathcal{X}, k_* \Omega_{\tilde{\mathcal{X}}^*}^\bullet)$ ;
- (iii)  $\mathbb{H}^n(\mathcal{X}, k_* \Omega_{\tilde{\mathcal{X}}^*}^\bullet) \cong \mathbb{H}^n(X, i^* k_* \Omega_{\tilde{\mathcal{X}}^*}^\bullet)$ .

These steps are the analogues of (2.4) and (2.5) in [St1]. Geometrically in step (iii) we may shrink  $\mathcal{X}$  to  $X$  by restricting to a product of discs of radius  $\epsilon$  and letting  $\epsilon \rightarrow 0$ . This is an analytic analogue of the Clemens retraction mapping [C], which is topological.

A central question in understanding (iii) is:

*What does the sheaf  $i^*k_*\Omega_{\tilde{\mathcal{X}}^*}^\bullet$  look like?*

The mapping  $k$  is induced from  $j : \tilde{S}^* \rightarrow S$  given by

$$s_i = \exp 2\pi\sqrt{-1}z_i, \quad z_i \in \mathbb{H}.$$

Shrinking  $S$  to  $\{s_0\}$  means taking  $|s_i| < \epsilon$  and letting  $\epsilon \rightarrow 0$ . The sheaf  $i^*j_*\mathcal{O}_{\tilde{S}^*}$  is supported at  $s_0$ ; intuitively it consists of functions  $f(s_1, \dots, s_k; \log s_1, \dots, \log s_k)$  that are Laurent series in the  $s_i$  and power series in the  $\log s_i$ . On  $\tilde{\mathcal{X}}^*$  the sheaf  $k_*i^*\mathcal{O}_{\tilde{\mathcal{X}}^*}$  consists of functions  $f(\mathbf{x}, \mathbf{log s})$  where  $\mathbf{x} = (x_1, \dots, x_{d+k})$ ,  $\mathbf{log s} = (\log s_1, \dots, \log s_k)$  and  $f(\mathbf{x}, \mathbf{s})$  is a Laurent series in the  $x_i$  where  $i \in I_1 \cup \dots \cup I_k$ , a power series in the remaining  $x_j$ 's, and  $\mathbf{log s}$  is a power series in the  $\log s_i$ .<sup>9</sup> Then  $k_*i^*\Omega_{\tilde{\mathcal{X}}^*}^\bullet$  are differentials

$$(VII.2) \quad \omega = \sum_I f_I(\mathbf{x}, \mathbf{log s}) dx_I$$

where the  $f_I(\mathbf{x}, \mathbf{log s})$  are as above. Here we have used (VII.1) to express the  $ds_i$ 's in terms of  $dx_j$ 's. If we only have differentials of this form without the  $\log s_i$ 's, then we are essentially in the same situation as in the original paper [HA]. The stalk of the cohomology sheaf  $\mathcal{H}^*(k_*i^*\mathcal{O}_{\tilde{\mathcal{X}}^*})$  would then be the tensor product of the exterior algebra of the logarithmic differentials corresponding to the product of the normal crossing divisors given by the factors in (VII.1). For a closed form  $\omega$  with coefficients  $f_I(\mathbf{x})$ , this comes by expanding  $f_I(\mathbf{x})$  in a power series and subtracting off exact forms to eliminate all terms except those in which  $f_I(\mathbf{x})$  has a  $1/x_i$  factor for  $i \in I$ . In this way we may reduce  $\omega$  to a logarithmic differential.

Putting in the  $\log s_i$ 's implies that we introduce the relation

$$(VII.3) \quad ds_i/s_i = \sum_{j \in I_i} dx_j/x_j,$$

<sup>9</sup>An argument, not given here but which is essentially the same in [St1], gives that we may assume  $f(\mathbf{x}, \mathbf{log s})$  to be a polynomial in the  $\log s_i$ . Here we recall the expressions (III.5) for a section of  $\mathbb{V} \rightarrow \Delta^{*\ell}$  relative to a holomorphic frame for  $\mathbb{V} \rightarrow \Delta^\ell$  that was discussed in Section III.

so that we effectively are in the space of relative differentials. Beyond the situation in [HA] what must be proved is that we can reduce any closed form to eliminate positive powers of the  $\log s_i$ 's. This is done using

$$d(\log s_i)^p = (\log s_i)^{p-1} ds_i/s_i.$$

When there is only one factor in the local product (VII.1) the essential calculation occurs already when we are in  $\mathbb{C}^{k+1}$  with the equation

$$x_1 \cdots x_k = s.$$

If

$$(VII.4) \quad \omega = \sum_{I,j} c_{I,j} (\log s)^j dx_I/x_I$$

satisfies  $d\omega = 0$ , then we must show that inductively we can reduce by exact differentials of this form to one where no positive power of  $\log s$  appears. If

$$\sum (\log s)^p dx_I/x_I = (\log s)^p \sum_I c_{I,p} dx_I/x_I$$

is the term with the highest power  $p > 0$  of  $\log s$ , then from  $d\omega = 0$  we infer that

$$\left( \sum c_{I,p} dx_I/x_I \right) \wedge ds/s = 0.$$

This implies that the term in parentheses is divisible by  $ds/s = d \log s$ , and we may subtract an exact form to reduce  $p$  to  $p - 1$ .

For a closed form

$$\omega = \sum f_{I,j}(\mathbf{x}) (\log s)^j dx_I/x_I$$

we again consider the term with the highest power  $(\log s)^p$ . Then considering the equation

$$d\omega \equiv 0 \pmod{(\log s)^p}$$

we are in the same situation as in [HA] and may reduce by exact forms to have constant coefficients so that  $\omega$  has the form (VII.4).

For the locally a product situation (VII.1), the essential case is when we have two factors given by equations

$$\begin{aligned} x'_1 \cdots x'_i &= s', \\ x''_1 \cdots x''_k &= s''. \end{aligned}$$

Then we consider forms

$$\omega = \sum c_{I,j,I',j'} (\log s')^j (\log s'')^{j'} dx'_I/x'_I \wedge dx''_I/x''_I.$$

We may then first successively reduce the positive powers of  $\log s'$  by exact forms as above, and then do the same for the positive powers of  $\log s''$ .

### VIII. FILTRATIONS IN THE SEVERAL PARAMETER CASE

We are seeking to define a mixed Hodge structure  $(V, W, F^\bullet)$  where

$$V_{\mathbb{C}} = \mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X).$$

Here the mapping  $\mathcal{X} \rightarrow S$  has the local form (I.7). The normal crossing divisor is  $\mathcal{Y} = \mathcal{Y}_1 + \cdots + \mathcal{Y}_\ell$  where locally  $\mathcal{Y}_i$  is given by  $s_i = 0$ .

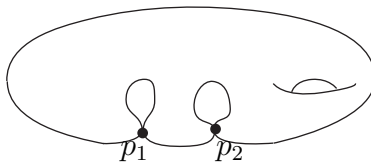
We have fixed an ordering of the discs, so that  $S = \Delta_1 \times \cdots \times \Delta_\ell$  where  $\Delta_i$  has coordinate  $s_i$ . Letting  $T_i = \Delta_1 \times \cdots \times \{0\} \times \cdots \times \Delta_\ell$  and  $T = T_1 + \cdots + T_\ell$ , we have a mapping  $(\mathcal{X}, \mathcal{Y}) \rightarrow (S, T)$ . The  $\mathcal{Y}_i$  are globally defined divisors in  $\mathcal{X}$ , and thus in  $\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y})$  the subcomplex  $W_k^i \Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y})$  of differentials that have  $\leq k$   $dx_i/x_i$ 's from the index set  $I_i$  is well defined.

Although  $\mathcal{X}$  is locally the product of normal crossing divisors  $U_i$  and a parameter space  $U_0$ , this is not the case globally. In this regard the situation is analogous to the 1-parameter case of  $\mathcal{X} \rightarrow \Delta$  where the mapping is locally given by

$$x_1 \cdots x_k = s.$$

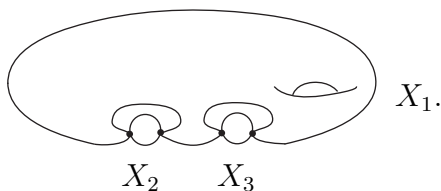
Then the central fibre  $X$  is only locally a normal crossing divisor. A simple and standard example that illustrates this is when  $X$  is an

irreducible nodal curve



In order to keep track of the book-keeping in the Poincaré residue and Gysin maps it is necessary to introduce further notation that labels the inverse images of the nodes in the normalization of  $X$ . If we have  $\omega \in H^0(\Omega_X^1(\log(p_1+p_2)))$ , then for each of the  $p_i$  there are two choices of the values of  $\text{Res}_{p_i} \omega$ , and one needs a notation to distinguish between these. There is a similar situation for the Gysin maps  $H^0(X^{[2]}) \rightarrow H^2(X^{[1]})$ .

From the theoretical point of view, one may apply further base change to  $\mathcal{X} \rightarrow \Delta$  to arrive at a situation where  $X$  is globally a normal crossing divisor with an ordering of the components. In the above picture the central fibre becomes



This is the usual procedure and is preferable from a theoretical, but not necessarily a computational, perspective.

In the several parameter case of  $\mathcal{X} \rightarrow \Delta^\ell$  similar complications arise. We have assumed an ordering of the factors in the base space so that the components  $\mathcal{Y}_i$  of the crossing divisor  $\mathcal{Y} = \mathcal{Y}_1 + \cdots + \mathcal{Y}_\ell$  are globally defined. Additional notations are needed to keep track of the fact that even though the local factors  $U_i$  of  $\mathcal{X}$  are well defined, the irreducible components of  $U_i$  depend on an ordering of the indices that label these components, which does not have intrinsic meaning. Such additional notations are introduced in [Fu1]. We shall not try to summarize them here; rather we shall work with the local situation (I.7) and remark which local constructions have global meaning. In fact, to simplify and

keep at the forefront the essential geometric content, we shall assume that

(i) we are in the case  $\ell = 2$  and  $\mathcal{X}$  has the local form<sup>10</sup>

$$(VIII.1) \quad \begin{aligned} x'_1 \cdots x'_{k'} &= s' \\ x''_1 \cdots x''_{k''} &= s''; \end{aligned}$$

- (ii) we shall suppress the signs necessary to do the book-keeping of the various restriction, Poincaré residue and Gysin mappings;  
 (iii) finally, when no ambiguity seems likely we shall omit the pullback mappings of differential forms.

Because of (i) we are in the case where an open set in  $\mathcal{X}$  is

$$U = U' \times U''$$

and where  $U'$  and  $U''$  correspond to the two factors in (VIII.1).

Setting  $\mathcal{X}' = \pi^{-1}(\Delta' \times \{0\})$  and  $\mathcal{X}'' = \pi^{-1}(\{0\} \times \Delta'')$ , we may use the identification

$$(VIII.2) \quad \mathcal{O}_X(U) \cong \pi'^* \mathcal{O}_{\mathcal{X}'}(U') \widehat{\otimes}_{\mathbb{C}} \pi''^* \mathcal{O}_{\mathcal{X}''}(U'')$$

to locally take the tensor product of Steenbrink's constructions in the 1-parameter case and see which of the resulting constructions have intrinsic meaning.<sup>11</sup> Some care is needed to make this work, and it seems preferable to first go to the basic principle underlying the construction.

Due to the localization discussed in Section VII, the basic local object is  $\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X$  whose hypercohomology gives the complex vector space  $V_{\mathbb{C}}$  for the to be constructed limiting mixed Hodge structure  $(V, W, F^\bullet)$ . As we have noted, it is possible to directly define the evident Hodge filtration on it, but for both geometric and formal reasons it is not possible to directly define a weight filtration on  $\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X$ . Among other things, there is no similar exact sequence to (II.5) that

<sup>10</sup>In the introduction we used the notation  $(s_1, s_2)$  instead of  $s', s''$ . Here we use the latter as we find that it may make the calculation less cluttered with indices.

<sup>11</sup>Note that the completed tensor product is over  $\mathbb{C}$ . We are thinking of power series in  $(x'_i, x''_j)$  as a completed tensor product of power series in the  $x'_i$  and  $x''_j$  separately.

leads to the Gauss-Manin connection. Rather as will be discussed below a spectral sequence construction is required, and this does not seem to immediately suggest a resolution of  $\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X$  in the several parameter case. Following the steps in the argument in the 1-parameter case, one may proceed as follows:

(i) Given a local section

$$\varphi = \sum_i \alpha'_i \wedge (dx'_i/x'_i) + \sum_j \alpha''_j \wedge (dx''_j/x''_j)$$

of  $\Omega_{\mathcal{X}/S}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X$ , any two local lifts of  $\varphi$  to  $\Omega_{\mathcal{X}}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X$  differ by a

$$\psi = \beta' \wedge ds' + \beta'' \wedge ds''.$$

Neither  $\psi \wedge ds'$  nor  $\psi \wedge ds''$  is zero, but

$$\psi \wedge ds' \wedge ds'' = 0.$$

This suggests that our resolution of  $\Omega_{\mathcal{X}/S}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X$  must start with  $\Omega_{\mathcal{X}}^{p+2}(\log \mathcal{Y})/(\text{something})$ .

(ii) We then lift  $\varphi$  to a section of  $\Omega_{\mathcal{X}}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X$  and extend this locally from  $X$  to  $\mathcal{X}$  to obtain a section of  $\Omega_{\mathcal{X}}^p(\log \mathcal{Y})$ . The difference between any two such extensions is of the form  $s'\gamma' + s''\gamma''$  where  $\gamma', \gamma''$  are sections of  $\Omega_{\mathcal{X}}^p(\log \mathcal{Y})$ . Then in contrast to the 1-parameter case  $\frac{ds'}{s'} \wedge \frac{ds''}{s''} \wedge (s'\gamma' + s''\gamma'')$  may not be holomorphic. For example, in the first term

$$-\frac{ds''}{s''} \wedge (ds' \wedge \gamma')$$

the logarithmic poles along  $U'$  are cleared by the same argument as in the 1-parameter case, but those along  $U''$  may remain.

(iii) We may intrinsically define

$$W'_{k'} \Omega_{\mathcal{X}}^\bullet(\log \mathcal{Y})$$

to be the forms that have  $\leq k'$   $dx'_i/x'_i$  terms, and similarly for  $W''_{k''} \Omega_{\mathcal{X}}^\bullet(\log \mathcal{Y})$ . Since these descriptions are in terms of the logarithmic differentials along each of the global divisors  $\mathcal{Y}'$  and  $\mathcal{Y}''$ ,



they are well defined. From the above we have

*There is a well-defined injection*

$$\Omega_{x/S}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X \rightarrow \Omega_x^{p+2}(\log \mathcal{Y})/W'_0 \Omega_x^{p+2}(\log \mathcal{Y}) + W''_0 \Omega_x^{p+2}(\log \mathcal{Y}).$$

This leads to the

$$(VIII.3) \quad \textbf{Definition: } A^{p;q',q''} = \Omega_x^{p+q'+q''+2}(\log \mathcal{Y})/W'_{q'} \Omega_x^{p+q+2}(\log \mathcal{Y}) + W''_{q''} \Omega_x^{p+q+2}(\log \mathcal{Y}).$$

Then the above inclusion becomes

$$(VIII.4) \quad \Omega_{x/S}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X \hookrightarrow A^{p,0,0}.$$

We may use the formalism suggested by (VIII.2) to define a resolution of  $\Omega_{x/S}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X$ . Namely, to use notations that will make more transparent the constructions, over the open set  $U = U' \times U''$  we set  $\mathcal{Y} = \mathcal{Y}' + \mathcal{Y}''$  and

$$(VIII.5) \quad \begin{aligned} \Omega'^{\bullet} &= \pi'^* \Omega_{U'/S'}^{\bullet}(\log \mathcal{Y}'), \\ \Omega''^{\bullet} &= \pi''^* \Omega_{U''/S''}^{\bullet}(\log \mathcal{Y}'') \\ \Omega^{\bullet} &= \Omega_{U/S}^{\bullet}(\log \mathcal{Y}). \end{aligned}$$

Then we have

$$(VIII.6) \quad \Omega^{\bullet} \cong \Omega'^{\bullet} \otimes \Omega''^{\bullet}.$$

On each of  $U'$  and  $U''$  we have the bi-complexes  $A'^{\bullet,\bullet}$  and  $A''^{\bullet,\bullet}$  as in (IV.3) for each factor, and in each case with resolutions  $\Omega'^{\bullet} \rightarrow A'^{\bullet,\bullet}$  and  $\Omega''^{\bullet} \rightarrow A''^{\bullet,\bullet}$  by using the corresponding total complexes. We then set

$$(VIII.7) \quad A^{\bullet} = A'^{\bullet} \otimes A''^{\bullet}$$

and use (VIII.6) to obtain a resolution of  $\Omega^{\bullet}$ .

To check this with (VIII.4), we set

$$\begin{aligned} W'_{q'} &= W'_{q'} \Omega_{U'/S'}^{\bullet}(\log \mathcal{Y}'), \\ W''_{q''} &= W''_{q''} \Omega_{U''/S''}^{\bullet}(\log \mathcal{Y}''). \end{aligned}$$

Then a local coordinate calculation gives

$$\Omega^{\bullet}/W'_0 + W''_0 \cong (\Omega'^{\bullet}/W'_0) \otimes (\Omega''^{\bullet}/W''_0),$$

and in general

$$\Omega^\bullet/W'_{q'} + W''_{q''} \cong (\Omega'^\bullet/W'_{q'}) \otimes (\Omega''^\bullet/W''_{q''}).$$

This checks that (VIII.4) is the initial term in a resolution

$$\Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X \rightarrow A^\bullet$$

where  $A^\bullet$  is the total complex associated to the complex given by the  $A^{p;q',q''}$  in (VIII.3). We note that locally

$$(VIII.8) \quad A^{p;q',q''} = \bigoplus_{p'+p''=p} A'^{p',q'} \otimes A''^{p'',q''},$$

and the total complex is

$$A^r = \bigoplus_{p+q'+q''=r} A^{p;q',q''}.$$

We will next discuss the differentials, and after this the weight filtration.

The differentials on  $A^{\bullet;\bullet;\bullet}$  are induced by

- $d : A^{p;q',q''} \rightarrow A^{p+1;q',q''}$  is the usual  $d$ ;
- $\delta' : A^{p;q',q''} \rightarrow A^{p;q'+1,q''}$  is  $\wedge ds'/s'$ , and similarly for  $\delta'' = \wedge ds''/s''$ ;
- $\delta = \delta' + \delta''$ ;
- $D = d + \delta : A^\bullet \rightarrow A^{\bullet+1}$  is the total differential.

In other words, with the local identification (VIII.7) the above differentials are induced from those on  $A'^\bullet$  and  $A''^\bullet$ , with the understanding that we take the total degree in the first factor as indicated by (VIII.8). The Hodge filtration on  $A^\bullet$  is induced by taking  $F^\bullet A'^\bullet \otimes F^\bullet A''^\bullet$ ; it is preserved by all of the above differentials.

The weight filtration on  $A^\bullet$  is also locally induced by taking the weight filtrations on  $A'^\bullet$  and  $A''^\bullet$ . Explicitly,

$$(VIII.9) \quad W_r A^{p;q',q''} = W_{r+2q'+2q''+2} \Omega^{p+q'+q''+2}(\log \mathcal{Y}) / W_{q'} \Omega_{\mathcal{X}}^{p+q'+q''+2}(\log \mathcal{Y}) \\ + W_{q''} \Omega_{\mathcal{X}}^{p+q'+q''+2}(\log \mathcal{Y}).$$

We note that passing to  $\text{Gr}^W$  kills  $\delta'$ ,  $\delta''$  and  $\delta$ ; thus, on  $\text{Gr}^W$  the induced differential is just the usual exterior derivative. This suggests that,

omitting indices,

$${}_W E_1 = \mathbb{H}(\mathrm{Gr}^W)$$

should be related to the de Rham cohomology of  $\Omega^\bullet$ -complexes on smooth projective varieties.<sup>12</sup>

Here we run into notational issues. If  $X$  were globally a product  $X' \times X''$  where

$$X' = X'_1 \cap \cdots \cap X'_{N'}, \quad X'' = X''_1 \cap \cdots \cap X''_{N''},$$

then we would have

$$(VIII.10) \quad \mathrm{Gr}^W \cong \bigoplus \left( (a'_{i'})_* \Omega_{[X'_{i'}]}^{j'}(-k') \right) \otimes \left( (a''_{i''})_* \Omega_{[X''_{i''}]}^{j''}(-k'') \right);$$

the induced differential would be

$$d_0 = d' \otimes 1'' + 1' \otimes d''$$

and  ${}_W E_1$  would be a cohomological mixed Hodge complex. As in the 1-parameter case when  $X$  is only locally a normal crossing divisor, we must use additional notations to label a choice of an ordering of the local irreducible components in the local product decomposition of  $X$ . This is carried out in [Fu1]. The general conclusions are the same for the several parameter case as for the 1-parameter one:

- the spectral sequence  ${}_F E_m$  for the Hodge filtration degenerates at  ${}_F E_1$ ;
- the spectral sequence  ${}_W E_m$  for the weight filtration degenerates at  ${}_W E_2$ ;
- the associated graded

$$\mathrm{Gr}_r^W \mathbb{H}^n \left( \Omega_{X/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X \right) \cong {}_W E_\infty^{n-r, r}$$

is a pure Hodge structure.

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<sup>12</sup>More precisely,  $\mathrm{Gr}^W A^\bullet$  is a sub-complex of a direct sum of the de Rham complexes on smooth projective varieties, the “sub” reflecting the matching conditions as discussed above.

And from this, taking into account that we are referring to [St2], [Fu2] for the  $\mathbb{Q}$ -structure, we have (cf. [Fu1], Theorem (6.10))

(VIII.11)

$(\mathbb{H}^n(\Omega_{\mathcal{X}/\mathcal{Y}}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X, W, F^\bullet))$  is a mixed Hodge structure.

## IX. MONODROMY AND POLARIZED LIMITING MIXED HODGE STRUCTURES

Here there are significant differences between the one and several parameter cases. The geometric reason is the obvious one: there are partial degenerations of the general smooth fibre corresponding to the faces of the monodromy cone. Along each face we get a variation of limiting mixed Hodge structures in the sense of [SZ], and when we take the limit at a boundary point of that face we obtain a further degeneration. The purely Hodge theoretic aspect of this is given in [CKS1]; in these notes we shall discuss the corresponding algebro-geometric picture. For this we need the strengthening of (VIII.11) stating that we obtain a *polarized limiting mixed Hodge structure*. For this monodromy enters in an essential way.

The first main observation is that the constructions in [St1] of an operator on  $A^\bullet$  that commutes with the differentials in the multi-complex  $\bigoplus_{p,q',q''} A^{p;q',q''}$  and induces the action of monodromy on  $\mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$  extends to the several parameter case. Thus we define

$$\begin{aligned} \nu' &: A^{p';q',q''} \rightarrow A^{p-1;q'+1,q''} \\ \nu'' &: A^{p;q',q''} \rightarrow A^{p-1;q',q''+1} \\ \nu &= \nu' + \nu'' \end{aligned} \tag{IX.1}$$

to be the projections. Because  $\mathcal{Y}'$  and  $\mathcal{Y}''$  are global divisors on  $\mathcal{X}$ , the above are well defined. In terms of the local product structure where

$$A^{p;q',q''} = \bigoplus_{p'+p''=p} A^{p',q'} \otimes A^{p'',q''}$$

we may write

$$\nu = \nu' \otimes 1'' + 1' \otimes \nu'' \tag{IX.2}$$

Then because of the commutativity properties of the corresponding operators in the 1-parameter case, we have that

*all of  $d, \delta', \delta'', \nu', \nu''$  mutually commute.*

The next step is to identify the actions  $[\nu'], [\nu'']$  and  $[\nu] = [\nu'] + [\nu'']$  induced on  $\mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$  with that of the monodromy operators  $N', N''$  and  $N = N' + N''$ .

As in the 1-parameter case, the action of monodromy is given, up to a constant, by the residues of the Gauss-Manin connection acting on the fibre at the origin of the Deligne extension of the bundle  $\mathbb{R}_\pi^n \mathbb{C}_{\mathcal{X}^*} \otimes \mathcal{O}_{S^*}$ . In the 1-parameter case the ingredients in the above identification are

- (i) the identification of  $N$  as the connecting map in the exact hypercohomology sequence of

$$(IX.3) \quad 0 \rightarrow \Omega_{\mathcal{X}/\Delta}^{\bullet-1}(\log X) \otimes \mathcal{O}_X \xrightarrow{ds/s} \Omega_{\mathcal{X}}^\bullet(\log X) \otimes \mathcal{O}_X \\ \rightarrow \Omega_{\mathcal{X}/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X \rightarrow 0;$$

- (ii) the resolution

$$\Omega_{\mathcal{X}/\Delta}^p(\log X) \otimes \mathcal{O}_X \rightarrow A^{p,\bullet}; \text{ and}$$

- (iii) the interpretation of the connecting map on hypercohomology in (IX.3) in terms of a mapping cone construction using the complexes  $A^{p,\bullet}$ .

In the several parameter case the situation is a little more involved. For step (i) the exact sequence (IX.3) is replaced by the filtration  $W_k(\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$  induced by the images of  $\pi^* \Omega_S^k(\log T) \otimes \mathcal{O}_X$  in  $\Omega_{\mathcal{X}/S}^{\bullet+k}(\log \mathcal{Y}) \otimes \mathcal{O}_X$ . The Gauss-Manin connection is then given by

$$(IX.4) \quad \begin{array}{ccc} E_1^{0,n} & \xrightarrow{d_1} & E_1^{1,n} \\ \wr & & \wr \end{array}$$

$$\mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X) \longrightarrow \Omega_S^1(\log T) \otimes \mathcal{O}_{S,s_0} \otimes_{\mathbb{C}} \mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$$

where the top row are terms in the spectral sequence arising from the above filtration on  $\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X$ .

For step (ii), we have the resolution

$$(IX.5) \quad \Omega_{X/S}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X \rightarrow A^{p;\bullet}$$

of  $\Omega_{X/S}^p(\log \mathcal{Y}) \otimes \mathcal{O}_X$  by the total complex associated to the double complex  $A^{p;q',q''}$ . What must be done to carry out step (iii) is to interpret the differential  $d_1$  in (IX.4) above in terms of the resolution (IX.5).

Rather than give a detailed calculation we shall give the conceptual ideas, and for this we shall use the shorthand notations (VIII.5) and (VIII.6) together with

$$\begin{aligned} \Phi'^{\bullet} &= \pi'^* \Omega_{U'}^{\bullet}(\log \mathcal{Y}') \\ \Phi''^{\bullet} &= \pi''^* \Omega_{U''}^{\bullet}(\log \mathcal{Y}'') \\ \Phi^{\bullet} &= \Omega_{U/S}^{\bullet}(\log \mathcal{Y}) \end{aligned}$$

and

$$\begin{aligned} \Psi'^{\bullet} &= \pi'^* \Omega_{\Delta'}^{\bullet}(\log o') \\ \Psi''^{\bullet} &= \pi''^* \Omega_{\Delta''}^{\bullet}(\log o'') \\ \Psi^{\bullet} &= \Psi'^{\bullet} \otimes \Psi''^{\bullet} \end{aligned}$$

where  $o'$  and  $o''$  are the origins in  $\Delta'$  and  $\Delta''$ . Briefly,  $\Phi'^{\bullet}$  and  $\Phi''^{\bullet}$  are the log-complexes on the factors  $U'$  and  $U''$  of  $U$ , and  $\Psi'^{\bullet}$  and  $\Psi''^{\bullet}$  are the pullbacks of the log complexes on the disc factors  $\Delta'$ ,  $\Delta''$  of  $\Delta$ . Thus we have

$$\begin{aligned} \Omega'^{\bullet} &\longleftrightarrow \left\{ \text{span of the } dx'_i/x'_i \text{ modulo } \sum dx'_i/x'_i \right\} \\ \Phi'^{\bullet} &\longleftrightarrow \left\{ \text{span of the } dx'_i/x'_i \text{ with no relation} \right\} \\ \Psi'^{\bullet} &\longleftrightarrow \left\{ ds'/s' = \sum dx'_i/x'_i \right\}. \end{aligned}$$

We then have

$$(IX.6) \quad \begin{array}{ccccccc} 0 & \rightarrow & \Psi'^{\bullet-1} & \rightarrow & \Phi'^{\bullet} & \rightarrow & \Omega'^{\bullet} \rightarrow 0 \\ 0 & \rightarrow & \Psi''^{\bullet-1} & \rightarrow & \Phi''^{\bullet} & \rightarrow & \Omega''^{\bullet} \rightarrow 0 \end{array}$$

where each row in (IX.6) corresponds to (IX.3) over the respective factors  $U'$ ,  $U''$  of  $U$ .

For the first step, the spectral sequence referred to above is the one associated to the filtration of  $\Phi'^{\bullet} \otimes \Phi''^{\bullet}$  by the sub-complex  $\Psi'^{\bullet} \otimes \Psi''^{\bullet}$ .

If  $\mathcal{X} \rightarrow \Delta' \times \Delta''$  were globally a product  $\mathcal{X}' \times \mathcal{X}'' \rightarrow \Delta' \times \Delta''$ , then the differential  $d_1$  in the spectral sequence (IX.4) could be expressed in terms of the connecting maps in the exact hypercohomology sequences associated to the exact sequences in (IX.6). This is not the case, but will become so with the additional data locally labeling the irreducible components of  $X \cap U'$  and  $X \cap U''$ .

For the next step, we consider the resolutions

$$(IX.7) \quad \begin{aligned} \Omega'^{\bullet} &\rightarrow A'^{\bullet} \\ \Omega''^{\bullet} &\rightarrow A''^{\bullet} \end{aligned}$$

whose tensor product gives the resolution  $\Omega^{\bullet} \rightarrow A^{\bullet}$ . The connecting maps in the exact hypercohomology sequences of (IX.6) may be expressed in terms of a cone construction for each of the rows in (IX.7). We may then use this in the first step to express the differential  $d_1$  in (IX.4) in terms of the projections  $\nu' : A^{p;q',q''} \rightarrow A^{p-1;q'+1,q''}$  and  $\nu'' : A^{p;q',q''} \rightarrow A^{p-1;q',q''+1}$ . When this is all written out we obtain the following conclusion:

(IX.8) *Via the weight spectral sequence for the complex  $A^{\bullet}$ , the monodromy operators  $N', N''$  and  $N = N' + N''$  are given by the induced actions  $[\nu'], [\nu'']$  and  $[\nu' + \nu''] = [\nu]$  on  $\mathbb{H}^n(A^{\bullet}) \cong \mathbb{H}^n(\Omega_{\mathcal{X}/\Delta' \times \Delta''}^{\bullet}(\log \mathcal{Y}) \otimes \mathcal{O}_X) = V_{\mathbb{C}}$ .*

Because of this we shall drop the notations  $[\nu'], [\nu'']$  and  $[\nu]$  and shall simply write  $N', N''$  and  $N$ . For the weight filtration  $W_k(V_{\mathbb{C}})$  we have

$$N : W_k(V_{\mathbb{C}}) \rightarrow W_{k-2}(V_{\mathbb{C}})$$

which for  $k \geq 0$  gives

$$(IX.9) \quad N^k : \mathrm{Gr}_k^W V_{\mathbb{C}} \rightarrow \mathrm{Gr}_{-k}^W V_{\mathbb{C}}.$$

The main result is then

(IX.10) *Assuming that we have a projective embedding  $\mathcal{X} \subset \mathbb{P}^A$ , the maps (IX.9) and isomorphisms.*

At this point the proof is an extension of the argument in the 1-parameter case given by [GN] and in chapter 11 in [PS]. In the  $\ell = 2$  case there is an action of  $\mathrm{sl}_2 \times \mathrm{sl}_2 \times \mathrm{sl}_2$  on the  ${}_W E_1$  term of the weight

spectral sequence and whose nil-negative elements are  $L = o_1(\mathcal{O}_X(1))$ ,  $\nu', \nu''$ . As discussed above, with the framework we have established the essential content of the calculations occur already when  $\mathcal{X} \rightarrow \Delta' \times \Delta''$  is a product  $\mathcal{X}' \times \mathcal{X}'' \rightarrow \Delta' \times \Delta''$ . For these, one uses the tri-primitive decomposition of  $V_{\mathbb{C}}$  as an  $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ -module and the Hodge-Riemann bilinear relations and construction of an algebraic Laplace operator as in loc. cit. Here the essential point is that in the product situation one has  $\Delta = \Delta' \otimes 1'' + 1' \otimes \Delta''$ ; i.e., there are no cross-terms.

X. ALGEBRO-GEOMETRIC INTERPRETATIONS OF  
 PROPERTIES OF THE MONODROMY CONE  
 AND LIMITING MIXED HODGE STRUCTURES

In the introduction we listed the properties (I.2)(i)–(ii) of the monodromy cone and (I.3)(i)–(ii) of the limiting mixed Hodge structure  $(V, W, F^\bullet)$  that are consequences of the Cattani-Kaplan-Schmid theory in the Hodge theoretic setting (I.1)<sub>HT</sub>. There we noted that in the algebro-geometric setting (I.1)<sub>AG</sub> these properties were proved by Deligne as consequences of the Weil conjectures and the comparison between  $\ell$ -adic and ordinary cohomology. In this section we will discuss how these properties follow from the complex algebro-geometric description of the limiting mixed Hodge structure in the several parameter family case. With the theory that has been developed above the basic ideas behind (I.2)(i) are the obvious ones:

- (a) the identification  $V_{\mathbb{C}} = \mathbb{H}^n(\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$ ;
- (b) the quasi-isomorphism

$$\Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X \rightarrow A^\bullet;$$

- (c) the interpretation of monodromy as arising from operators defined locally on the complex of sheaves  $A^\bullet$ .

The properties (I.2)(ii)–(iii) and (I.3)(i)–(iii) deal with relations among faces of the monodromy cone. They may also be derived from the theory developed above, but for this some further considerations are required.



We begin with a discussion of (I.2)(i), the independence of the weight filtration  $W(N)$  of  $N \in \sigma$ . This basic result together with (I.2)(ii) are due to Cattani-Kaplan [CK] and they provided the gateway to the subsequent analysis of limiting mixed Hodge structures in the several parameter (I.1)<sub>HT</sub> case ([CKS1], [Kas1], [Kas2], and [KK]). In this discussion we shall take the 2-parameter case and shall use the notations from Section VIII where we have

$$\mathcal{X} \rightarrow \Delta' \times \Delta''$$

with the local description (VIII.1). In terms of the definition (VII.3) of the multi-complex  $A^{p,q',q''}$  whose associated single complex is the  $A^\bullet$  above, the monodromy operators  $N', N''$  and  $N = N' + N''$  are induced from the hypercohomology spectral sequence from the operators  $\nu', \nu''$  and  $\nu = \nu' + \nu''$  defined in (IX.1).

With the notation

$$N_\lambda = \lambda' N' + \lambda'' N'', \quad \lambda', \lambda'' \in \mathbb{C}$$

we shall prove the following basic result relating the two filtrations on  $\mathbb{H}^n(\Omega_{\mathcal{X}/\Delta' \times \Delta''}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$  that arise from the local filtration  $W_r(A^\bullet)$  and the global monodromy filtration  $W(N)$ :

(X.1) For  $\lambda', \lambda'' \in \mathbb{R}^{>0}$  we have

$$W(N_\lambda) = W(N).$$

*Proof.* The argument is a little subtle. Setting  $\nu_\lambda = \lambda' \nu' + \lambda'' \nu''$ , on the complex  $A^\bullet$  with  $\mathbb{H}^n(A^\bullet) = \mathbb{H}^n(\Omega_{\mathcal{X}/\Delta' \times \Delta''}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$  there are defined

- the weight filtration  $W_r(A^\bullet)$  given by (VIII.9),
- the nilpotent operators  $\nu_\lambda : A^\bullet \rightarrow A^\bullet$  that satisfy

$$(X.2) \quad \nu_\lambda : W_r(A^\bullet) \rightarrow W_{r-2}(A^\bullet).$$

Given a *finite-dimensional* vector space  $E', E''$  and nilpotent endomorphisms  $A' \in \text{End}(E')$  and  $A'' \in \text{End}(E'')$ , then we have the elementary observation that for  $\lambda', \lambda'' \neq 0$

$$(X.3) \quad W(A'^\bullet \otimes \text{Id}'' + \text{Id}' \otimes A''^\bullet)(E' \otimes E'') = W(A'^\bullet)(E') \otimes W(A''^\bullet)(E''),$$

where  $\text{Id}'$  and  $\text{Id}''$  are the identity operators on  $E', E''$ . From the local tensor product description  $A^\bullet = A'^\bullet \otimes A''^\bullet$  one might then initially suspect that (X.2) should hold for all  $\lambda', \lambda'' \in \mathbb{C}^*$ . This is false, as we shall see below by example.<sup>13</sup> Thus, the condition  $\lambda', \lambda'' \in \mathbb{R}^{>0}$  must enter for global reasons.

We first note a further elementary linear algebra fact. Given a finite dimensional vector space  $E$  and commuting nilpotent endomorphisms  $A', A'' \in \text{End}_Q(E)$ , we have (cf. (1.7) in [CK]):

(X.4) *If the conditions*

$$\begin{aligned} A' &: W_k(A'') \rightarrow W_{k-2}(A'') \\ A'' &: W_k(A') \rightarrow W_{k-2}(A') \end{aligned}$$

*are satisfied, then*

$$W(A') = W(A'').$$

From (X.2) we have

$$N_\lambda : W_k(\mathbb{H}^n(A^\bullet)) \rightarrow W_{k-2}(\mathbb{H}^n(A^\bullet))$$

for any  $\lambda', \lambda''$ . Thus, using (X.4) it will suffice to show

(X.5) *for  $\lambda', \lambda'' \in \mathbb{R}^{>0}$  we have*

$$\boxed{W(N_\lambda)(\mathbb{H}^n(A^\bullet)) = W(\mathbb{H}^n(A^\bullet))}.$$

This is **the** basic result relating the two filtrations  $W$  and  $W(N_\lambda)$  on  $\mathbb{H}^n(\Omega_{\mathcal{X}/\Delta' \times \Delta''}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$ . Its proof consists in observing that the same argument for (IX.10) that was given for  $N = N' + N''$  works for  $N_\lambda$  provided that  $\lambda', \lambda'' \in \mathbb{R}^{>0}$ . Indeed, the eigenspace decomposition as complex vector spaces of the  $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$  acting on  ${}_W E_1$  is unchanged if we use  $\nu_\lambda = \lambda' \nu'$  and  $\lambda'' \nu''$  where  $\lambda', \lambda'' \in \mathbb{C}^*$ . If we want the decomposition of  ${}_W E_1$  as *real* vector spaces, then we need to have  $\lambda', \lambda'' \in \mathbb{R}^*$ . But it is only for  $\lambda', \lambda'' \in \mathbb{R}^{>0}$  that Hodge-Riemann II is satisfied, and this is crucial for the proof. We shall not write out the details here, but rather we shall explain why the calculation should work out.

<sup>13</sup>In case  $\mathcal{X} = \mathcal{X}' \times \mathcal{X}'' \rightarrow \Delta' \times \Delta''$  is *globally* a product, by (X.3) this is true.

If we have a product situation  $\mathcal{X}' \times \mathcal{X}'' \rightarrow \Delta' \times \Delta''$ , then we may use the calculations in Chapter 11 of [PS] and [GN], together with the standard Clebsch-Gordon decomposition of the action of an  $\mathfrak{sl}_2$  on the tensor product to deduce (X.5) in this case. One point to note in carrying this out is that  $F^p A^\bullet = \bigoplus_{p'+p''=p} F^{p'} A'^\bullet \otimes F^{p''} A''^\bullet$ , so that we are not in strictly a tensor product situation. These calculations relate to the action of  $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$  on the term  ${}_W E_1$  of the weight spectral sequence and on the differential  $d_{1W} E_1 \rightarrow {}_W E_1$  as a map of  $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ -modules.

In the general case the situation is similar to the product one considered above with the additional notational complication that involves a labeling of the local irreducible pieces in the local product of normal crossing divisors. We shall not write out the details, referring to [Fr1] where similar calculations are carried out.

**Remark:** For  $V_{\mathbb{C}} = \mathbb{H}^n(\Omega_{\mathcal{X}/\Delta' \times \Delta''}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$ , from (X.4) we have

$$N', N'' : W_r(V_{\mathbb{C}}) \rightarrow W_{r-2}(V_{\mathbb{C}}).$$

However, thinking of  $V_{\mathbb{C}}$  as the cohomology  $\mathbb{H}^n(X_s, \mathbb{C})$  of a general fibre of  $\mathcal{X} \rightarrow \Delta' \times \Delta''$ , in Figure 1 of the introduction the weight filtrations  $W(N')$  and  $W(N'')$  on  $V_{\mathbb{C}}$  reflect quite different geometric phenomena involving the limiting mixed Hodge structures that arise as  $X_s$  degenerates to the two axes. Concerning  $N_\lambda$  where  $\lambda', \lambda'' \in \mathbb{Q}^{>0}$ , we first note that the weight filtration of a nilpotent endomorphism does not change when we scale the endomorphism. Thus we may assume that

$$\lambda' = n' \in \mathbb{Z}^{>0}, \quad \lambda'' = n'' \in \mathbb{Z}^{>0}.$$

When we apply the base change

$$\tilde{s}' = (s')^{n'}, \quad \tilde{s}'' = (s'')^{n''}$$

to the family  $\mathcal{X} \rightarrow \Delta' \times \Delta''$ , for the new family  $\tilde{\mathcal{X}} \rightarrow \tilde{\Delta}' \times \tilde{\Delta}''$  the general fibre is the same but the logarithm of monodromy is now  $N$  is now  $N_\lambda$ . It might seem plausible geometrically that one might show

from this that the weight filtration  $W(N_\lambda)$  is independent of  $\lambda$ . But this is *not* a topological result; Hodge theory is required.

We now turn to a brief, informal discussion of (I.3)(ii) in the  $\ell = 2$  algebro-geometric case. The basic idea is to first note how the result in the case of a product  $\mathcal{X}' \times \mathcal{X}'' \rightarrow \Delta' \times \Delta''$  may be deduced from the isomorphism

$$(X.6) \quad \mathbb{H}^n \left( \Omega_{\mathcal{X}' \times \mathcal{X}'' / \Delta' \times \Delta''}^\bullet (\log(\mathcal{Y}' + \mathcal{Y}'')) \otimes \mathcal{O}_X \right) \cong \bigoplus_{p+q=n} \mathbb{H}^p \left( \Omega_{\mathcal{X}' / \Delta'}^\bullet (\log \mathcal{Y}') \otimes \mathcal{O}_X \right) \otimes \mathbb{H}^q \left( \Omega_{\mathcal{X}'' / \Delta''}^\bullet (\log \mathcal{Y}'') \otimes \mathcal{O}_X \right)$$

that results from the local isomorphism

$$(X.7) \quad (A^\bullet, W) \cong (A'^\bullet, W') \otimes (A''^\bullet, W'')$$

where  $A'^\bullet, A''^\bullet$  are as above and  $W', W''$  are the respective weight filtrations. Using the identification (X.6) we have that

$W$  is the weight filtration of  $N' \otimes \text{Id}''$  relative to  $W(\text{Id}' \otimes N'')$ .

Implicit here are the assertions that  $W(N'') = W''$  and  $W(N') = W'$ , and finally that  $W = W(N' \otimes \text{Id}'' + \text{Id}' \otimes N'')$ .

In case  $\mathcal{X} \rightarrow \Delta' \times \Delta''$  is only locally a product (VIII.1), we do not have (X.7). Once we choose an ordering of the irreducible components in each of the two factors in the local product (VIII.1), we locally have (X.7). What is globally defined are the filtrations  $W'$  and  $W''$  on  $A^\bullet$  and where the filtration  $W$  on  $A^\bullet$  is described in terms of  $W'$  and  $W''$  by (VIII.3) and (VIII.9). Also, we may intrinsically define the complexes

$$(A^\bullet, s' = \text{constant})$$

by setting  $s' = \text{constant}$  in the definition (VIII.3) of  $A^\bullet$ . Effectively, we are restricting  $A^\bullet$  to the disc  $\Delta_{s''}$  in Figure 1 in the introduction. Then the groups

$$\mathbb{H}^n(A^\bullet, s' = \text{constant})$$

will vary with  $s'$  and form a variation of limiting mixed Hodge structures over  $\Delta'^*$ . The associated graded to this variation of limiting mixed Hodge structures gives a variation of polarized Hodge structures

over  $\Delta'^*$  with monodromy logarithm  $N'$ . The local identification (X.7) together with the facts that  $W'$  and  $W''$  are globally defined and that on the corresponding hypercohomology groups we have

- $W = W(N)$  on  $\mathbb{H}^n(A^\bullet)$ ;
- $W'' = W(N'')$  on  $\mathbb{H}^n(A^\bullet, s' = \text{constant})$ ;
- $N'$  preserves the filtration  $W''$  on  $\mathbb{H}^n(A^\bullet, s' = \text{constant})$ ;
- the weight filtration  $W(\text{Gr}^{W''} N')$  induced by the action  $\text{Gr}^{W''} N'$  of  $N'$  on  $\text{Gr}^{W''} \mathbb{H}^n(A^\bullet, s' = \text{constant})$  is the filtration induced on this space by  $W$

gives the result (I.2)(ii). The details of this will be provided in a later version of these notes.

## XI. DEFORMATION THEORY AND LIMITING MIXED HODGE STRUCTURES

For the situation (I.1)<sub>AG</sub> which has the local normal form (I.7) around the central fibre  $X$ , in the case  $\ell = 1$  and where  $X$  is a global normal crossing divisor it was suggested and partially established in [Fr1], and then proved in [St2], that

(XI.1) *the limiting mixed Hodge structure depends only on the 1<sup>st</sup> order neighborhood of  $X$  in  $\mathcal{X}$ .*

More precisely, in the case  $\ell = 1$  the family (I.1)<sub>AG</sub> gives a tangent vector

$$\xi \in T_X \text{Def}(X).$$

As shown in [GG], in terms of  $\xi$  we may define

- a complex of sheaves  $(\Lambda_X^\bullet, d)$  where in case there is a family  $\mathcal{X} \rightarrow \Delta$

$$(XI.2) \quad \Lambda_X^\bullet = \Omega_{\mathcal{X}/\Delta}^\bullet(\log X) \otimes \mathcal{O}_X;$$

- sheaves  $\tilde{\Lambda}_X^p$  where there is an exact sequence

$$(XI.3) \quad 0 \rightarrow \Lambda_X^{p-1} \xrightarrow{ds/s} \tilde{\Lambda}_X^p \rightarrow \Lambda_X^p \rightarrow 0.$$

The complex vector space for the mixed Hodge structure is

$$V_{\mathbb{C}} = \mathbb{H}^n(\Lambda_X^\bullet).$$

The Hodge filtration is induced by the bête filtration on the complex  $(\Lambda_X^\bullet, d)$ . For the weight filtration the connecting map in the exact hypercohomology sequence of (XI.3) gives the monodromy operator  $N$ , and the weight filtration is given by  $W(N)$ . Although the arguments in the literature seem to require that  $\xi$  be unobstructed, i.e., there is a family  $\mathcal{X} \rightarrow \Delta$  with tangent  $\xi$ , proof analysis shows that this is not necessary.

The analysis in these notes shows that

(XI.4) *the statement (XI.2) remains true in the several parameter case.*

More precisely, there should be given a subspace

$$T \subset T_X \text{Def}(X)$$

with the properties that  $T \cong \mathbb{C}^\ell$ , and that any  $\xi \in \mathbb{C}^{*\ell}$  is to 1<sup>st</sup> order a smoothing deformation. In this situation we may construct the several parameter analogues of  $\Lambda_X^\bullet$  and  $\tilde{\Lambda}_X^\bullet$ , and then the above constructions of the Hodge and weight filtrations will go through to define these filtrations on  $\mathbb{H}^n(\Lambda_X^\bullet)$ . The  $\mathbb{Q}$ -structure may be defined by extending the arguments in [St2]; cf. [Fu1] and [Fu2].

Finally, we observe that intuitively the deformations of  $X$  may be thought of as composed of two types:.

(i) deformations where a component of the singular locus of  $X$  is smoothed.

For example,  $X$  might be a nodal curve and the deformation smooths some, but not necessarily all, of the nodes.

(ii) deformations of  $X$  where in the local product of normal crossing divisor situation (I.7) one of the factors is smoothed.

Both types of deformation occur, and we shall give an interesting example of the first type.

**Example:** This is the example in Figure 1 in Section I of a genus 2 curve degenerating to one with three nodes. With the choice  $\delta_1, \delta_2, \delta_3$

indicated there, the monodromy matrices will be

$$N_i = \left( \underbrace{\begin{pmatrix} 0 & \widehat{N}_i \\ 0 & 0 \end{pmatrix}}_2 \right) \}_2$$

where

$$\widehat{N}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{N}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widehat{N}_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For  $N_\lambda = \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 N_3$  with the evident notation for  $\widehat{N}_\lambda$ , we have

$$\widehat{N}_\lambda = \begin{pmatrix} \lambda_1 + \lambda_3 & \lambda_3 \\ \lambda_3 & \lambda_2 + \lambda_3 \end{pmatrix},$$

$$\det \widehat{N}_\lambda = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3.$$

This can be zero if all  $\lambda_i \in \mathbb{C}^*$ ; e.g., if  $\lambda_3 = \frac{-\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$ . Thus  $W(N_\lambda)$  is not independent of  $\lambda$  when all  $\lambda_i \in \mathbb{C}^*$  (or even  $\lambda_i \in \mathbb{Q}^*$ ). The conditions for  $N_\lambda$  to be positive definite are

$$\begin{cases} \lambda_1 + \lambda_3 > 0 \\ \lambda_2 + \lambda_3 > 0 \\ \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 > 0. \end{cases}$$

This is a cone  $\sigma_{\text{pol}}$ , which is *not* polyhedral but does, of course, contain the quadrant where all  $\lambda_i \in \mathbb{R}^{>0}$ .

This is also an example where  $X$  is not a global normal crossing divisor, and so we must use additional notations to do the computation of  $\mathbb{H}^1(\Omega_{X/\Delta^3}^\bullet(\log \mathcal{Y}) \otimes \mathcal{O}_X)$  in terms of the normalizations of the strata of  $X$ . Here

$$\widetilde{X} = X^{[1]} = \mathbb{P}^1 \amalg \mathbb{P}^1,$$

$$X^{[2]} = \{z_1, z_2, z_3\} = \text{the three nodes.}$$

We have the exact sequences

$$0 \rightarrow \mathcal{O}_X \rightarrow (a_1)_* \mathcal{O}_{X^{[1]}} \xrightarrow{\text{diff}^0} \mathcal{O}_{X^{[2]}} \rightarrow 0$$

$$0 \rightarrow \Omega_{X^{[1]}}^1(\log X^{[2]})^{\text{opp}} \rightarrow \Omega_{X^{[1]}}^1(\log X^{[1]}) \xrightarrow{\text{diff}^1} \mathcal{O}_{X^{[2]}} \rightarrow 0$$

where  $\text{diff}^0 =$  difference of values,  $\text{diff}^1 =$  difference of the residues, and  $\Omega_{X^{[1]}}^1(\log X^{[2]})^{\text{opp}}$  are the forms with opposite residues at the nodes. Then

$$V_{\mathbb{C}} = \mathbb{H}^1 \left( \mathcal{O}_X \xrightarrow{d} \Omega_{X^{[1]}}^1(\log X^{[2]})^{\text{opp}} \right)$$

and for  $N = N_1 + N_2 + N_3$

$$\begin{aligned} \ker N &= \text{im } N \cong H^1(\mathcal{O}_X) \cong H^0(\mathcal{O}_{X^{[2]}})/H^0(\mathcal{O}_{X^{[1]}}), \\ V/\ker N &\cong H^0(\Omega_{X^{[1]}}^1(\log X^{[2]})^{\text{opp}}). \end{aligned}$$

We have written this out to give an indication, in this very simple case, of what one needs to do to treat the case when  $\mathcal{X}$  is only locally a product of normal crossing varieties.

#### APPENDIX A. LOCAL CONSIDERATIONS

In this appendix we will collect some known results concerning complexes of logarithmic differentials and of relative logarithmic differentials. We will use the following terminology and notations:

- $X$  will be a normal crossing variety, which we define to be a  $d$ -dimensional complex analytic variety that locally has an embedding in  $\mathbb{C}^{d+1}$  defined by

$$(A.1) \quad x_1 \cdots x_k = 0$$

where  $x_1, \dots, x_{d+1}$  are coordinates in  $\mathbb{C}^{d+1}$ ;

- $X_k$  will be the locus of  $k$ -fold singular points, which in terms of (A.1) are defined by

$$x_1 = \cdots = x_k = 0;$$

- $X^{[k]}$  denotes the normalization of  $X_k$  with the corresponding map

$$a_k : X^{[k]} \rightarrow X.$$

**Remark:** It will simplify our notations and in the end make no essential difference to assume that globally

$$X = X_1 \cup \cdots \cup X_N$$



is the union of smooth compact complex manifolds  $X_i$  that intersect transversely. Then

$$X_k = \bigcup_{|I|=k} X_I$$

where  $I$  runs over subsets of  $1, \dots, n+1$  and

$$X_I = \bigcap_{i \in I} X_i.$$

We note that  $X_1 = X$  and that

$$X^{[k]} = \prod_{|I|=k} X_I.$$

- $\mathcal{X}$  will be an  $(n+1)$ -dimensional complex manifold on which  $X \subset \mathcal{X}$  is a normal crossing divisor;
- $\mathcal{X} \xrightarrow{\pi} S$  will be a proper mapping to the disc  $S = \{|s| < 1\}$  which is locally given by

$$(A.2) \quad x_1 \cdots x_k = s.$$

The log complexes and complexes of differentials that will be defined are

- (i) the log-complex  $\Omega_{\mathcal{X}}^{\bullet}(\log X)$ ;
- (ii) the relative log complex  $\Omega_{\mathcal{X}/S}^{\bullet}(\log X)$ ;
- (iii) the Kähler differentials  $\Omega_X^{\bullet}$ , and in case  $X$  is *d-semi-stable* ([Fr1]) the abstract relative log complex

$$\Lambda_X^{\bullet},$$

which has the property that, if there is a mapping  $\mathcal{X} \rightarrow S$  as above then

$$(A.3) \quad \Lambda_X^{\bullet} = \Omega_{\mathcal{X}/S}^{\bullet}(\log X) \otimes \mathcal{O}_X.$$

The point of (iii) is that  $\Lambda_X^{\bullet}$  may be defined purely in terms of a 1<sup>st</sup> order smoothing deformation  $\xi \in T_X^0 \text{Def}(X)$ .

The way to think of this is that locally  $\xi$  defines the  $s$  in (A.2). In a different set of coordinates  $x'_i = u_i x_i$  where  $u_i$  is a unit, we will then have

$$(A.4) \quad u_1 \cdots u_k \equiv 1 \text{ modulo } s^2.$$

We will return to this following a discussion of (i), (ii), (iii) above. In each of these cases we will give the generators and relations in local coordinates; the conclusions that are drawn will have intrinsic meaning.

We now turn to the local formulas for (i)–(iii).

(i) Locally the ideal of  $X$  and ideal of the singular locus  $X_{\text{sing}} = X_2$  of  $X$  are generated respectively by

- $f = x_1 \cdots x_k$  and the
- $f_i = x_2 \cdots \hat{x}_i \cdots x_k$  for  $i = 1, \dots, k$ .

Then  $\Omega_X^\bullet(\log X)$  is the  $\mathcal{O}_X$ -module generated by

$$(A.5) \quad dx_1/x_1, \dots, dx_k/x_k; dx_{k+1}, \dots, dx_{d+1}.$$

Intrinsically, denoting by  $\Omega_X^\bullet(*X) = j_*\Omega_{\mathcal{X}^*}^\bullet$  where  $j : \mathcal{X}^* \rightarrow \mathcal{X}$  is the inclusion, the complex of meromorphic differentials with poles on  $X$  is given by

$$\Omega_X^\bullet(\log X) = \{\omega \in \Omega_X^\bullet(*X) : f\omega \text{ and } fd\omega \text{ are in } \Omega_X^\bullet\}.$$

There is a weight filtration  $W_q\Omega_X^\bullet(\log X)$  of  $\Omega_X^\bullet(\log X)$  by sub-complexes where, for an index set  $I$  setting

- $x_I = \prod_{i \in I} x_i$ ,
- $dx_I = \bigwedge_{i \in I} dx_i$ .

$W_q\Omega_X^\bullet(\log X)$  is generated by the

$$(A.6) \quad dx_I/x_I \wedge \varphi$$

where  $I \subset \{1, \dots, k\}$ ,  $|I| \leq q$  and  $\varphi$  is holomorphic. We may describe  $W_q\Omega_X^\bullet(\log X)$  verbally as “the forms with  $\leq q$  of the  $dx_i/x_i$  terms.” The *Poincaré residue map*

$$\text{Gr}_q^W \Omega_X^\bullet(\log X) \xrightarrow{\sim} (a_q)_* \Omega_{X^{[q]}}^{\bullet-q}$$

is defined by the map on the form (A.6) given by

$$dx_I/x_I \wedge \varphi \rightarrow \varphi|_{X_I}.$$

As a consequence, we have for the cohomology sheaves

$$(A.7) \quad \mathcal{H}^p(\Omega_X^\bullet(\log X)) \cong (a_p)_* \mathbb{C}_{X^{[p]}}.$$

(ii) The relative log complex  $\Omega_{\mathcal{X}/S}^\bullet(\log X)$  has the same  $\mathcal{O}_{\mathcal{X}}$ -generators (A.5) and with the defining relation

$$(A.8) \quad ds/s = \sum_i dx_i/x_i = 0.$$

By (A.4), a local change of coordinates in  $\mathcal{X}$  leaves (A.8) unchanged modulo the ideal  $(f)$  of  $X$ . There is an exact sequence of  $\mathcal{O}_{\mathcal{X}}$ -modules

$$(A.9) \quad 0 \rightarrow \Omega_{\mathcal{X}/S}^{\bullet-1}(\log X) \xrightarrow{ds/s} \Omega_{\mathcal{X}}^\bullet(\log X) \rightarrow \Omega_{\mathcal{X}/S}^\bullet(\log X) \rightarrow 0.$$

For the stalks at the origin of the cohomology sheaves of the complex  $\Omega_{\mathcal{X}/S}^\bullet(\log X)$ , denoting by  $\{u_1, u_2, \dots\}$  the span of elements  $u_1, u_2, \dots$  in a vector space  $U$ , we have

$$(A.10) \quad \begin{cases} \mathcal{H}^0(\Omega_{\mathcal{X}/S}^\bullet(\log X))_0 \cong \mathbb{C}\{f\}, \\ \mathcal{H}^1(\Omega_{\mathcal{X}/S}^\bullet(\log X))_0 \cong \mathbb{C}\{f\} \otimes \{dx_1/x_1, \dots, dx_k/x_k : \sum dx_i/x_i = 0\}, \\ \mathcal{H}^p(\Omega_{\mathcal{X}/S}^\bullet(\log X))_0 \cong \wedge^p \mathcal{H}^1(\Omega_{\mathcal{X}/S}^\bullet(\log X))_0, \quad p \geq 1. \end{cases}$$

(iii) The Kähler differentials  $\Omega_X^\bullet$  are an intrinsically defined complex on  $X$ . Locally,  $\Omega_X^\bullet$  is generated as an  $\mathcal{O}_X$ -module with generators  $dx_1, \dots, dx_{d+1}$  and defining relations

$$f = 0, \quad df = 0.$$

We thus have the exact sequence of  $\mathcal{O}_X$ -modules

$$(A.11) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \otimes \{dx_1, \dots, dx_{d+1}\} \rightarrow \Omega_X^1 \rightarrow 0$$

where  $g \in \mathcal{O}_X$  maps to  $gdf = g(\sum f_i dx_i)$ .

The sheaf of Kähler differentials is coherent but not locally free. In fact, for each  $x \in X$  we have

$$(A.12) \quad \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)_x \cong \mathcal{O}_{X,x}$$

with the sequence (A.11) giving a generator  $c_x$ .

There is a natural restriction map

$$\Omega_X^1 \rightarrow (a_1)_* \Omega_{X^{[1]}}^1$$

whose kernel  $\tau_X^1$  consists of the torsion differentials generated by the

$$\varphi_i =: f_i dx_i$$

with the defining relation

$$\sum \varphi_i = 0.$$

The Kähler differentials arise in deformation theory via the identification

$$T_X \text{Def}(X) \cong \mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X).$$

Via the local to global spectral sequence for Ext, we have

$$\mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \rightarrow H^0(\mathbb{E}xt_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)),$$

and the condition that  $\xi \in T_X \text{Def}(X)$  be to 1<sup>st</sup> order a smoothing deformation of  $X$  is that the restriction maps  $\xi \rightarrow \xi_x$  be surjective for each  $x \in X$ . In this case  $\xi$  defines an  $s$  in (A.2) that is well-defined modulo  $s^2$  under a change of the local defining equation of  $X$ . Then as noted there we may define a complex  $\Lambda_X^\bullet$  of  $\mathcal{O}_X$ -modules that for any family  $\mathcal{X} \rightarrow S$  gives  $\Omega_{\mathcal{X}/S}^1(\log X) \otimes \mathcal{O}_X$  where  $\xi$  corresponds to the tangent vector  $d/ds \in T_X \text{Def}(X)$ .

Returning to the discussion of (i), the exact sequences

$$0 \rightarrow \Omega_X^p \xrightarrow{ds/s} \Omega_X^{p+1}(\log X) \xrightarrow{ds/s} \Omega_X^{p+2}(\log X) \rightarrow \dots$$

give

(A.13)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Gr}_0^W \Omega_X^p(\log X) & \longrightarrow & \text{Gr}_1^W \Omega_X^{p+1}(\log X) & \longrightarrow & \text{Gr}_2^W \Omega_X^{p+1}(\log X) \longrightarrow \dots \\ & & \wr \parallel & & \wr \parallel & & \wr \parallel \\ 0 & \longrightarrow & \Omega_X^p / \tau_X^p & \longrightarrow & (a_1)_\nu \Omega_{X[1]}^p & \longrightarrow & (a_2)_* \Omega_{X[2]}^p \longrightarrow \dots \end{array}$$

where  $\Omega_X^p$  is the sheaf of Kähler differentials on  $X$  and  $\tau_X^p$  is the  $\mathcal{O}_X$ -subsheaf generated by the torsion differential  $\tau_X^1$ .

As in [Fr1], using the hypercohomology of the double complex of sheaves

$$\mathcal{L}^{p,q} = (a_p)_* \Omega_{X[p]}^q$$

leads to (A.7). More relevant for our purposes, one may ask for the interpretation of the associated graded for the weight filtration on

$\Omega_{X/S}^\bullet(\log X)$  induced by the weight filtration on  $\Omega_X^\bullet(\log X)$ . Using (A.9) and (A.12) we find that

$$\mathrm{Gr}_p^W \Omega_{X/S}^{p+q}(\log X) \cong \mathrm{coker} \left\{ (a_{p-1})_* \Omega_{X^{[p-1]}}^p \rightarrow (a_p)_* \Omega_{X^{[p]}}^p \right\}.$$

This suggests that this may not be the correct definition of the weight filtration for  $\Omega_{X/S}^\bullet(\log X)$ .

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