

POSITIVITY OF VECTOR BUNDLES AND HODGE THEORY

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ABSTRACT. It is well known that positivity properties of the curvature of a vector bundle have implications on the algebro-geometric properties of the bundle, such as numerical positivity, vanishing of higher cohomology leading to existence of global sections etc. It is also well known that bundles arising in Hodge theory tend to have positivity properties. From these considerations several issues arise:

- (i) For a bundle that is semi-positive but not strictly positive; what further natural conditions lead to the existence of sections of its symmetric powers?
- (ii) In Hodge theory the Hodge metrics generally have singularities; what can be said about these and their curvatures, Chern forms etc.?
- (iii) What are some algebro-geometric applications of positivity of Hodge bundles?

The purpose of this partly expository paper is fourfold. One is to summarize some of the general measures and types of positivity that have arisen in the literature. A second is to introduce and give some applications of *norm positivity*. This is a concept that implies the different notions of metric semi-positivity that are present in many of the standard examples and one that has an algebro-geometric interpretation in these examples. A third purpose is to discuss and compare some of the types of metric singularities that arise in algebraic geometry and in Hodge theory. Finally we shall present some applications of the theory from both the classical and recent literature.

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This paper is partly based on joint work in progress with Radu Laza and Colleen Robles (cf. [GGLR17]).

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I. INTRODUCTION AND NOTATION AND TERMINOLOGY

I.A. Introduction.

The general purpose of this paper is to give an account of some aspects and applications of the concept of positivity of holomorphic vector bundles, especially those that appear in Hodge theory. The applications use the positivity of Hodge line bundle (Theorem I.A.7)* (the meaning of the asterisk is explained in footnote 1 below), the semi-positivity of the cotangent bundle to the image of a period mapping (Theorem VI.B.1), and the semi-positivity of the Hodge vector bundle (Theorem I.A.27). Also discussed is the numerical positivity of the Hodge bundles.

The overall subject of positivity is one in which there is an extensive and rich literature and in which there is currently active research occurring with interesting results appearing regularly. We shall only discuss a few particular topics and shall use [Dem12a] as our reference for general background material concerning positivity in complex analytic geometry as well as an account of some recent work, and refer to [Pă16] as a source for a summary of some current research and as an overall guide to the more recent literature. The recent paper [Den18] also contains an extensive bibliography. For Hodge theory we shall use [PS08] and [CMSP17] as our main general references.

Following this introduction and the establishment of general notations and terminology in Sections I.B and I.C, in Section II we shall give a synopsis of some of the standard measures and types of positivity of holomorphic vector bundles. In Section II.H we present the further result (I.A.10) below. It is this result that enables us to largely replace the cohomological notion of weak positivity in the sense of Viehweg [Vie83a], [Vie83b] with purely differential geometric considerations.

One purpose of this paper is to give an informal account of some of the results and their applications concerning the positivity of the Hodge line bundles and to discuss topics that are directly related to and/or grew out of them (cf. [GGLR17] and the references cited there). One application to algebraic geometry of Theorems 1.2.2 and

1.3.10 there is the following:¹

(I.A.1) *Let \mathcal{M} be the KSBA moduli space for algebraic surfaces X of general type and with given $p_g(X)$, $q(X)$ and K_X^2 .² Then the Hodge line bundle $\Lambda_e \rightarrow \overline{\mathcal{M}}$ is defined over the canonical completion $\overline{\mathcal{M}}$ of \mathcal{M} . Moreover it is free, and therefore $\text{Proj}(\Lambda_e)$ exists and may be used to define the Satake-Baily-Borel completion $\overline{\mathcal{P}}$ of the image of $\mathcal{P} = \Phi(\mathcal{M})$ of the period mapping $\Phi : \mathcal{M} \rightarrow \Gamma \backslash D$.³*

The period mapping extends to

$$\Phi_e : \overline{\mathcal{M}} \rightarrow \overline{\mathcal{P}}$$

and set-theoretically the image of the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ consists of the associated graded to the limiting mixed structures along the boundary strata of $\partial\mathcal{M}$. As has been found in applying this result to the surfaces analyzed in [FPR15a], [FPR15b], [FPR17] and to similar surfaces arising from discussions with the authors of those papers, the extended period mapping may serve as an effective method for organizing and understanding the boundary structure of $\overline{\mathcal{M}}$ and in providing a road map in how to desingularize it.⁴

The construction of $\overline{\mathcal{P}}$ is general and in brief outline proceeds as follows:

(i) One begins with a variation of Hodge structure over a smooth quasi-projective variety B given by a period mapping

$$(I.A.2) \quad \Phi : B \rightarrow \Gamma \backslash D.$$

Here B has a smooth completion \overline{B} where $Z = \overline{B} \setminus B$ is a reduced normal crossing divisor with irreducible component Z_i around which the local monodromies T_i are assumed to be unipotent. We may also assume that Φ has been extended across any Z_i for which T_i are of finite order; then Φ is proper and the image $\Phi(B) = \mathcal{P} \subset \Gamma \backslash D$ is a closed analytic subvariety ([Som78]).⁵

(ii) Along the smooth points Z_I^* of the strata $Z_I := \bigcap_{i \in I} Z_i$ of Z there is a limiting mixed Hodge structure ([CKS86]), and passing to the associated graded gives a variation of Hodge structure

$$\Phi_I : Z_I^* \rightarrow \Gamma_I \backslash D_I.$$

¹Significant amplification and complete proofs of the results in [GGLR17] are currently being prepared. The results (I.A.1), (I.A.7) and (1.A.15) are here stated in general but complete proofs have to date only been given in special cases. For this reason whenever reference is made to these results we have inserted an asterisk to have (I.A.1)*, (I.A.7)*, and (1.A.15)*. Below we shall give a short algebro-geometric argument for all the results the case when $\dim B = 2$ in (I.A.2) below.

²We shall use [Kol13] as our general reference for the topic of moduli.

³The definition of a Satake-Baily-Borel completion will be explained below. Here we are mainly considering that part of the period mapping that arises from the polarized Hodge structure on $H^2(X)$. When $p_g(X) \geq 2$ we are in a non-classical situation, and the new phenomena that arise in this case are the principal aspect of interest.

⁴In contrast to the case of curves, $\overline{\mathcal{M}}$ seems almost never to be smooth even when \mathcal{M} is.

⁵The papers [BBT18] and [BKT18] give a very interesting “model-theoretic” proof of the quasi-projectivity of \mathcal{P} .

Extending Φ_I across the boundary components of Z_I^* around which the monodromy is finite, from the image of the extension of $\Phi_{I,e} : Z_{I,e}^* \rightarrow \Gamma_i \setminus D_I$ we obtain a complex analytic variety \mathcal{P}_I . Then as a set

$$\overline{\mathcal{P}} = \mathcal{P} \cup \left(\bigcup_I \mathcal{P}_I \right),$$

where on the right-hand side there are identifications made corresponding to strata Z_I^* , Z_J^* where $Z_I \cap Z_J \neq \emptyset$ and where Φ_I, Φ_J have both been extended across intersection points.

(iii) This defines $\overline{\mathcal{P}}$ as a set, and it is not difficult to show that the resulting $\overline{\mathcal{P}}$ has the structure of a compact Hausdorff topological space and that the extended period mapping

$$(I.A.3) \quad \Phi_e : \overline{B} \rightarrow \overline{\mathcal{P}}$$

is proper with compact analytic subvarieties $F_x := \Phi_e^{-1}(x)$, $x \in \overline{\mathcal{P}}$, as fibres. If we define

$$\mathcal{O}_{\overline{\mathcal{P}},x} = \left\{ \begin{array}{l} \text{ring of functions } f \text{ that are continuous in} \\ \text{a neighborhood } \mathcal{U} \text{ of } x \in \overline{\mathcal{P}} \text{ and which are} \\ \text{holomorphic in } \Phi_e^{-1}(\mathcal{U}) \text{ and constant on fi-} \\ \text{bres of } \Phi_e \end{array} \right\}$$

then the issue is to show that *there are enough functions in $\mathcal{O}_{\overline{\mathcal{P}},x}$ to define the structure of a complex analytic variety on $\overline{\mathcal{P}}$.*

(iv) Using [CKS86] the *local* structure of Φ_e along $F_x \subset \Phi_e^{-1}(\mathcal{U})$ can be analyzed; this will be briefly recounted in Section V below, the main point being to use implications of the relative weight filtration property (RWFP) of limiting mixed Hodge structures.

(v) This leaves the issue of the *global* structure of Φ_e along F_x . We note that there is a map

$$\mathfrak{m}_x / \mathfrak{m}_x^2 \rightarrow H^0 \left(N_{F_x/\overline{B}}^* \right)$$

where $\mathfrak{m}_x \subset \mathcal{O}_{\overline{\mathcal{P}},x}$ is the maximal ideal and $N_{F_x/\overline{B}}^* \rightarrow F_x$ is the co-normal bundle of F_x in \overline{B} . Thus one expects some “positivity” of $N_{F_x/\overline{B}}^*$. From $F_x \subset Z_{I,e}^* \subset \overline{B}$ we obtain

$$0 \rightarrow N_{Z_{I,e}^*/\overline{B}}^* \Big|_{F_x} \rightarrow N_{F_x/\overline{B}}^* \rightarrow N_{F_x/Z_{I,e}^*}^* \rightarrow 0.$$

Since $\Phi_{I,e} : Z_{I,e}^* \rightarrow \Gamma_I \setminus D_I$ is defined and maps F_x to a point, we obtain some positivity of $N_{F_x/Z_{I,e}^*}^*$. Denoting by $\mathfrak{m}_{I,x}$ the maximal ideal in $\mathcal{O}_{\Gamma_I \setminus D_I,x}$ at $x = \Phi_{I,e}(F_x)$, from the local knowledge of Φ_e along F_x one is able to infer that the positivity of $N_{F_x/Z_{I,e}^*}^*$ that arises from $\mathfrak{m}_{I,x} / \mathfrak{m}_{I,x}^2$ lifts to sections in $H^0 \left(N_{F_x/\overline{B}}^* \right)$; thus the main issue is that of the positivity of $N_{Z_{I,e}^*/\overline{B}}^* \Big|_{F_x}$.

(vi) Here, as we hope will be explained in the currently being revised and expanded version of [GGLR17], some apparently new Hodge theoretic considerations arise. In the literature, and in this work, there have been numerous consequences drawn from

the positivity properties of the Hodge line bundle, and also from some of the positivity properties of the cotangent bundle of the smooth points of the image $\mathcal{P} = \Phi(B)$ of a period mapping. However we are not aware of similar applications of positivity properties of the bundles constructed from the *extension data* \mathcal{E} associated to a limiting mixed Hodge structure.⁶ As will be explained in loc. cit., there is an ample line bundle $\mathcal{L} \rightarrow \mathcal{E}_1$ where \mathcal{E}_1 is the lowest level extension data and an isomorphism

$$(I.A.4) \quad \nu^* \mathcal{L} \cong N_{Z_{I,e/\overline{B}}^*|_{F_x}}$$

where

$$(I.A.5) \quad \nu : F_x \rightarrow \mathcal{E}$$

associates to each point of F_x the extension data associated to the LMHS at that point. Put another way, assuming Φ_* is generically injective, since the associated graded to the limiting mixed Hodge structure is constant along F_x one may expect non-trivial variation in the extension data to the LMHS's, i.e. in ν in (I.A.5) (or possibly some jet of ν) should be non-constant. The map (I.A.4) then converts sections of the ample bundle $\mathcal{L} \rightarrow \mathcal{E}_1$ into sections of $N_{F_x/\overline{B}}^*$.⁷

It may happen that the lowest level extension data is constant, e.g., if the LMHS is of Hodge-Tate type. In this case the higher level extension data enters, and it is of a different character. Namely, it gives rise to the exponentials of iterated integrals of quantities constructed from the “coordinates” in the extension data.⁸

In summary, in addition to the classical uses of positivity of the Hodge line bundle, arising from the SBB completion of the images of period mappings are a new type of positive line bundles and related objects that lead to additional Hodge theoretic invariants arising from families of algebraic varieties.

Returning to the discussion of (I.A.2), a central ingredient in its proof consists of the positivity properties of the Hodge line bundle Λ_e . Referring to [GGLR17] and to Sections IV.B, V, and VI.A below for details, the Chern form ω_e of Λ_e is a singular differential form on a desingularization \overline{B} of $\overline{\mathcal{M}}$, and the exterior differential system

$$(I.A.6) \quad \omega_e = 0$$

defines a complex analytic fibration whose quotient captures the polarized Hodge structure on $H^2(X)$ when X is smooth (or has canonical singularities), and when $X = \lim X_t$ is a specialization of smooth X_t 's it captures the associated graded to the limiting mixed Hodge structure.⁹ As discussed above it is the extension data in

⁶Interestingly the constructions require having a *limiting* mixed Hodge structure, not just a mixed Hodge structure.

⁷In the classical case of curves this extension data is given by the Jacobian variety of a smooth, but generally reducible, curve and $\mathcal{L} \rightarrow \mathcal{E}$ is the “theta” line bundle given by the polarization. It is interesting and we think noteworthy that it appears that constructions akin to classical theta functions can be given in non-classical situations.

⁸The iterated integrals occur only in the non-classical case.

⁹Of course one must make sense of (I.A.6) where ω_e has singularities; this is addressed in Section VI.E below.

the LMHS that is not detected by the extended period mapping. In this way the Satake-Baily-Borel completion is a minimal completion of the image of the period mapping.

The proof of Theorem (I.A.1)* uses Lie theory, differential geometry and complex analysis. If completed it would have the following purely algebro-geometric consequence for which we know of no purely algebraic argument. Given

$$\begin{array}{ccc} \mathcal{X} & \subset & \overline{\mathcal{X}} \\ \downarrow & & \downarrow \bar{f} \\ B & \subset & \overline{B} \end{array}$$

where $\bar{f} : \overline{\mathcal{X}} \rightarrow \overline{B}$ is a family of general type algebraic surfaces with smooth fibres over the smooth Zariski open set B of the complete variety \overline{B} and having semi-log-canonical singularities over $\overline{B} \setminus B$ and where we assume that $\omega_{\overline{\mathcal{X}}/\overline{B}}$ is Cartier, we have the

THEOREM I.A.7: $\det(f_*\omega_{\overline{\mathcal{X}}/\overline{B}})$ is free.¹⁰

It is relatively easy to show that $\det(f_*\omega_{\overline{\mathcal{X}}/\overline{B}})$ is nef, and it is also not difficult to give conditions when it is big. The issue of showing it is free seems to be more subtle.

The positivity of the Hodge line bundle raises naturally the issue of the degree of positivity of the Hodge vector bundle. It is this question that provides some of the background motivation for this paper. A vector bundle $E \rightarrow X$ over a complex manifold will be said to be *semi-positive* if it has a Hermitian metric whose curvature form

$$(I.A.8) \quad \Theta_E(e, \xi) \geq 0$$

(cf. Sections II.A and II.B for notations). We shall usually write this condition as $\Theta_E \geq 0$. The Hodge vector bundle is semi-positive, but the curvature form is generally not positive, even at a general $x \in X$ and general $e \in E_x$. An interesting algebro-geometric question is: *How positive is it?*

We shall say that a general bundle $E \rightarrow X$ is *strongly semi-positive* if there is a metric where (I.A.8) is satisfied and where

$$(I.A.9) \quad \text{Tr } \Theta_E = \Theta_{\det E} > 0$$

on an open set. For example, the Hodge vector bundle is strongly semi-positive if the end piece $\Phi_{*,n}$ of the differential of the period mapping is injective at a general point, a condition that is frequently satisfied in practice.¹¹ One of the main observations in

¹⁰This result is proved when $\dim B = 2$ (cf. Section VI.A below); we are not aware of an algebraic proof of this theorem even in this case.

¹¹For families of algebraic curves and surfaces this condition is the same as the differential being injective. In general, using the augmented Hodge bundle there is a similar interpretation using the full differential Φ_* .

this paper is

(I.A.10) *If X is compact and $E \rightarrow X$ is strongly semi-positive, then for some $r_0 \leq \text{rank } E$ we have for $m \geq r_0$*

$$\Theta_{\text{Sym}^m E} > 0$$

on an open set.

As a corollary,

$\text{Sym}^m E$ is big for $m \geq r_0$.

As the proof of (I.A.10) will show, although $\text{Sym}^m E$ “gets more sections” as m increases, in general $\text{Sym}^m E$ will never become entirely positive. We will see in Section VI.E that the $\text{Sym}^m E$ have an intrinsic amount of “flatness” no matter what metric we use for E .¹²

Among the algebro-geometric measures of positivity of a vector bundle $E \rightarrow X$ one may single out

- (i) E is nef (see Section II.D);
- (ii) some $\text{Sym}^m E$ is big;
- (iii) some $\text{Sym}^m E$ is free (i.e., $\text{Sym}^m E$ is semi-ample).

There are natural curvature conditions that imply both (i) and (ii), but other than the assumption of strict positivity which implies ampleness, we are not aware of natural curvature conditions that imply (iii). For a specific question concerning this point, suppose that $L \rightarrow X$ is a line bundle and $Z \subset X$ is a reduced normal crossing divisor. Assume that h is a smooth metric in $L \rightarrow X$ whose Chern form has the properties

- (a) $\omega \geq 0$;
- (b) for $\xi \in T_x X$, $\omega(\xi) = 0 \iff x \in Z$ and $\xi \in T_x Z \subset T_x X$.

Briefly, $\omega \geq 0$ and the exterior differential system $\omega = 0$ defines $Z \subset X$. Then by (a) we see that L is nef, and as a consequence of (b) L is big (cf. II.G). Now ω defines a Kähler metric ω^* on $X^* := X \setminus Z$, and one may pose the

(I.A.11) **Question:** Are there conditions on the curvature R_{ω^*} of ω^* that imply that L is free?

We note that by (b) the Chern form ω defines a norm in the normal bundle $N_{Z/X}$. It seems reasonable to ask if the curvature form of this norm may be computed from $\lim_{x \rightarrow Z} R_{\omega^*}(x)$, and if there are sign properties of this curvature form that imply freeness?¹³

Remark that the construction of the completion $\overline{\mathcal{P}}$ of the image of a period mapping (I.A.2) could have been accomplished directly if one could show that the canonically extended Hodge line bundle $\Lambda_e \rightarrow \overline{B}$, which satisfies (i) and (ii), is free.¹⁴ Here one

¹²This is not surprising as one can always add a trivial bundle to one that is strongly semi-positive and it will remain strongly semi-positive. Somewhat more subtly, a strongly semi-positive bundle may have “twisted” sub-bundles on which the curvature form vanishes.

¹³In a special case this question has been treated in the interesting paper [Mok12]. We refer to (I.A.39) at the end of this introduction for similar questions concerning the Hodge vector bundles.

¹⁴Of course the semi-ampleness of $\Lambda_e \rightarrow \overline{B}$ is a posteriori a consequence of (I.A.2).

has the situation of the above question in the case where h and ω have singularities along Z . However, as discussed in Section V these singularities are mild; in fact the singularities tend to “increase” the positivity of the Chern form of Λ_e .

In Section III we introduce the notation of *norm positivity*.¹⁵ This means that $E \rightarrow X$ has a metric whose curvature matrix is of the form

$$\Theta_E = -{}^t\bar{A} \wedge A$$

where A is a matrix of $(1, 0)$ forms arising from a holomorphic bundle map

$$(I.A.12) \quad A : E \otimes TX \rightarrow G$$

for G a holomorphic vector bundle having a Hermitian metric. The curvature form is then

$$(I.A.13) \quad \Theta_E(e, \xi) = \|A(e \otimes \xi)\|_G^2$$

where $\|\cdot\|_G$ is the norm in G . Many of the bundles that arise naturally in algebraic geometry, such as the Hodge bundle and any globally generated bundle, have this property: The mapping A in (I.A.12) generally has algebro-geometric meaning, e.g., as the differential of a map. Such bundles are semi-positive,¹⁶ and their degree of positivity will have an algebro-geometric interpretation.

Remark: One may speculate as to the reasons for what might be called “the unreasonable effectiveness of curvature in algebraic geometry.” After all, algebraic geometry is in some sense basically a 1st order subject—the Zariski tangent space being the primary infinitesimal invariant—whereas differential geometry is a 2nd order subject—the principle invariants are the 2nd fundamental form and the curvature. One explanation is that for bundles that have the norm positivity property, the curvature form measures the *size* of the 1st order quantity A in (I.A.12).¹⁷

Another principal topic in this paper concerns the singularities of metrics and their curvatures and Chern forms. This also is a very active area that continues to play a prominent role in complex algebraic geometry (cf. [Dem12a] and [Pă16]). Reflecting the fact that in families of algebraic varieties there are generally interesting singular members, the role of the singularities of Hodge metrics is central in Hodge theory. Here there are two basic principles:

- (i) the singularities are mild, and
- (ii) singularities increase positivity.

The first is explained in Section IV.B. Intuitively it means that although the Chern polynomials are singular differential forms, they behave in many essential aspects as if they are smooth. In particular, although they have distribution coefficients they may be multiplied and restricted to particular subvarieties as if they were smooth forms. This last property is central to the arguments involved in of (I.A.1)* above.

¹⁵The origin of the term is that a quantity (such as a Hermitian form) will be non-negative if it is expressed as a norm.

¹⁶Both in the sense of (I.A.8) and in the sense of Nakano positivity ([Dem12a]).

¹⁷In this regard in the Hodge-theoretic situation we note that curvature R_{ω^*} in (I.A.11) is a 2nd, rather than a 3rd, order invariant.

The first major steps in the general analysis of the singularities of the Chern forms of Hodge bundles for several parameter variations of Hodge structure were taken by Cattani-Kaplan-Schmid ([CKS86]), with subsequent refinements and amplifications by a number of people including Kollár ([Kol87]). In the paper [GGLR17] further steps are taken, ones that may be thought of as further refining the properties of the wave front sets of the Chern forms.¹⁸ This will be explained in Section IV.B and will be applied in Section V where an alternate proof of one of the two main ingredients in the proof of (I.A.15)* below will be given.

The result (I.A.1)* is an application to a desingularization of a KSBA moduli space of a result that we now explain, referring to Sections I.B and I.C for explanations of notation and terminology. Let B be a smooth projective variety with smooth completion \overline{B} such that $\overline{B} \setminus B = Z = \bigcup Z_i$ is a reduced normal crossing divisor. We denote by $Z_I = \bigcap_{i \in I} Z_i$ the strata of Z and by $Z_I^* = Z_{I,\text{reg}}$ the smooth points of Z_I . We consider a variation of Hodge structure over B given by a period mapping

$$(I.A.14) \quad \Phi : B \rightarrow \Gamma \setminus D.$$

We assume that the local monodromies T_i around the Z_i are unipotent with logarithms N_i , an assumption that may always be achieved by passing to a finite covering of B .¹⁹ We also suppose that the end piece

$$\Phi_{*,n} : TB \rightarrow \text{Hom}(F^n, F^{n-1}/F^n)$$

of the differential of Φ is generically injective.²⁰ Finally assuming as we may that all $N_i \neq 0$, the image

$$\Phi(B) := \mathcal{P} \subset \Gamma \setminus D$$

of the period mapping (I.A.14) is a closed analytic subvariety of $\Gamma \setminus D$. Beginning with [Som78] there have been results stating that under certain conditions \mathcal{P} is a quasi-projective variety and the Hodge line bundle $\Lambda \rightarrow \mathcal{P}$ is at least big.²¹ The following result from [GGLR17] serves to extend and clarify the previous work in the literature:

THEOREM I.A.15*: *There exists a canonical completion $\overline{\mathcal{P}}$ of \mathcal{P} as a compact complex analytic space that has the properties*

- (i) *the Hodge bundle extends to $\Lambda_e \rightarrow \overline{\mathcal{P}}$ and there it is ample; and*
- (ii) *$\overline{\mathcal{P}}$ is a Satake-Baily-Borel completion of \mathcal{P} .*

¹⁸The use of micro-local analysis in the theory of \mathcal{D} -modules is well known. Here the subtlety from [CKS86] is that the wave front sets are whole half-spaces (or quadrants in the case of a normal crossing divisor), and the sectorial analysis as one approaches the axes is the central issue.

¹⁹We will use [PS08], [CMSP17] and [CKS86] as general references for Hodge theory, including limiting mixed Hodge structures.

²⁰By using the *augmented Hodge line bundle* $\bigotimes_{p=0}^{\lfloor n-1/2 \rfloor} \det(F^{n-p})$ rather than just the Hodge line bundle $\Lambda = \det F^n$, this assumption may be replaced by the injectivity of Φ_* (cf. [GGLR17]).

²¹Cf. [BBT18] and [BKT18].

The second statement means the following: The period mapping (I.A.14) extends to

$$(I.A.16) \quad \Phi_e : \overline{B} \rightarrow \overline{\mathcal{P}}.$$

Along the non-singular strata Z_I^* the extended period mapping Φ_e induces variations of graded polarized limiting mixed Hodge structures, and passing to the associated graded of these mixed Hodge structures gives period mappings

$$(I.A.17) \quad \Phi_I : Z_I^* \rightarrow \Gamma_I \backslash D_I.$$

Then *the restriction to Z_I^* of Φ_e in (I.A.16) may be identified with Φ_I* . Setting $\mathcal{P}_I = \Phi_I(Z_I^*) \subset \Gamma_I \backslash D_I$, as a set $\overline{\mathcal{P}} = \mathcal{P} \amalg (\coprod_I \mathcal{P}_I)$. The precise meaning of this is explained in [GGLR17]; among other things it means that $\text{Proj}(\Lambda_e \rightarrow \overline{B})$ exists and on the boundary strata exactly detects the variation of the associated graded to the limiting mixed Hodge structures.²² The exterior differential system (I.A.6) defined by the singular differential form ω_e may be made sense of on \overline{B} (cf. Section VI.E), and there it defines a fibration by complex analytic subvarieties whose quotient is just $\overline{\mathcal{P}}$. The restriction property of ω_e referred to above may be summarized as saying that

$$(I.A.18) \quad \omega_e|_{Z_I^*} \text{ is defined and is equal to } \omega_I$$

where ω_I is the Chern form of the Hodge line bundle associated to (I.A.17).

An implication of the above is

$$(I.A.19) \quad \omega_e \text{ is defined on } \overline{\mathcal{P}} \text{ and there } \omega_e > 0.^{23}$$

One aspect of this is that given B and \overline{B} as above the EDS

$$(I.A.20) \quad \begin{cases} \omega = 0 & \text{on } B \\ \omega_I = 0 & \text{on the } Z_I^* \end{cases}$$

²²We remark that in the non-classical case when $\Gamma \backslash D$ is not an algebraic variety, the construction of $\overline{\mathcal{P}}$ is necessarily accomplished by gluing together local extensions of \mathcal{P} at the points of $\partial\mathcal{P} = \overline{\mathcal{P}} \setminus \mathcal{P}$. This requires both a *local* analysis, based on [CKS86], of local neighborhoods in $\overline{\mathcal{P}}$ along the fibres F of the set-theoretically extended period map together with *global* analysis of those fibres, and especially of the above mentioned positivity of the co-normal bundle $N_{F/\overline{B}}^*$. As mentioned above, global issues necessitate apparently new Hodge-theoretic constructions arising from the extension data in a limiting mixed Hodge structure. Even in the classical case this seems to give a new perspective on the Satake-Baily-Borel compactification.

²³In general, given a fibration $f : A \rightarrow B$ between manifolds A, B and a smooth differential form Ψ_A on A , the necessary and sufficient conditions that

$$\Psi_A = f^* \Psi_B$$

for a smooth differential form Ψ_B on B are that both Ψ_A and its exterior derivative $d\Psi_A$ restrict to zero on

$$\ker\{f_* : TA \rightarrow TB\} \subset TA.$$

The above results extend this to a situation where Ψ_A is a possibly singular differential form.

on the smooth strata defines on each a complex analytic fibration and a natural question is

(I.A.21) *For $I \subset J$ so that we have $Z_J^* \subset \overline{Z_I^*}$, does the closure of a fibre of $\omega_I = 0$ in Z_I^* intersect Z_J^* in a fibre of $\omega_J = 0$? Are the limits of the fibres of Φ_I contained in the fibres of Φ_J ?*

In other words, *do the period mappings given by Φ on B and Φ_I on Z_I^* fit together in an analytic way?* That this is the case is proved in Section 3 of [GGLR17] (cf. Step 1 in the proof of Theorem 3.15 there). Although not directly related to the positivity of Hodge bundles, because of the subtle way in which the relative weight filtration property of a several parameter degeneration of polarized Hodge structures enters in the argument we think the result is of interest in its own right and in Section VI.A we have given a different proof from that in [GGLR17] of the main special case of a positive answer to (I.A.21).

In Section VI we shall give applications of some of the bundles that naturally arise in Hodge theory. As explained above, one such application would be the use of the Hodge line bundle in proving that its “Proj” defines the Satake-Baily-Borel completion of the image of a period mapping. The next application is to the cotangent bundle $\Omega_{\mathcal{P}^0}^1$ over the smooth points \mathcal{P}^o of \mathcal{P} . It is classical [CMSP17] that the induced metric on $\mathcal{P}^0 \subset \Gamma \backslash D$ is Kähler and has holomorphic sectional curvatures

$$R(\xi) \leq -C, \quad \xi \in T\mathcal{P}^0$$

bounded above by a constant $-C$ where $C > 0$. On the other hand the more basic curvature form for $T\mathcal{P}^0$ is given by the holomorphic bi-sectional curvatures $R(\eta, \xi)$, and we shall prove (cf. Theorem VI.B.1) that they satisfy

(I.A.22) $R(\eta, \xi) \leq 0$ and $R(\eta, \xi) < 0$ is an open \mathcal{U} set in $T\mathcal{P}^0 \times_{\mathcal{P}^0} T\mathcal{P}^0$.
Moreover, \mathcal{U} projects onto each factor in $\mathcal{P}^0 \times \mathcal{P}^0$.

An interesting point here is that the induced curvature on the horizontal sub-bundle $(TD)_h \subset TD$ is a difference of non-negative terms, each of which has the norm positivity property (I.A.13). On *integrable* subspaces I of $(TD)_h$ the positive term drops out leading to (I.A.22).²⁴

A consequence of (I.A.22) is

(I.A.23) *the curvature form $\Theta_{T^*\mathcal{P}^0} \geq 0$ and it is positive on an open set.*

To apply (I.A.23) two further steps are required. One is that curvatures decrease on holomorphic sub-bundles, which in the case at hand is used in the form

$$(I.A.24) \quad \Theta_I \leq \Theta_{(TD)_h} \Big|_I.$$

²⁴In general the curvature matrices of Hodge bundles are differences of non-negative curvature operators, and the paper [Zuo00] isolated the important point that on the kernels of Kodaira-Spencer maps one of the two terms drops out and on these subspaces the curvature forms have a sign. Here the relevant bundles are constructed from the $\text{Hom}(F^p/F^{p+1}, F^{p-1}/F^p)$ bundles, and the relevant Kodaira-Spencer maps vanish in the integrable subspaces (cf. Proposition 2.1 in loc. cit.). This issue is discussed below in some detail in Section VI.B.

The second involves the singularities that arise both at the singular point $\mathcal{P}_{\text{sing}} = \mathcal{P} \setminus \mathcal{P}^0$, and at the boundary $\partial\mathcal{P} = \overline{\mathcal{P}} \setminus \mathcal{P}$ of \mathcal{P} . Once these are dealt with we find the following, which are variants and very modest improvements of results in the literature due to Zuo in [Zuo00] and others (cf. [Bru16b] for a recent paper on this general topic and [Den18] for related results and further bibliography).

(I.A.25) \mathcal{P} is of log general type,

and there exists an m_0 such that

(I.A.26) $\text{Sym}^m \Omega_{\overline{\mathcal{P}}}^1(\log)$ is big for $m \geq m_0$.

The precise meaning of these statements will be explained in Section VI.B below.²⁵

The theme of this paper is the positivity of vector bundles, especially those arising from Hodge theory, and some applications of this positivity to algebraic geometry. Above we have discussed applications of the positivity of the Hodge *line* bundle and of the cotangent bundle at the smooth points to the image of a period mapping. We next turn to an application of positivity of the Hodge *vector* bundle to the *Iitaka conjecture*, of which a special case is

(I.A.27) Let $f : X \rightarrow Y$ be a morphism between two smooth projective varieties and assume that

- (i) the general fibre $X_y = f^{-1}(y)$ has Kodaira dimension $\kappa(X_y) = \dim X_y$,
- (ii) $\text{Var } f = \dim Y$.²⁶

Then the Kodaira dimension is sub-additive

(I.A.28) $\kappa(X) \geq \kappa(Y) + \kappa(X_y)$.

This result was proved with one assumption, later seen to be unnecessary, by Viehweg ([Vie83a], [Vie83b]). His work built on [Fuj78], [Uen74], [Uen78], [Kaw82], [Kaw83], and [Kaw85], and was extended by [Kol87]. The sub-additivity of the Kodaira dimension plays a central role in the classification of algebraic varieties, and over the years refinements of the result and alternative approaches to its proof have stimulated a very active and interesting literature (cf. [Kaw02] and the more recent papers [Sch15], [Pă16]). It is not our purpose here to survey this work but rather it is to focus on some of the Hodge theoretic aspects of the result.

One of these aspects is that the original proofs showed the necessity for understanding the singularities of the Chern form ω of the Hodge line bundle. Here [CKS86] is fundamental; as noted there by those authors their work was in part motivated by establishing what was required for the proof of the Iitaka conjecture.

In this particular discussion we shall mainly concentrate on the Hodge-theoretic aspect of the problem where everything is smooth. The issues that arise from the

²⁵In fact, as will be discussed there \mathcal{P} is of *stratified-log-general* type, a notion that is a refinement of log-general type.

²⁶This means that at a general point the Kodaira-Spencer map $\rho_y : T_y Y \rightarrow H^1(TX_y)$ is injective.

singularities may be dealt with using Theorem IV.B.8. In Section VI.B following general remarks on what is needed to establish the result we first observe that if generic local Torelli²⁷ holds for $f : X \rightarrow Y$, then (I.A.28) is a direct consequence of Theorem III.B.7 applied to the Hodge vector bundle.

However, in general the assumption $\kappa(X_y) = \dim X_y$ pertains to the $H^0(K_{X_y}^m)$ for $m \gg 0$, not to $H^0(K_{X_y})$ itself. So the question becomes

How can Hodge theory be used to study the pluricanonical series
 $H^0(K_{X_y}^m)$?²⁸

To address this an idea in [Kaw82] was expanded and used in [Vie83b] with further refinement and amplification in [Kol87]. It is this aspect that we shall briefly discuss here with further details given in Section VI.C.

First a general comment. Given a smooth variety W of dimension n and an ample line bundle $L \rightarrow W$, there are two variations of Hodge structure associated to the linear systems $|mL|$ for $m \gg 0$:

- (a) the smooth divisors $D \in |mL|$ carry polarized Hodge structures;
- (b) for each $s \in H^0(L^m)$ with smooth divisor $(s) = D \in |mL|$ there is a cyclic covering $\widetilde{W}_s \xrightarrow{\pi} W$ branched over D and with a distinguished section $\tilde{s} \in H^0(\widetilde{W}_s, \pi^*L)$ where $\tilde{s}^m = \pi^*s$.

We may informally think of (b) as the correspondence

$$(I.A.29) \quad (W, s^{1/m}) \longleftrightarrow (\widetilde{W}_s, \tilde{s})$$

obtained by extracting an m^{th} root of s .

We note that for each $\lambda \in \mathbb{C}^*$ there is an isomorphism

$$(I.A.30) \quad \widetilde{W}_s \xrightarrow{\sim} \widetilde{W}_{\lambda s},$$

so that we should think of (b) as giving the smooth fibres in a family

$$\{\widetilde{W}_{[s]} : [s] \in \mathbb{P}H^0(L^m)\}.$$

A variant of the construction (b) is then given by

$$(c) \quad \begin{array}{c} \widetilde{W} = \text{desingularization of } \bigcup_{[s] \in \mathbb{P}H^0(L^m)} \widetilde{W}_{[s]} \\ \downarrow \\ \mathbb{P}H^0(L^m). \end{array}$$

It is classical that a suitable form of local Torelli for the end piece of the differential of the period mapping holds for the families of each of the types (a) and (c) when

²⁷This means that the end piece $\Phi_{*,n}$ of the differential of the period mapping is generically injective.

²⁸Here, as will be explained below, we are referring to both the traditional use of the positivity of the Hodge line bundle, and also to the semi-positivity of the Hodge vector bundle which may be used in place of the concept of weak positivity introduced by Viehweg [Vie83a] and [Vie83b].

$m \gg 0$ (cf. [CMSP17]). Moreover, the methods used to prove these results may be readily extended to show that

(I.A.31) *local Torelli holds for both families of types (a) and (c) when W also varies.*

In the case $L = K_W$ there is the special feature

(I.A.32) *$H^0(K_W^m)$ is related to the $H^{n,0}$ -term of the polarized Hodge structure on $H^n(\widetilde{W})$ arising in the construction (c).*

This means that, setting $\mathbb{P} = \mathbb{P}H^0(K_W^m)$

$H^0(K_W^m)$ is a summand of $F \otimes \mathcal{O}_{\mathbb{P}}(-1)$ where $F \rightarrow \mathbb{P}$ is the Hodge vector bundle associated to the family.

As a consequence

(I.A.33) *$H^0(K_W^m) \otimes H^0(\mathcal{O}_{\mathbb{P}}(1))$ is a summand of $H^0(F)$.*

There is also a metric Hodge theoretic interpretation of the pluricanonical series. For $\psi \in H^0(K_W^m)$ the Narashimhan-Simka [NS68] Finsler-type norm

$$(I.A.34) \quad \|\psi\| = \int_W (\psi \wedge \bar{\psi})^{1/m}$$

has the Hodge-theoretic interpretation

$$(I.A.35) \quad \|\psi\| = \int_{\widetilde{W}_\psi} \tilde{\psi} \wedge \bar{\tilde{\psi}}$$

where the RHS is a constant times the square of the Hodge length of $\tilde{\psi} \in H^0(K_{\widetilde{W}_\psi})$. The curvature properties of $\|\psi\|$ arising from its Hodge-theoretic interpretation were used by Kawamata [Kaw82] in his proof of the Iitaka conjecture when Y is a curve. The metric properties of direct images of the pluricanonical systems have recently been an active and highly interesting subject; cf. [Pă16] for a summary of some of this work and references to the literature (cf. also [Den18]). It is possible that growing out of this work the Kawamata argument could be extended to give a proof of the full Iitaka conjecture. We will comment further on this at the end of Section VI.C where a precise conjecture (VI.C.30) will be formulated.

Returning to the discussion of (I.A.28), the essential point is to show that

(I.A.36) *$f_*\omega_{X/Y}^m$ has lots of sections for $m \gg 0$.*

The idea is that “positivity \implies sections” and “assumptions (i) and (ii) in (I.A.27) $\implies f_*\omega_{X/Y}^m$ has positivity.” As noted above, when $m = 1$ and local Torelli holds using Theorem III.B.7 there is sufficient positivity to achieve (I.A.36). The issue then becomes how to use the Hodge-theoretic interpretation of the pluricanonical series and semi-positivity of Hodge vector bundles to also produce sections of $f_*\omega_{X/Y}^m$.

If we globalize the cyclic covering construction taking W to be a typical X_y we arrive at a commutative diagram that is essentially

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ g \downarrow & \searrow \tilde{f} & \downarrow f \\ \mathbb{P} & \xrightarrow{p} & Y \end{array}$$

where the fibres are given by

- $\mathbb{P}_y = p^{-1}(y) = \mathbb{P}H^0(\omega_{X_y}^m)$,
- $\tilde{X}_y = \tilde{f}^{-1}(y)$, where
- $\pi : \tilde{X}_y \rightarrow X_y$ is the family of cyclic coverings $\tilde{X}_{y, [\psi]} \rightarrow X_y$ as $[\psi] \in \mathbb{P}H^0(\omega_{X_y}^m)$ varies.²⁹

If it were possible to say that from (I.A.32)

$$(I.A.37) \quad f_*\omega_X^m \text{ is a direct factor of } \tilde{f}_*\omega_{\tilde{X}/Y}$$

then local Torelli-type statements would imply positivity of $f_*\omega_{X/Y}^m$. However, due to the twisting that occurs as λ varies in the scaling identification (I.A.30) essentially what happens is that

$$(I.A.38) \quad f_*\omega_{X/Y}^m \text{ is a direct factor of } \tilde{f}_*\omega_{\tilde{X}/\mathbb{P}} \otimes p_*\mathcal{O}_{\mathbb{P}}(1).$$

Now $\mathcal{O}_{\mathbb{P}}(1)$ is positive along the fibres of $\mathbb{P} \rightarrow Y$, but the hoped for positivity of $f_*\omega_{X/Y}^m$ means that $\mathcal{O}_{\mathbb{P}}(1)$ tends to be negative in the directions normal to \mathbb{P}_y in \mathbb{P} . Thus the issue has an additional subtlety that, as will be explained in Section VI.C, necessitates bringing in properties of the maps

$$\mathrm{Sym}^k \tilde{f}_*\omega_{\tilde{X}/\mathbb{P}}^m \rightarrow \tilde{f}_*\omega_{\tilde{X}/\mathbb{P}}^{km}.$$

A final question that arises is this:

(I.A.39) *Let $f : X \rightarrow Y$ be a map between smooth, projective varieties, and assume that the locus of $y \in Y$ where $X_y = f^{-1}(y)$ is singular is a reduced normal crossing divisor. Then*

- (i) *if $\det f_*\omega_{X/Y}^m$ is non-zero, is it free?*
- (ii) *if $f_*\omega_{X/Y}^m$ is non-zero, is it free?*

We note that (i) for $m = 1$ follows from (I.A.1)*. So far as we know, (ii) is not known for $m = 1$.

Turning to Section VI.D, from the description of the Satake-Baily-Borel completion $\overline{\mathcal{P}}$ of the image \mathcal{P} of a period mapping, we may say *that the extended Hodge line bundle does not detect extension data in limiting mixed Hodge structures*. One may ask whether the same is or is not the case for the extended Hodge vector bundle. In Section VI.C we give examples to show that in fact this vector bundle may detect both discrete and continuous extension data in limiting mixed Hodge structures.

²⁹See (VI.C.19) for the complete definition and explanation of this diagram.

As noted above the Hodge line bundle lives on the canonical completion $\overline{\mathcal{M}}$ of the KSBA moduli space for surfaces of general type. Interestingly, as will be seen by example in a sequel to [GGLR17] the Hodge vector bundle does *not* live on $\overline{\mathcal{M}}$. This is in contrast to the case of curves where the Hodge vector bundle is defined on the moduli space $\overline{\mathcal{M}}_g$.

In the final section VI.E we shall revisit the exterior differential system (I.A.6) defined by a Chern form, this time for the Chern form ω_E of the line bundle $\mathcal{O}_{\mathbb{P}E}(1) \rightarrow \mathbb{P}E$. For a bundle $E \rightarrow X$ with $\Theta_E \geq 0$ so that $\omega_E \geq 0$ on $\mathbb{P}E$, the failure of strict positivity, or the degree of flatness, of the bundle E is reflected by the foliation given by the integral varieties of the exterior differential system

$$\omega_E = 0.$$

The result here is Proposition (VI.E.2), and it suggests a conjecture giving conditions under which equality might hold in the inequality

$$\kappa(E) \leq n(E)$$

between the Kodaira-Iitaka dimension $\kappa(E)$ and numerical dimension $n(E)$ of the bundle.

I.B. General notations.

- X, Y, W, \dots will be compact, connected complex manifolds.

In practice they will be smooth, projective varieties.

- $E \rightarrow X$ is a holomorphic vector bundle with fibres E_x , $x \in X$ and rank $r = \dim E_x$;
- $A^{p,q}(X, E)$ denotes the global smooth E -valued (p, q) forms;
- we will not distinguish between a bundle and its sheaf of holomorphic sections; the context should make the meaning clear;
- $L \rightarrow X$ will be a line bundle.

Associated to $L \rightarrow X$ are the standard notions

- (i) $\varphi_L : X \dashrightarrow \mathbb{P}H^0(X, L)^*$ is the rational mapping given for $x \in X$ by

$$(I.B.1) \quad \varphi_L(x) = [s_0(x), \dots, s_N(x)]$$

where s_0, \dots, s_N is a basis for $H^0(X, L)$; in terms of a local holomorphic trivialization of $L \rightarrow X$ the $s_i(x)$ are given by holomorphic functions which are used to give the homogeneous coordinates on the right-hand side of (I.B.1);

- (ii) the line bundle $L \rightarrow X$ is *big* if one of the equivalent conditions
- $h^0(X, L^m) = Cm^d + \dots$ where $C > 0$, $\dim X = d$ and \dots are lower order terms;
 - $\dim \varphi_L(X) = \dim X$
- is satisfied;

- (iii) $L \rightarrow X$ is *free*³⁰ if one of the equivalent conditions

³⁰The term *semi-ample* is also used.

- for some $m > 0$, the evaluation maps

$$(I.B.2) \quad H^0(X, L^m) \rightarrow L_x^m$$

are surjective for all $x \in X$;

- $\varphi_{mL}(x)$ is a morphism; i.e., for all $x \in X$ some $s_i(x) \neq 0$;
- the linear system $|mL| := \mathbb{P}H^0(X, L^m)^*$ is base point free for $m \gg 0$

is satisfied.

If the map (I.B.2) is only surjective for a general $x \in X$, we say that $L^m \rightarrow X$ is *generically globally generated*.

(iv) $L \rightarrow X$ is *nef* if $\deg(L|_C) \geq 0$ for all curves $C \subset X$; here $L|_C = L \otimes_{\mathcal{O}_X} \mathcal{O}_C$ is the restriction of L to C ;

(v) we will say that $L \rightarrow X$ is *strictly nef* if $\deg(L|_C) > 0$ for all curves $C \subset X$;

- for a vector bundle $E \rightarrow X$, we denote the k^{th} symmetric product by

$$S^k E := \text{Sym}^k E;$$

- $\mathbb{P}E \xrightarrow{\pi} X$ is the projective bundle of 1-dimensional quotients of the fibres of $E \rightarrow X$; thus for $x \in X$

$$(\mathbb{P}E)_x = \mathbb{P}E_x^*;$$

- $\mathcal{O}_{\mathbb{P}E}(1) \rightarrow \mathbb{P}E$ is the tautological line bundle; then

$$\pi_* \mathcal{O}_{\mathbb{P}E}(m) = S^m E$$

gives

$$H^0(X, S^m E) \cong H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(m))$$

for all m .³¹

I.C. Notations from Hodge theory. We shall follow the generally standard notations and conventions as given in [CMSP17] and are used in [GGLR17]. Further details concerning the structure of limiting mixed Hodge structures (LMHS) will be given in Section IV.

- B will denote a smooth quasi-projective variety;
- a *variation of Hodge structure* (VHS) parametrized by B will be given by the equivalent data

(a) a *period mapping*

$$\Phi : B \rightarrow \Gamma \backslash D$$

where D is the period domain of weight n polarized Hodge structures (V, Q, F^\bullet) with fixed Hodge numbers $h^{p,q}$, and where the infinitesimal period relation (IPR)

$$\Phi_* : TB \rightarrow I \subset T(\Gamma \backslash D)$$

is satisfied;

³¹The higher direct images $R_\pi^q \mathcal{O}_{\mathbb{P}E}(m) = 0$ for $q > 0$, $m \geq -r$; we shall not make use of this in this paper.

(b) $(\mathbb{V}, F^\bullet, \nabla; B)$ where $\mathbb{V} \rightarrow B$ is a local system with Gauss-Manin connection

$$\nabla : \mathcal{O}_B(\mathbb{V}) \rightarrow \Omega_B^1(\mathbb{V})$$

and $F^\bullet = \{F^n \subset F^{n-1} \subset \dots \subset F^0\}$ is a filtration of $\mathcal{O}(\mathbb{V})$ by holomorphic sub-bundles satisfying the IPR in the form

$$\nabla F^p \subset \Omega_B^1(F^{p-1}),$$

and where at each point $b \in B$ the data $(\mathbb{V}_b, F_b^\bullet)$ defines a polarized Hodge structure (PHS) of weight n .

In the background in both (a) and (b) is a bilinear form Q that polarizes the Hodge structures; we shall suppress the notation for it when it is not being explicitly used.

- The parameter space B will have a smooth projective completion \overline{B} with the properties

$Z := \overline{B} \setminus B$ is a reduced normal crossing divisor $Z = \bigcup Z_i$ having strata $Z_I := \bigcap_{i \in I} Z_i$ with $Z_I^* \subset Z_I$ denoting the non-singular points $Z_{I,\text{reg}}^*$ of Z_I , and where the local monodromies T_i around the irreducible branches Z_i of Z are unipotent with logarithms N_i ;

- the *Hodge vector bundle* $F \rightarrow B$ has fibres $F_b := F_b^n$;
- the *Hodge line bundle* $\Lambda := \det F = \wedge^{h^{n,0}} F$;
- the polarizing forms induce Hermitian metrics in F and Λ ;
- the differential of the period mapping is

$$\Phi_* : TB \rightarrow \bigoplus_{p \geq \lfloor \frac{n}{2} \rfloor} \text{Hom}(F^p, F^{p-1}/F^p)$$

where $F^p \rightarrow B$ denotes the Hodge filtration bundles;

- setting $F = F^n$ and $G = F^{n-1}/F^n$ the *end piece* of Φ_* is

$$\Phi_{*,n} : TB \rightarrow \text{Hom}(F, G);$$

- the Hodge filtration bundles have canonical extensions

$$F_e^p \rightarrow \overline{B};$$

using the Hodge metrics on B , the holomorphic sections of $F_e^p \rightarrow \overline{B}$ are those whose Hodge norms have logarithmic growth along Z ;

- the *geometric case* is when the VHS arises from the cohomology along the fibres in a smooth projective family

$$\mathcal{X} \xrightarrow{f} B;$$

- such a family has a completion to

$$\overline{\mathcal{X}} \xrightarrow{\bar{f}} \overline{B}$$

where this map has the Abramovich-Karu ([AK00]) form of semi-stable reduction (cf. Section 4 in [GGLR17] for more details and for the notations to be used here); then the canonical extension of the Hodge vector bundle is given by

$$F_e = \bar{f}_* \omega_{\overline{\mathcal{X}}/\overline{B}}.$$

We conclude this introduction with an observation and a question. For line bundles $L \rightarrow X$ over a smooth projective variety X of dimension d , there are three important properties:

- (i) L if nef;
- (ii) L is big;
- (iii) L if free (and therefore it is semi-ample).

Clearly (iii) \implies (i) and (ii), and (iii) $\implies |mL|$ gives a birational morphism for $m \gg 0$.

In this paper we have considered the case where $L \rightarrow X$ has a metric h that may be singular, with Chern ω that defines a $(1, 1)$ current representing $c_1(L)$. The singularities of h are generally of the following types:

- h vanishes along a proper subvariety of X ;
- h becomes infinite, either logarithmically or analytically (in the sense explained below) along a proper subvariety of X .

We note that

$$\omega \geq 0 \text{ (as a current)} \implies \text{(i),}$$

and that in a Zariski open $X^0 \subset X$ where h is a smooth metric

$$\omega^d > 0 \implies \text{(ii).}$$

The Kodaira theorem states that if $X^0 = X$, then h is ample. As discussed above in relation to the question (I.A.11), for a number of purposes, including applications to Hodge theory, it would be desirable to have conditions on ω that imply (iii).

II. MEASURES OF AND TYPES OF POSITIVITY

In this section we will

- (i) define two measures of positivity—the Kodaira-Iitaka dimension and the numerical dimension;
- (ii) define two types of positivity—metric positivity and numerical positivity.

Each of these will be done first for line bundles and then for vector bundles, and the definitions will be related via the canonical association

$$E \rightarrow X \rightsquigarrow \mathcal{O}_{\mathbb{P}E}(1) \rightarrow \mathbb{P}E$$

of the tautological line bundle $\mathcal{O}_{\mathbb{P}E}(1) \rightarrow \mathbb{P}E$ to the vector bundle $E \rightarrow X$.³²

For each of the types of positivity there will be two notions, *strict positivity* denoted > 0 and *semi-positivity* denoted ≥ 0 . For metric positivity there will be a third, denoted by $E_{\text{met}}^0 > 0$ and $L_{\text{met}}^0 > 0$ respectively for vector bundles and line bundles, and which means that we have ≥ 0 everywhere and > 0 on an open set. We also use the term *strong semi-positivity* to mean that $E_{\text{met}} \geq 0$ and $\det E_{\text{met}}^0 > 0$. Finally,

³²We will use the convention whereby $\mathbb{P}E$ is the bundle of 1-dimensional *quotients* of the fibres of E ; thus the fibre

$$\mathbb{P}E_x = \mathbb{P}(E_x^*).$$

since metric positivity will be the main concept used in this paper, in some of the later sections we will just write $E > 0$ and $E^0 > 0$ to mean $E_{\text{met}} > 0$ and $E_{\text{met}}^0 > 0$.

II.A. Kodaira-Iitaka dimension. In algebraic geometry positivity traditionally suggests “sections,” and one standard measure of the amount of sections of a line bundle $L \rightarrow X$ is given by its *Kodaira-Iitaka dimension* $\kappa(L)$. This is defined by

$$(II.A.1) \quad \kappa(L) = \max_m \dim \varphi_{mL}(X)$$

where

$$\varphi_{mL} : X \dashrightarrow \mathbb{P}H^0(mL)^*$$

is the rational mapping given by the linear system $|mL|$. If $h^0(mL) = 0$ for all m we set $\kappa(L) = -\infty$. From [Dem12a] we have

$$(II.A.2) \quad h^0(mL) \leq O(m^{\kappa(L)}) \quad \text{for } m \geq 1,$$

and $\kappa(L)$ is the smallest exponent for which this estimate holds.³³ We will sometimes write (II.A.2) as

$$h^0(mL) \sim Cm^{\kappa(L)}, \quad C > 0.$$

We note that

$$\kappa(L) = \dim X \iff L \rightarrow X \text{ is big.}$$

II.B. Metric positivity. Given a Hermitian metric h in the fibres of a holomorphic vector bundle $E \rightarrow X$ there is a canonically associated Chern connection

$$D : A^0(X, E) \rightarrow A^1(X, E)$$

characterized by the properties ([Dem12a])

$$(II.B.1) \quad \begin{cases} D'' = \bar{\partial} \\ d(s, s') = (Ds, s') + (s, Ds') \end{cases}$$

where $s, s' \in A^0(X, E)$ and (\cdot, \cdot) denotes the Hermitian inner product in E . The curvature

$$\Theta_E := D^2$$

is linear over the functions; hence it is pointwise an algebraic operator. Using (II.B.1) it is given by a *curvature operator*

$$\Theta_E \in A^{1,1}(X, \text{End } E)$$

which satisfies

$$(\Theta_E e, e') + (e, \Theta_E e') = 0$$

where $e, e' \in E_x$. Relative to a local holomorphic frame $\{s_\alpha\}$, $h = \|h_{\alpha\bar{\beta}}\|$ is a Hermitian matrix and the corresponding connection and curvature matrices are given by

$$\begin{cases} \theta = h^{-1}\partial h \\ \Theta_E = \bar{\partial}(h^{-1}\partial h) = \left\| \sum_{\alpha, \beta, i, j} \Theta_{\bar{\beta}i\bar{j}}^\alpha s_\alpha \otimes s_\beta^* \otimes dz^i \wedge d\bar{z}^j \right\| \end{cases} .$$

³³Including $\kappa(L) = -\infty$ where we set $m^{-\infty} = 0$ for $m > 0$.

For line bundles the connection and curvature matrices are respectively $\theta = \partial \log h$ and $\Theta_L = -\partial\bar{\partial} \log h$. If $h = e^{-\varphi}$, then

$$\Theta_L \geq 0 \iff (i/2)\partial\bar{\partial}\varphi \geq 0 \iff \varphi \text{ is plurisubharmonic.}$$

Definition: The *curvature form* is given for $x \in X$, $e \in E_x$ and $\xi \in T_x X$ by

$$(II.B.2) \quad \Theta_E(e, \xi) = \langle (\Theta_E(e), e), \xi \wedge \bar{\xi} \rangle.$$

When written out in terms of the curvature matrix $\Theta_E(e, \xi)$ is the bi-quadratic form

$$\sum_{\alpha, \beta, i, j} \Theta_{\beta\bar{i}j}^{\alpha} e_{\alpha} \bar{e}_{\beta} \xi_i \bar{\xi}_{\bar{j}}.$$

The bundle $E \rightarrow X$ is *positive*,³⁴ written $E_{\text{met}} > 0$, if there exists a metric such that $\Theta_E(e, \xi) > 0$ for all non-zero e, ξ . For simplicity we will write $\Theta_E > 0$. If we have just $\Theta_E(e, \xi) \geq 0$, then we shall say that $E \rightarrow X$ is *semi-positive* and write $E_{\text{met}} \geq 0$. It is *strongly semi-positive* if $E_{\text{met}} \geq 0$ and $(\det E)_{\text{met}} > 0$ on an open set.

The bundle is *Nakano positive* if there exists a metric such that for all non-zero $\psi \in E_x \otimes T_x X$ we have

$$(II.B.3) \quad (\Theta_E(\psi), \psi) > 0.$$

The difference between positivity and Nakano positivity is that the former involves only the decomposable tensors in $E \otimes TX$ whereas the latter involves all tensors. In [Dem12a] there is the concept of m -positivity that involves the curvature acting on tensors of rank m and which interpolates between the two notions defined above.

Positivity and semi-positivity have functoriality properties ([Dem12a]). For our purposes the two most important are

(II.B.4) the tensor product of positive bundles is positive, and similarly for semi-positive;

(II.B.5) the quotient of a positive bundle is positive, and similarly for semi-positive.

The second follows from an important formula that we now recall (cf. [Dem12b]). If we have an exact sequence of holomorphic vector bundles

$$(II.B.6) \quad 0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0,$$

then a metric in E induces metrics in S, Q and there is a canonical *second fundamental form*

$$\beta \in A^{1,0}(X, \text{Hom}(S, Q))$$

that measures the deviation from being holomorphic of the C^∞ splitting of (II.B.6) given by the metric. For $j : Q \hookrightarrow E$ the inclusion given by the C^∞ splitting and $q \in Q_x, \xi \in T_x X$ the formula is (loc. cit.)

$$(II.B.7) \quad \Theta_Q(q \otimes \xi) = \Theta_E(j(q) \otimes \xi) + \|\beta^*(q) \otimes \xi\|^2$$

³⁴We should say *metrically positive*, but since this is the main type of positivity used in this paper we shall drop the “metrically.”

where by definition the last term is $-\langle (\beta^*(q), \beta^*(q))_S, \xi \wedge \bar{\xi} \rangle$ and $(\cdot, \cdot)_S$ is the induced metric in S . The minus sign is because the Hermitian adjoint β^* is of type $(0,1)$.

Examples.

(i) The universal quotient bundle $Q \rightarrow G(k, n)$ with fibres $Q_\Lambda = \mathbb{C}^n/\Lambda$ over the Grassmannian $G(k, n)$ of k -planes $\Lambda \subset \mathbb{C}^n$ has a metric induced by that in \mathbb{C}^n , and with this metric

$$\Theta_Q \geq 0 \text{ and } \Theta_Q > 0 \iff k = n - 1.$$

Similarly, the dual $S^* \rightarrow G(k, n)$ of the universal sub-bundle has $\Theta_{S^*} \geq 0$ and $\Theta_{S^*} > 0 \iff k = 1$.

Geometrically, for a k -plane $\Lambda \in \mathbb{C}^n$ we have the usual identification

$$T_\Lambda G(k, n) \cong \text{Hom}(\Lambda, \mathbb{C}^n/\Lambda).$$

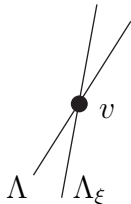
Then for $\xi \in \text{Hom}(\Lambda, \mathbb{C}^n/\Lambda)$ and $v \in \Lambda$

$$\Theta_{S^*}(v, \xi) = 0 \iff \xi(v) = 0.$$

Here the RHS means that for the infinitesimal displacement Λ_ξ of Λ given by ξ we have

$$v \in \Lambda \cap \Lambda_\xi.$$

The picture for $G(2, 4)$ viewed as the space of lines in \mathbb{P}^3 is



There are similar semi-positivity properties for any globally generated vector bundle, since such bundles are induced from holomorphic mappings to a Grassmannian, and positivity and semi-positivity have the obvious functoriality properties.

(ii) The Hodge bundle $F \rightarrow B$ with the metric given by the Hodge-Riemann bilinear relation satisfies $\Theta_F \geq 0$ [Gri70], but unless $h^{n,0} = 1$ very seldom do we have $\Theta_F > 0$.

These examples will be further discussed in Section III.A.

(iii) In the geometric case of a family $\mathcal{X} \xrightarrow{\pi} B$ with smooth fibres, the Narashimhan-Simka [NS68] Finsler type metrics (I.A.34) in $\pi_*\omega_{\mathcal{X}/B}^m$ have the Hodge theoretic interpretation (I.A.35). As a consequence

There is a metric h_m in $\mathcal{O}_{\mathbb{P}^f_\omega_{\mathcal{X}/Y}^m}(1)$ whose Chern form $\omega_m \geq 0$.*

Some care must both be taken here as although h_m is continuous it is not smooth and so $\omega_m = (i/2)\bar{\partial}\partial \log h_m$ and the inequality $\omega_m \geq 0$ must be taken in the sense of currents (cf. [Dem12b] and [Pă16]). Metrics of this sort were used in [Kaw82] and have been the subject of numerous recent works, including [Ber09], [BPă12], [PT14], [MT07], [MT08], and also [Pă16] where a survey of the literature and further references are given.

II.C. Interpretation of the curvature form. Given a holomorphic vector bundle $E \rightarrow X$ there is the associated projective bundle $\mathbb{P}E \xrightarrow{\pi} X$ of 1-dimensional quotients of the fibres of E ; thus $(\mathbb{P}E)_x = \mathbb{P}E_x^*$. Over $\mathbb{P}E$ there is the tautological line bundle $\mathcal{O}_{\mathbb{P}E}(1)$. A metric in $E \rightarrow X$ induces one in $\mathcal{O}_{\mathbb{P}E}(1) \rightarrow \mathbb{P}E$, and we denote by ω_E the corresponding curvature form. Then $\Omega_E := (i/2\pi)\omega_E$ represents the Chern class $c_1(\mathcal{O}_{\mathbb{P}E}(1))$ in $H^2(\mathbb{P}E)$.

Since $\omega_E|_{(\mathbb{P}E)_x}$ is a positive $(1,1)$ form, the vertical sub-bundle

$$V := \ker \pi_* : T\mathbb{P}E \rightarrow TX$$

to the fibration $\mathbb{P}E \rightarrow X$ has a C^∞ horizontal complement $H = \omega_E^\perp$. Thus as C^∞ bundles

$$\begin{cases} T\mathbb{P}E \cong V \oplus H, \text{ and} \\ \pi_* : H \xrightarrow{\sim} \pi^*TX. \end{cases}$$

In more detail, using the metric we have a complex conjugate linear identification $E_x^* \cong E_x$, and using this we shall write points in $\mathbb{P}E$ as $(x, [e])$ where $e \in E_x$ is a non-zero vector. Then we have an isomorphism

$$(II.C.1) \quad \pi_* : H_{(x,[e])} \xrightarrow{\sim} T_x X.$$

Using this identification and normalizing to have $\|e\| = 1$, the interpretation of the curvature form is given by the equation

$$(II.C.2) \quad \Theta_E(e, \xi) = \langle \omega_E, \xi \wedge \bar{\xi} \rangle =: \omega_E(\xi)$$

where $\xi \in T_x X \cong H_{(x,[e])}$ and the RHS is evaluated at $(x, [e])$. Thus

$$(II.C.3) \quad \Theta_E > 0 \iff \omega_E > 0,$$

and similarly for ≥ 0 . There are the evident extensions of (II.C.3) to open sets in $\mathbb{P}E$ lying over open sets in X . For semi-positive vector bundles we summarize by saying that *the curvature form Θ_E measures the degree of positivity of ω_E in the horizontal directions.*

For later use we conclude with the observation that using $\mathcal{O}_X(E) \cong \pi_* \mathcal{O}_{\mathbb{P}E}(1)$, given $s \in \mathcal{O}_{X,x}(E)$ there is the identification of $(1,1)$ forms

$$(II.C.4) \quad (-\partial\bar{\partial} \log \|s\|^2)(x) = \omega_E(x, [s(x)])$$

where the RHS is the $(1,1)$ form ω_E evaluated at the point $(x, [s(x)]) \in \mathbb{P}E$ in the total tangent space (both vertical and horizontal directions).

II.D. Numerical positivity.

In this section we shall discuss various measures of numerical positivity, one main point being that these will apply to bundles arising from Hodge theory. The basic reference here is [Laz04]. A conclusion will be that the Hodge vector bundle F is numerically semi-positive; i.e., $F_{\text{num}} \geq 0$ in the notation to be introduced below.

II.D.1. *Definition of numerical positivity.* We first recall the definition of the cone $\mathcal{C} = \oplus \mathcal{C}_d$ of positive polynomials $P(c_1, \dots, c_r)$ where c_i has weighted degree i . For this we consider partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ with $0 \leq \lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_1 \leq r$, $\sum \lambda_i = n$, of $n = \dim X$. For each such λ the *Schur polynomial* s_λ is defined by the determinant

$$(II.D.1) \quad s_\lambda = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdot & \cdot & c_{\lambda_1+n-1} \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ c_{\lambda_n-n+1} & \cdot & \cdot & \cdot & c_{\lambda_n} \end{vmatrix}.$$

Then ([Laz04]) \mathcal{C} is generated over $\mathbb{Q}^{>0}$ by the s_λ . It contains the Chern monomials

$$c_1^{i_1} \cdots c_r^{i_r}, \quad i_1 + 2i_2 + \dots + ri_r \leq n$$

as well as some combinations of these with negative coefficients, the first of which is $c_1^2 - c_2$.

For each $P \in \mathcal{C}_d$ and d -dimensional subvariety $Y \subset X$ we consider

$$(II.D.2) \quad \int_Y P(c_1(E), \dots, c_r(E)) = P(c_1(E), \dots, c_r(E))[Y]$$

where the RHS is the value of the cohomology class $P(c_1(E), \dots, c_r(E)) \in H^{2d}(X)$ on the fundamental class $[Y] \in H_{2d}(X)$.

Definition: $E \rightarrow X$ is *numerically positive*, written $E_{\text{num}} > 0$, if (II.D.2) is positive for all $P \in \mathcal{C}_d$ and subvarieties $Y \subset X$.

We may similarly define $E_{\text{num}} \geq 0$.

A non-obvious result ([Laz04]) is

(II.D.3) *For line bundles $L \rightarrow X$, we have*

$$L_{\text{num}} \geq 0 \iff L \text{ is nef.}$$

The essential content of this statement is

$$c_1(L)[C] \geq 0 \text{ for all curves } C \implies c_1(L)^d[Y] \geq 0 \text{ for all } d\text{-dimensional subvarieties } Y \subset X.$$

This is frequently formulated as saying that if L is nef then it is in the closure of the ample cone.

II.D.2. *Relation between $E_{\text{num}} > 0$ and $\mathcal{O}_{\mathbb{P}E}(1)_{\text{num}} > 0$.*

For the fibration $\mathbb{P}E \xrightarrow{\pi} X$ there is a Gysin or *integration over the fibre* map

$$\pi_* : H^{2(d-r+1)}(\mathbb{P}E) \rightarrow H^{2d}(X).$$

It is defined by moving cohomology to homology via Poincaré duality, taking the induced map on homology and then again using Poincaré duality. In de Rham cohomology the mapping is given by what the name suggests. The d^{th} *Segre polynomial* is defined by

$$(II.D.4) \quad S_d(E) = \pi_* \left(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{d-r+1} \right).$$

Then ([Laz04]): (i) $S_d(E)$ is a polynomial in the Chern classes $c_1(E), \dots, c_r(E)$, and (ii) $S_d(c_1, \dots, c_r) \in \mathcal{C}_d$. That it is a polynomial in the Chern classes is a consequence of the *Grothendieck relation*

$$(II.D.5) \quad c_1(\mathcal{O}_{\mathbb{P}E}(1))^r - c_1(\mathcal{O}_{\mathbb{P}E}(1))^{r-1} \pi^* e_1(E) + \dots + (-1)^r \pi^* c_r(E) = 0.$$

The first few Segre polynomials are

$$\begin{cases} S_1 = c_1 \\ S_2 = c_1^2 - c_2 \\ S_3 = c_1 c_2 \\ S_4 = c_1^4 - 2c_1^2 c_2 + c_1 c_3 - c_4. \end{cases}$$

An important implication is

$$(II.D.6) \quad \mathcal{O}_{\mathbb{P}E}(1)_{\text{num}} > 0 \implies E_{\text{num}} > 0.$$

Proof. By Nakai-Moishezon (cf. (II.G.4) below), $\mathcal{O}_{\mathbb{P}E}(1)$ and hence E are ample. Then $E_{\text{num}} > 0$ by the theorem of Bloch-Gieseker ([Laz04]). \square

As will be noted below, the converse implication is not valid.

II.E. Numerical dimension. Let $L \rightarrow X$ be a line bundle with $L_{\text{num}} \geq 0$; i.e., L is nef.

Definition (cf. [Dem12a]): The *numerical dimension* is the largest integer $n(L)$ such that

$$c_1(L)^{n(L)} \neq 0.$$

In practice in this paper there will be a semi-positive $(1,1)$ form ω such that $[(i/2\pi)\omega] = c_1(L)$, and then $n(L)$ is the largest integer such that $\omega^{n(L)+1} \equiv 0$ but

$$\omega^{n(L)} \neq 0$$

on an open set.

Relating the Kodaira-Iitaka and numerical dimensions, from [Dem12a] we have

$$(II.E.1) \quad \kappa(L) \leq n(L)$$

with equality if $n(L) = \dim X$, but where equality may not hold if $n(L) < \dim X$.

Example (loc. cit.): Let C be an elliptic curve and $p, q \in C$ points such that $p - q$ is not a torsion point. Then the line bundle $[p - q]$ has a flat unitary metric, but $h^0(C, m[p - q]) = 0$ for all $m > 0$. For any nef line bundle $L' \rightarrow X'$ we set

$$X = C \times X', \quad L = [p - q] \boxtimes L'.$$

Then $\kappa(X) = -\infty$ while $n(L)$ may be any integer with $n(L) \leq \dim X - 1$.

In Section VI.E we will discuss that understanding the reason we may have the strict inequality $\kappa(L) < n(L)$ seems to involve some sort of flatness as in the above example.³⁵

For a vector bundle $E \rightarrow X$ with $E_{\text{num}} \geq 0$ we have the

³⁵Cf. [CD17a] and [CD17b] for a related discussion involving certain Hodge bundles.

Definition: The *numerical dimension* $n(E)$ of the vector bundle $E \rightarrow X$ is given by $n(\mathcal{O}_{\mathbb{P}E}(1))$.

Since $\mathcal{O}_{\mathbb{P}E}(1)$ is positive on the fibers of $\mathbb{P}E \rightarrow X$ we have

$$r - 1 \leq n(E) \leq \dim \mathbb{P}E = \dim X + r - 1.$$

Conjecture VI.E.6 below suggests conditions under which equality will hold.

From (II.D.4) we have

$$(II.E.2) \quad n(E) \text{ is the largest integer with } S_{n(E)-r+1}(E) \neq 0,$$

which serves to define the numerical dimension of a semi-positive vector bundle in terms of its Segre classes. We remark that as noted above

$$(II.E.3) \quad \mathcal{O}_{\mathbb{P}E}(1)_{\text{num}} \geq 0 \implies S_q(E) \geq 0 \text{ for any } q \geq 0;$$

we are not aware of results concerning the converse implication.

II.F. Tangent bundle. When $E = TX$ and the metric on $E \rightarrow X$ is given by a Kähler metric on X the curvature form has the interpretation

$$(II.F.1) \quad \Theta_{TX}(\xi, \eta) = \left\{ \begin{array}{l} \text{holomorphic bi-sectional curvature} \\ \text{in the complex 2-plane } \xi \wedge \eta \\ \text{spanned by } \xi, \eta \in T_x X \end{array} \right\}.$$

When $\xi = \eta$ we have

$$(II.F.2) \quad \Theta_{TX}(\xi, \xi) = \left\{ \begin{array}{l} \text{holomorphic sectional curvature in} \\ \text{the complex line spanned by } \xi \end{array} \right\}.$$

Of particular interest and importance in Hodge theory and in other aspects of algebraic geometry is the case when TX has some form of negative curvature.

PROPOSITION II.F.3 ([BKT13]): *Assume there is $c > 0$ such that (i) $\Theta_{TX}(\xi, \xi) \leq -c$ for all ξ , and (ii) $\Theta_{TX}(\lambda, \eta) \leq 0$ for all λ, η . Then there exists ξ, η such that $\Theta_{TX}(\xi, \eta) \leq -c/2$ for all η .*

In other words, if the holomorphic sectional curvatures are negative and the holomorphic bi-sectional curvatures are non-positive, then they are negative on an open set in $G(2, TX)$, the Grassmann bundle of 2-planes in TX , and this open set maps onto X . Noting that $\text{Gr}(2, TX)$ maps to an open subset of the horizontal sub-bundle in the fibration $\mathbb{P}TX \rightarrow X$, from (II.G.2) below we have the

COROLLARY II.F.4: *If the assumptions in II.F.3 are satisfied, then T^*X is big.*

II.G. First implications. In this section for easy reference we will summarize the first implications of the two types of positivity on the Kodaira-Iitaka dimension and numerical dimension. These are either well known or easily inferred from what is known.

Case of a line bundle $L \rightarrow X$

$$(II.G.1) \quad L_{\text{met}} > 0 \implies L \text{ ample.}$$

This is the Kodaira theorem which initiated the relation between metric positivity and sections.

The next is the Grauert-Riemenschneider conjecture, established by Siu and Demailly (cf. [Dem12a] and the references there):

$$(II.G.2) \quad L_{\text{met}}^0 > 0 \implies \kappa(L) = \dim X.$$

Here we recall that $L_{\text{met}}^0 > 0$ means that there is a metric in $L \rightarrow X$ whose curvature form $\omega \geq 0$ and where $\omega > 0$ on an open set. The result may be phrased as

$$L_{\text{met}}^0 > 0 \implies L \text{ is big.}$$

For this paper this variant of the Kodaira theorem will play a central role as bundles constructed from the extended Hodge vector bundle tend to be big and perhaps free,³⁶ but just exactly what their “Proj” is seems to be an interesting issue. Because of this for later use, in the case when X is projective we now give a

Proof of (II.G.2). Let $H \rightarrow X$ be a very ample line bundle chosen so that $H - K_X$ is ample. Setting $F = L + H$ we have $F_{\text{met}} > 0$. For $D \in |F|$ smooth using the Kodaira vanishing theorem we have

$$\begin{cases} h^q(X, mF) = 0, & q > 0, \\ h^q(D, mF|_D) = 0, & q > 0. \end{cases}$$

We note that the vanishing theorems will remain true if we replace L by a positive multiple.

Let $D, \dots, D_m \in |H|$ be distinct smooth divisors. From the exact sequence $0 \rightarrow m(F - H) \rightarrow mF \rightarrow \bigoplus_{j=1}^m mF|_{D_j}$ we have

$$0 \rightarrow H^0(X, m(F - H)) \rightarrow H^0(X, mF) \rightarrow \bigoplus_{j=1}^m H^0(D_j, mF).$$

This gives

$$h^0(X, mL) = h^0(X, m(F - H)) \geq h^0(X, mF) - mh^0(D, mF).$$

Using the above vanishing results

$$h^0(X, mL) \geq \chi(X, mF) - m\chi(D, mF).$$

³⁶The issue of additional conditions that will imply that an L which is nef and big is also free is a central one in birational geometry (cf. [Ko-Mo]). The results there seem to involve assumptions on $mL - K_X$. In Hodge theory K_X is frequently one of the things that one wishes to establish properties of.

For $d = \dim X$ and letting \sim denote modulo lower order terms, from the Riemann-Roch theorem we have

$$\begin{aligned}\chi(X, mF) &\sim \frac{m^d}{d!} F^d \\ m\chi(D, mF) &\sim \frac{m^d}{d!} (dF^{d-1} \cdot H).\end{aligned}$$

For $m \gg 0$ this gives

$$h^0(X, mL) \geq \frac{m^d}{d!} (F^d - dF^{d-1} \cdot H) + o(m^d).$$

From

$$\begin{aligned}F^d - dF^{d-1} \cdot H &= (L + H)^d - d(L + H)^{d-1} \cdot H \\ &= (L + H)^{d-1} \cdot (L - (d-1)H)\end{aligned}$$

replacing L by a multiple we may make this expression positive.³⁷ □

The next result is

$$(II.G.3) \quad L_{\text{met}} > 0 \implies L_{\text{num}} > 0, \text{ and similarly for } \geq 0 \quad (\text{obvious}).$$

The inequality in (II.G.3) is sometimes phrased as

$$L_{\text{met}} \geq 0 \implies L \text{ is nef.}$$

The theorem of Nakai-Moishezon is

$$(II.G.4) \quad L_{\text{num}} > 0 \iff L \text{ is ample.}$$

The next inequality was noted above:

$$(II.G.5) \quad \kappa(L) \leq n(L), \text{ with equality if } n(L) = \dim X;$$

Case of a vector bundle $E \rightarrow X$

$$(II.G.6) \quad E_{\text{met}} > 0 \implies E \text{ ample, and } E_{\text{met}}^0 > 0 \implies \kappa(E) = \dim X + r - 1.$$

In words, $E_{\text{met}}^0 > 0$ implies that E is big.

We next have

$$(II.G.7) \quad E_{\text{met}} > 0 \implies E_{\text{num}} > 0.$$

This result may be found in [Laz04]; the proof is not obvious from the definition. We are not aware of any implication along the lines of

$$E_{\text{met}} \geq 0 \implies E_{\text{num}} \geq 0.$$

Next we have

$$(II.G.8) \quad \kappa(E) \leq n(E) \text{ with equality if } n(E) = \dim X + r - 1 \text{ (this follows from (II.G.5)).}$$

³⁷This argument is due to Catanese; cf. [Dem12a].

This leads to

$$(II.G.9) \quad E_{\text{met}} \geq 0 \text{ and } E_{\text{num}} > 0 \implies \kappa(E) = \dim X + r - 1.$$

This follows from (II.E.2) and (II.G.8).

It is *not* the case that

$$E_{\text{num}} > 0 \implies E \text{ ample};$$

there is an example due to Fulton of a numerically positive vector bundle over a curve that is not ample (cf. [Laz04]).

We will conclude this section with a sampling of well-known results whose proofs illustrate some of the traditional uses of positivity.

PROPOSITION II.G.10: *If $E \rightarrow X$ is a Hermitian vector bundle with $\Theta_E > 0$, then $H^0(X, E^*) = 0$.*

Proof. Using (II.C.4) applied to $E^* \rightarrow X$, if $s \in H^0(X, E^*)$ then when we evaluate $\partial\bar{\partial} \log \|s\|^2$ at a strict maximum point where the Hessian is definite we obtain a contradiction. If the maximum is not strict then the usual perturbation argument may be used. \square

PROPOSITION II.G.11: *If $E \rightarrow X$ is a Hermitian vector bundle of rank $r \leq \dim X$ and with $\Theta_E > 0$, then every section $s \in H^0(X, E)$ has a zero.*

Proof. The argument is similar to the preceding proposition, only this time we assume that s has no zero and evaluate $\partial\bar{\partial} \log \|s\|^2$ at a minimum. This (1,1) form is positive in the pullback to X of the vertical tangent space, and it is negative in the pullback of the horizontal tangent space. The assumption $r \leq \dim X$ then guarantees that it has at least one negative eigenvalue. \square

PROPOSITION II.G.12: *If $E \rightarrow X$ is a Hermitian vector bundle with $\Theta_E \geq 0$ and $s \in H^0(X, E)$ satisfies $\Theta_E(s) = 0$, then $Ds = 0$.*

Proof. Let ω be a Kähler form on X . Then

$$\partial\bar{\partial}(s, s) = (Ds, Ds) + (s, \Theta_E(s)) = (Ds, Ds) \geq 0,$$

and if $\dim X = d$ using Stokes theorem we have

$$0 = \int_X \omega^{d-1} \wedge (i/2) \partial\bar{\partial} \|s\|^2 = \int_X \omega^{d-1} \wedge (i/2) (Ds, Ds)$$

which gives the result. \square

II.H. A further result.

As discussed above, for many purposes positivity is too strong (see the examples in Section II.B) and semi-positivity is too weak (adding the trivial bundle to a semi-positive bundle gives one that is semi-positive). One desires a more subtle notion than just $\Theta_E \geq 0$. With this in mind, a specific guiding question for this paper has been the

QUESTION: Suppose that one has $\Theta_E \geq 0$ and for $\wedge^r E = \det E$ we have $\Theta_{\det E} > 0$ on an open set; that is, E is strongly semi-positive. Does this enable one to produce lots of sections of $\text{Sym}^m E$ for $m \gg 0$?

The following is a response to this question:

THEOREM II.H.1: Suppose that $E \rightarrow X$ is a Hermitian vector bundle of rank r that is strongly semi-positive. Then $\text{Sym}^m E \rightarrow X$ is big for any $m \geq r$.

Proof. Setting $S^r E = \text{Sym}^r E$, we have $\Theta_{S^r E} \geq 0$. Let ω_r be the curvature form for $\mathcal{O}_{\mathbb{P}S^r(E)}(1)$. Then $\omega_r \geq 0$, and we will show that

$$(II.H.2) \quad \omega_r > 0 \text{ on an open set.}$$

For this it will suffice to find one point $p = (x, [e_1 \cdots e_r]) \in \mathbb{P}S^r(E)_x$ where (II.H.2) holds. Let $x \in X$ be a point where $(\text{Tr } \Theta_E)(x) > 0$ and let e_1, \dots, e_r be a unitary basis for E_x . Then some

$$\langle (\Theta_E(e_i), e_i), \xi \wedge \bar{\xi} \rangle > 0.$$

We may assume that the Hermitian matrix

$$\langle (\Theta_E(e_i), e_j), \xi \wedge \bar{\xi} \rangle = \delta_{ij} \lambda_i, \quad \lambda_i \geq 0$$

is diagonalized. Then

$$\begin{aligned} \langle (\Theta_{S^r E}(e_1 \cdots e_r), e_1 \cdots e_r), \xi \wedge \bar{\xi} \rangle &= \sum_i \left\langle \left(\Theta_{S^r E}(e_i) e_1 \overset{i}{\cdots} e_r, e_1 \cdots e_r \right), \xi \wedge \bar{\xi} \right\rangle \\ &= \sum_i \lambda_i > 0. \end{aligned}$$

The same argument works for any $m \geq r$. □

Using this we are reduced to proving the

LEMMA II.H.3: Let $F \rightarrow X$ be a Hermitian vector bundle with $\Theta_F \geq 0$, and where this is a point $x \in X$ and $f \in F_x$ such that the $(1, 1)$ form $\Theta_F(f, \cdot)$ is positive. Then for the curvature form ω_F of the line bundle $\mathcal{O}_{\mathbb{P}F}(1) \rightarrow \mathbb{P}F$, we have $\omega_F > 0$ at the point $(x, [f]) \in \mathbb{P}F$.

Proof. Let f_1^*, \dots, f_r^* be a local holomorphic frame for $F^* \rightarrow X$ and

$$\sigma = u([a_1, \dots, a_r]; x) \sum_i a_i f_i^*$$

a local holomorphic section of $\mathcal{O}_{\mathbb{P}E}(1) \rightarrow \mathbb{P}F$, where a_1, \dots, a_r are variables defined modulo the scaling action $(a_1, \dots, a_r) \rightarrow \lambda(a_1, \dots, a_r)$ and $u([a_1, \dots, a_r], x)$ is holomorphic. We have

$$\|\sigma\| = |u|^2 \sum_{i,j} h_{i\bar{j}} \cdot a_i \bar{a}_j$$

where $h_{i\bar{j}} = (f_i^*, f_j^*)$ is the metric in $F \rightarrow X$; up to a constant

$$\omega_F = \partial \bar{\partial} \log \|\sigma\|^2.$$

We may choose our frame and scaling parameter so that at the point $(x, [f])$

$$(II.H.4) \quad h_{i\bar{j}}(x) = \delta_{ij}, \quad dh_{ij}(x) = 0 \quad \text{and} \quad \|\sigma(x, [f])\| = 1.$$

Computing $\partial\bar{\partial}\log\|\sigma\|^2$ and evaluating at the point where (II.H.4) holds any cross-terms involving $dh_{i\bar{j}}(x)$ drop out and we obtain

$$\omega_F = \sum_{ij} (\partial\bar{\partial}h_{i\bar{j}})(x) a_i \bar{a}_j + \left(\sum_{i,j} da_i \wedge \bar{d}a_i - \left(\sum_{i,j} a_i \cdot da_i \right) \wedge \left(\overline{\sum_j a_j da_j} \right) \right).$$

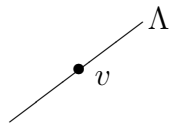
When we take the scaling action into account and use Cauchy-Schwarz it follows that that $\omega_F > 0$ in $T_{(x,[f])}\mathbb{P}F$. \square

Remark that the point $(x, [e_1 \cdots e_r])$ corresponding to a decomposable tensor in $S^r E_x$ is very special. Easy examples show that we do not expect to have $\omega_r > 0$ everywhere. In fact, the exterior differential system (EDS)

$$\omega_r = 0$$

is of interest and will be discussed in Section VI.E in the situation when the curvature has the norm positivity property to be introduced in Section III.A.

Example II.H.5: We will illustrate the mechanism of how passing to $S^r E$ increases the Kodaira-Iitaka dimension of the bundle. Let $E \rightarrow G(2, 4)$ denote the dual of the universal sub-bundle. As above, points of $G(2, 4)$ will be denoted by Λ and thought of as lines in \mathbb{P}^3 . For $v \in \Lambda$ we denote by $[v]$ the corresponding line in \mathbb{C}^4 . Points of $\mathbb{P}E$ will be (Λ, v)



and then the fibre of $\mathcal{O}_{\mathbb{P}E}(1)$ at (Λ, v) is $[v]$. The fibre $E_\Lambda \cong \Lambda^*$, and we have

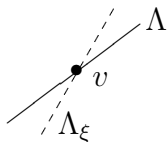
$$\begin{array}{ccccccc} H^0(G(2, 4), E) & \longrightarrow & E_\Lambda & \longrightarrow & 0 \\ \parallel & & \parallel & & \\ \mathbb{C}^{4*} & \longrightarrow & \Lambda^* & \longrightarrow & 0. \end{array}$$

The tangent space

$$T_\Lambda G(2, 4) \cong \text{Hom}(\Lambda, \mathbb{C}^4/\Lambda)$$

is isomorphic to the horizontal space $H_{(\Lambda, v)} \subset T_{(\Lambda, v)}\mathbb{P}E$. As previously noted, for $\xi \in T_\Lambda G(2, 4)$ we have

$$(II.H.6) \quad \Theta_E(v, \xi) = \omega(\xi) = 0 \iff \xi(v) = 0$$



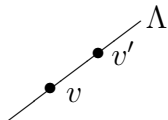
Here Λ_ξ is the infinitesimal displacement of Λ in the direction ξ .

We observe that

$$(II.H.7) \quad \varphi_{\mathcal{O}_{\mathbb{P}E}(1)} : \mathbb{P}E \rightarrow \mathbb{P}^3$$

is the tautological map $(\Lambda, v) \rightarrow [v]$, and consequently the fibre of (II.H.7) through v is the \mathbb{P}^2 of lines in \mathbb{P}^3 through v . The tangent space to this fibre are the ξ 's as pictured above. We note that $\dim \mathbb{P}E = 5$ while $\kappa(\mathcal{O}_{\mathbb{P}E}(1)) = n(\mathcal{O}_{\mathbb{P}E}(1)) = 3$.

Points of $\mathbb{P}S^2E$ are (Λ, v, v')



and unless $v = v'$ we have

$$\Theta_E(v \cdot v', \xi) \neq 0$$

for any non-zero $\xi \in T_\Lambda G(2, 4)$. Thus for ω_2 the curvature form of $\mathcal{O}_{\mathbb{P}S^2E}(1)$ we have

$$\omega_2 > 0 \text{ at } (\Lambda, v \cdot v')$$

unless $v = v'$; consequently S^2E is big.

We shall give some further observations and remarks concerning the question posed at the beginning of this section.

PROPOSITION II.H.8: *If $E \rightarrow X$ is generically globally generated and $\det E$ is big, then $S^k E$ is big for some $k > 0$.*

Proof. By standard arguments passing to a blowup of X and pulling E back, we may reduce to the case where E is globally generated. Let $N = h^0(X, E)$ and denote by $Q \rightarrow G(N - r, N)$ the universal quotient bundle over the Grassmannian. We then have a diagram

$$(II.H.9) \quad \begin{array}{ccccc} \mathbb{P}E & \xrightarrow{\alpha} & \mathbb{P}Q & \xrightarrow{\beta} & \mathbb{P}^{N-1} \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{f} & G(N - r, N) & & \end{array}$$

where

- $f^*Q = E$;
- $\beta \circ \alpha = \varphi_{\mathcal{O}_{\mathbb{P}E}(1)}$

where $\varphi_{\mathbb{P}E}(1)$ is the map induced by $H^0(\mathcal{O}_{\mathbb{P}E}(1))$. As a metric in $E \rightarrow X$ we use the one induced by the standard metric on $Q \rightarrow G(N - r, N)$. Then we claim that

- $\Theta_E \geq 0$;
- $\text{Tr } \Theta_E > 0$ on an open set.

The first of these is clear. For the second, $\det Q$ is an ample line bundle over $G(N - r, r)$ and $\det E = f^* \det Q$. Thus

$$\Theta_{\det E} = \text{Tr } \Theta_E \geq 0$$

and for $d = \dim X$

$$(\text{Tr } \Theta_E)^d \equiv 0$$

contradicts the assumption that $\det E$ is big. The proposition now follows from Theorem II.H.1. \square

This proposition will be used in connection with the following well known

PROPOSITION II.H.10: *If $S^k E$ is big for some $k > 0$, then there exist arbitrarily large ℓ such that $S^{k\ell} E$ is generically globally generated.*

Proof. As a general comment, for any holomorphic vector bundle $F \rightarrow X$

$$\left\{ \begin{array}{l} F \text{ is generically} \\ \text{globally generated} \end{array} \right\} \iff \left\{ \begin{array}{l} \mathcal{O}_{\mathbb{P}F}(1) \rightarrow \mathbb{P}F \text{ is generically} \\ \text{globally generated} \end{array} \right\}.$$

Since sufficiently high powers of a big line bundle are generically globally generated, and since by definition $F \rightarrow X$ is big if the line bundle $\mathcal{O}_{\mathbb{P}F}(1)$ is big, we have

$$F \text{ big} \implies S^n F \rightarrow X \text{ is generically globally generated.}$$

Taking $F = S^k E$, our assumption then implies that there are arbitrarily large ℓ such that $S^\ell(S^k E)$ is generically globally generated, consequently the direct summand $S^{k\ell} E$ of $S^\ell(S^k E)$ is also generically globally generated. \square

COROLLARY II.H.11: *If $E \geq 0$ and $\det E > 0$, then $S^m E$ is big for arbitrarily large m .*

Proof. By Theorem II.H.1, $S^m E$ is generically globally generated and $\det S^m E > 0$ so that (II.H.8) applies. \square

Remark II.H.12: For a line bundle $L \rightarrow X$ we consider the properties

- (i) L is nef;
- (ii) L is big;
- (iii) L is free (frequently expressed by saying that L is *semi-ample*).

For vector bundles $E \rightarrow X$ one has the corresponding properties using $\mathcal{O}(1)$. In (ii) and (iii) one generally uses $\text{Sym}^{m\mathbb{P}E} E$ rather than just E itself.

Although all these properties are important, in some sense (iii) is the most stable (note that (iii) \implies (i)). Repeating a discussion from the introduction, suppose that we have in $L \rightarrow X$ a Hermitian metric with Chern form ω such that

$$(II.H.13) \quad \omega \geq 0 \text{ and } \omega > 0 \text{ on } X \setminus Z \text{ where } Z \subset X \text{ is a normal crossing divisor.}$$

Then ω defines a Kähler metric ω^* on $X^* := X \setminus Z$, and then as in (I.A.11) one may ask the

- (II.H.14) **Question:** *Are there properties of the metric ω^* , especially those involving its curvature R_{ω^*} , that imply that $L \rightarrow X$ is free?*

III. NORM POSITIVITY

III.A. Definition and first properties.

As previously noted, for many purposes including those arising from Hodge theory strict positivity of a holomorphic vector bundle in the sense of (II.B.2) and (II.B.3) is too strong, whereas semi-positivity is too weak. The main observation of this section is that for many bundles that arise naturally in algebraic geometry the curvature has a

special form, one that implies semi-positivity in both of the above senses and where in examples the special form has Hodge-theoretic and algebro-geometric interpretations.

Definition III.A.1: Let $E \rightarrow X$ be a Hermitian vector bundle with curvature Θ_E . Then Θ_E has the *norm positivity* property if there is a Hermitian vector bundle $G \rightarrow X$ and a holomorphic bundle mapping

$$(III.A.2) \quad A : TX \otimes E \rightarrow G$$

such that for $x \in X$ and $e \in E_x, \xi \in T_x X$

$$(III.A.3) \quad \Theta_E(e, \xi) = \|A(\xi \otimes e)\|_G^2.$$

Here we are identifying E_x with E_x^* using the metric, and $\|\cdot\|_G^2$ denotes the square norm in G . In matrix terms, relative to unitary frames in E and G there will be a matrix A of $(1,0)$ forms such that the curvature matrix is given by

$$(III.A.4) \quad \Theta_E = -{}^t\bar{A} \wedge A.$$

We note that (III.A.3) will hold for any tensors in $TX \otimes E$, not just decomposable ones. As a consequence $E \rightarrow X$ is semi-positive in both senses (II.B.2) and (II.B.3).

The main implications of norm positivity will use the following observation:

(III.A.5) *If the curvatures of Hermitian bundles $E, E' \rightarrow X$ have the norm positivity property, then the same is true for $E \oplus E' \rightarrow X$ and $E \otimes E' \rightarrow X$, as well as Hermitian direct summands of these bundles.*

Proof. If we have $A : TX \otimes E \rightarrow G$ and $A' : TX \otimes E' \rightarrow G'$, then $\Theta_{E \otimes E'} = (\Theta_E \otimes \text{Id}_{E'}) \oplus (\text{Id}_E \otimes \Theta_{E'})$, and

$$(A \otimes \text{Id}_{E'}) \oplus (\text{Id}_E \otimes A') : TX \otimes E \otimes E' \rightarrow (G \otimes E') \oplus (E \otimes G')$$

leads to norm positivity for $\Theta_{E \otimes E'}$. The argument for \oplus is evident. \square

III.B. A result using norm positivity.

The idea is this: For this discussion we abbreviate

$$T = T_x X, \quad E = E_x, \quad G = G_x$$

and identify $E \cong E^*$ using the metric. We have a linear mapping

$$(III.B.1) \quad A : T \otimes E \rightarrow G,$$

and using (III.A.3) non-degeneracy properties of this mapping will imply positivity properties of Θ_E . Moreover, in examples the mapping A will have algebro-geometric meaning so that algebro-geometric assumptions will lead to positivity properties of the curvature.

The simplest non-degeneracy property of (III.B.1) is that A is injective; this seems to not so frequently happen in practice. The next simplest is that A has injectivity properties in each factor separately. Specifically we consider the two conditions

$$(III.B.2) \quad A : T \rightarrow \text{Hom}(E, G) \text{ is injective;}$$

(III.B.3) for general $e \in E$, the mapping $A(e) : T \rightarrow G$ given by

$$A(e)(\xi) = A(\xi \otimes e), \quad \xi \in T$$

is injective.

The geometric meanings of these are:

(III.B.4) (III.B.2) is equivalent to having

$$\Theta_{\det E} = \text{Tr } \Theta_E > 0$$

at x ; and

(III.B.5) (III.B.3) is equivalent to having

$$\omega > 0$$

at $(x, [e]) \in (\mathbb{P}E)_x$.

This gives the

PROPOSITION III.B.6: *If $E \rightarrow X$ has a metric whose curvature has the norm positivity property, then*

- (i) (II.B.2) $\implies \det E$ is big;
- (ii) (III.B.3) $\implies E$ is big.

A bit more subtle is the following result, which although it is a consequence of Theorem II.H.1 and (III.B.3), for later use we shall give another proof.

THEOREM III.B.7: *If the rank r bundle $E \rightarrow X$ has a metric whose curvature has the norm positivity property, then*

$$(II.B.2) \implies S^r E \text{ is big.}$$

COROLLARY III.B.8: *With the assumptions in (III.B.7) the map*

$$H^0(X, S^m(S^r E)) \rightarrow S^m(S^r E)$$

is generically surjective for $m \gg 0$.³⁸

Proof of Theorem III.B.7. Keeping the above notations and working at a general point in $\mathbb{P}E$ over $x \in X$, given $\xi \in T_x X$ and a basis e_1, \dots, e_r of E_x from (III.B.3) we have

$$(III.B.9) \quad \sum_{i=1}^r \|A(\xi \otimes e_i)\|_G^2 \neq 0.$$

Then using (III.A.5) for the induced map

$$A : T_x X \otimes S^r E_x \rightarrow S^{r-1} E_x \otimes G_x$$

from (III.B.9) for ω_r the canonical (1,1) form on $\mathbb{P}S^r E$ at the point $(x, [e_1 \cdots e_r])$

$$\langle \omega_r, \xi \wedge \bar{\xi} \rangle > 0. \quad \square$$

³⁸Then this also implies that the map

$$H^0(X, S^m(S^r E)) \rightarrow S^{mr}$$

is generically surjective.

Remark: Viehweg ([Vie83a]) introduced the notion of *weak positivity* for a coherent sheaf. For vector bundles this means that for any ample line bundle $L \rightarrow X$ there is a $k > 0$ such that the evaluation mapping

$$H^0(X, S^\ell(S^k E \otimes L)) \rightarrow S^\ell(S^k E \otimes L)$$

is generically surjective for $\ell \gg 0$. He then shows that for the particular bundles that arise in the proof of the Iitaka conjecture if one has $\det E > 0$ on an open set, and then an intricate cohomological argument gives that E is weakly positive. As will be explained in Section V.A of this paper, (III.B.8) may be used to circumvent the need for weak positivity in this case.

We note that the ample line bundle $L \rightarrow X$ is not needed in (III.B.8). We also point out that

$$E_{\text{met}} \geq 0 \implies E \text{ is weakly positive (cf. [Pă16])}.$$

This is plausible since $S^k E \geq 0$ and $L > 0 \implies S^k E \otimes L > 0$. In loc. cit. this result is extended to important situations where the metrics have certain types of singularities.

We conclude this section with a discussion of the Chern forms of bundles having the norm positivity property, including the Hodge vector bundle.

PROPOSITION III.B.10: *The linear mapping A induces*

$$\wedge^q A : \wedge^q T \rightarrow \wedge^q G \otimes S^q E,$$

and up to a universal constant

$$c_q(\Theta) = \|\wedge^q A\|^2.$$

Proof. The notation means

$$\|\wedge^q A\|^2 = (\wedge^q A, \wedge^q A)$$

where in the inner product we use the Hermitian metrics in G and E , and we identify

$$\wedge^q T^* \otimes \overline{\wedge^q T^*} \cong (q, q)\text{-part of } \wedge^{2q} (T^* \otimes \overline{T^*}).$$

Then letting A^* denote the adjoint of A we have

$$\wedge^q \Theta = \wedge^q A \otimes \wedge^q A^* = \wedge^q A \otimes (\wedge^q A)^*$$

and

$$c_q(\Theta) = \text{Tr } \wedge^q(\Theta) = (\wedge^q A, \wedge^q A). \quad \square$$

In matrix terms, if

$$A = \dim G \times \dim T \text{ matrix with entries in } E$$

then

$$\wedge^q A = \left\{ \begin{array}{l} \text{matrix whose entries are the } q \times q \text{ minors of } A, \\ \text{where the terms of } E \text{ are multiplied as polynomials.} \end{array} \right\}$$

It follows that up to a universal constant for the Hodge vector bundle F

$$c_q(\Theta_F) = \sum_a \Psi_\alpha \wedge \bar{\Psi}_\alpha$$

where the Ψ_α are $(q, 0)$ forms. In particular, any monomial $c_I(\Theta_F) \geq 0$.

The vanishing of the matrix $\wedge^q \Phi_{*,n}$ is *not* the same as $\text{rank } \Phi_{*,n} < q$. In fact,

$$(III.B.11) \quad \text{rank } \Phi_{*,n} < q \iff c_1(\Theta_F)^q = 0.$$

In general we have the

PROPOSITION III.B.12: *If $E \rightarrow X$ has the norm positivity property, then $P(\Theta_E) \geq 0$ for any $P \in \mathcal{C}$.*

A proof of this appears in [Gri69].

In the geometric case when we have a VHS arising from a family of smooth varieties we have the period mapping Φ with the end piece of differential being

$$\Phi_{*,n} : T_b B \rightarrow \text{Hom}(H^0(\Omega_{X_b}^n), H^1(\Omega_{X_b}^{n-1}))$$

and the algebro-geometric interpretation of (III.B.11) is standard; e.g., $\Phi_{*,n}$ injective is equivalent to local Torelli holding for the $H^{n,0}$ -part of the Hodge structure.

We conclude this subsection with a result that pertains to a question that was raised above.

PROPOSITION III.B.13: *If $\Phi : B \rightarrow \Gamma \backslash D$ has no trivial factors, and if $h^{n,0} \leq \dim B$ and $H^0(\bar{B}, F_e) \neq 0$, then*

$$c_{h^{n,0}}(F) \neq 0.$$

Proof. We will first prove the result when $B = \bar{B}$. We let $s \in H^0(F, B)$ and assume that $c_{h^{n,0}}(F) = 0$. Then s is everywhere non-zero and we may go to a minimum of $\|s\|^2$. From Proposition II.G.12 we have $D\sigma = 0$, which implies that the norm $\|s\|$ is constant and

$$\nabla s = 0$$

where ∇ is the Gauss-Manin connection. Using the arguments [Gri70] we may conclude that the variation of Hodge structure has a trivial factor.

If $B \neq \bar{B}$, the arguments given in Section IV below may be adapted to show that the proof still goes through. The point is the equality of the distributional and formal derivatives that arise in integrating by parts. \square

IV. SINGULARITIES

In recent years the use of singular metrics and their curvatures in algebraic geometry has become widespread and important. Here we shall discuss some aspects of this development; the main objective is to define mild singularities and show that the singularities that arise in Hodge theory have this property.

One may roughly divide singularities into three classes:

- (i) metrics with *analytic singularities* as defined in [Dem12a] and [Pă16]; these arise in various extensions of the Kodaira vanishing theorem. We shall only briefly discuss these in part to draw a contrast with the next type of singularities that will play a central role in this paper;
- (ii) metrics with *logarithmic singularities*; these arise in Hodge theory especially in [CKS86], and also in [Kol87], [Zuo00], [Bru16b] and [GGLR17], and for those that do arise in Hodge theory we shall show that they are *mild* as defined below;
- (iii) metrics with *PDE singularities*; these arise in several places including the applications to moduli where important classes of varieties have canonical special metrics. We shall not discuss these here but refer to the survey papers [Don16] and [Don15] and the Bourbaki talk [Dem16] for summaries of results and guides to the literature.

IV.A. Analytic singularities. For $E \rightarrow X$ a holomorphic vector bundle over a compact complex manifold, these singularities arise from metrics of the form

$$(IV.A.1) \quad h = e^{-\varphi} h_0$$

where h_0 is an ordinary smooth metric in the bundle and φ is a *weight function*, which in practice and will almost always locally be of the form

$$(IV.A.2) \quad \varphi = \varphi' + \varphi''$$

where φ' is plurisubharmonic (psh) and φ'' is smooth.

Definition: The metric (IV.A.1) has *analytic singularities* if it locally has the form (IV.A.2) with

$$\varphi' = \alpha \left(\log \sum_{j=1}^m |f_j(z)|^{\alpha_j} \right), \quad \alpha, \alpha_j > 0$$

and where the $f_j(z)$ are holomorphic functions.

Definition: Given a weight function φ of the form (IV.A.2), we define the subsheaf $I(\varphi) \subset \mathcal{O}_X$ by

$$I(\varphi) = \left\{ f \in \mathcal{O}_X : \int e^{-\varphi} |f|^2 < \infty \right\}.$$

Here the integral is taken over a relatively compact open set in the domain of definition of f using any smooth volume form on X .

PROPOSITION IV.A.3 (Nadel; cf. [Dem12a]): $I(\varphi)$ is a coherent sheaf of ideals.

Example IV.A.4 ([Dem12a]): For

$$\varphi = \log (|z_1|^{\alpha_1} + \cdots + |z_n|^{\alpha_n}), \quad \alpha_j > 0$$

the corresponding ideal is

$$I(\varphi) = \left\{ z_1^{\beta_1} \cdots z_n^{\beta_n} : \sum_j (\beta_j + 1)/\alpha_j > 1 \right\}$$

where the RHS is the ideal generated by the monomials appearing there.

Thus for $\beta_j = 1$ and $\sum_j \alpha_j = 2 + \epsilon$

$$I(\varphi) = \mathfrak{m}_0$$

is the maximal ideal at the origin in \mathbb{C}^n . The blow up using this ideal $I(\varphi)$ is the usual blow up $\tilde{\mathbb{C}}_0^n \xrightarrow{\pi} \mathbb{C}^n$ of \mathbb{C}^n at the origin.

Variants of this construction give powers \mathfrak{m}_0^k of the maximal ideal, and using z_1, \dots, z_m for $m \leq n$ for suitable choices of the α_j and β_j leads to weighted blowups of \mathbb{C}^n along the coordinate subspaces $\mathbb{C}^{n-m} \subset \mathbb{C}^n$. The use of weight functions with analytic singularities provides a very flexible analytic alternative to the traditional technique of blowing up along subvarieties.

Example IV.A.5: Quite different behavior occurs for the weight function

$$\varphi = \log((-\log |z_1|) \cdots (-\log |z_k|)).$$

In this case $I(\varphi) = \mathcal{O}_X$. We will encounter weight functions of this type in Hodge theory.

Example IV.A.6: If $L \rightarrow X$ is a line bundle and $s \in H^0(X, L)$ has divisor $(s) = D$, we may define a metric $\| \cdot \|$ in the line bundle by writing any local section $s' \in \mathcal{O}_X(L)$ as

$$s' = fs$$

where f is a meromorphic function and then setting

$$\|s'\| = |f| = e^{\log |f|}.$$

By the Poincaré-Lelong formula ([Dem12a]), formally $\partial\bar{\partial} \log \|s'\| = 0$ but as currents the Chern form is the (1,1) current

$$\Omega_L = [D]$$

given by integration over the effective divisor D .

We will be considering the case when $E = L$ is a line bundle with singular metrics of the form (IV.A.1), (IV.A.2). In this case the *curvature form* is given by

$$(IV.A.7) \quad \omega_h = (i/2)\partial\bar{\partial} \log h = (i/2)\partial\bar{\partial}\varphi - (i/2)\partial\bar{\partial} \log h_0.$$

For φ given by (IV.A.2) where φ' is psh, the singular part of the curvature form is the (1,1) current

$$(i/2)\partial\bar{\partial}\varphi' \geq 0$$

where the inequality is taken in the sense of currents as explained in [Dem12a] and [Pă16].

THEOREM IV.A.8 (Nadel vanishing theorem): *If $\omega_h > 0$ in the sense of currents, then*

$$H^q(X, (K_X + L) \otimes I(\varphi)) = 0, \quad q > 0.$$

Application: Assuming that $\omega_L > 0$ and using a cutoff function to globalize the function φ in (IV.A.4), we may replace L by mL to make $(i/2)\partial\bar{\partial}\varphi + \omega_{K_X} + m\omega_L > 0$. Then the Nadel vanishing theorem gives $H^1(X, mL \otimes I_x) = 0$ from which we infer that

$$H^0(X, mL) \rightarrow mL_x \rightarrow 0.$$

Thus mL is globally generated, and similar arguments show that for $m \gg 0$ the map

$$\varphi_{mL} : X \rightarrow \mathbb{P}^{N_m}$$

is an embedding.

This proof of the Kodaira embedding theorem illustrates the advantage of the flexibility provided by the choice of the weight function φ . Instead of blowing up as in the original Kodaira proof, the use of weight functions achieves the same effect with greater flexibility.

Another use of singular metrics is given by the

THEOREM IV.A.9 (Kawamata-Viehweg vanishing theorem): *If $L \rightarrow X$ is big, then*

$$H^q(X, K_X + L) = 0, \quad q > 0.$$

Proof (cf. [Dem12a]). Let $H \rightarrow X$ be a very ample line bundle and $D \in |H|$ a smooth divisor. From the cohomology sequence of

$$0 \rightarrow mL - H \rightarrow mL \rightarrow mL|_D \rightarrow 0$$

and

$$h^0(X, mL) \sim m^d, \quad h^0(D, mL|_D) \sim m^{d-1}$$

where $d = \dim X$ we have $h^0(mL - H) \neq 0$ for $m \gg 0$. If $E \in |mL - H|$ from Example IV.A.6 there exists a singular metric in $mL - H$ with Chern form

$$\Omega_{mL-H} = [E] \geq 0$$

where the RHS is the (1,1) current defined by E . If $\omega_H > 0$ is the curvature form for a positively curved metric in $H \rightarrow X$, then

$$\omega_L = \frac{1}{m} ([E] + \omega_H) \geq \left(\frac{1}{m}\right) \omega_H > 0,$$

and Nadel vanishing gives the result. \square

Remark: A standard algebro-geometric proof of Kawamata-Viehweg vanishing uses the branched covering method to reduce it to Kodaira vanishing. The above argument again illustrates the flexibility gained by the use of singular weight functions (we note that D plays a role similar to that of the branch divisor in the branched covering method).

IV.B. Logarithmic and mild singularities.

As our main applications will be to Hodge theory, in this section we will use the notations from Section I.C. We recall from that section that

- B is a smooth quasi-projective variety;
- \overline{B} is a smooth projective completion of B ;
- $Z = \overline{B} \setminus B$ is a divisor with normal crossings

$$Z = \cup Z_i$$

where $Z_I := \bigcap_{i \in I} Z_i$ is a stratum of Z and $Z_I^* = Z_{I, \text{reg}}$ are the smooth points of Z_I ;

- $E \rightarrow \overline{B}$ is a holomorphic vector bundle.

A neighborhood \mathcal{U} in \overline{B} of a point $p \in Z$ will be

$$\mathcal{U} \cong \Delta^{*k} \times \Delta^\ell$$

with coordinates $(t, w) = (t_1, \dots, t_k; w_1, \dots, w_\ell)$.

We now introduce the co-frame in terms of which we shall express the curvature forms in \mathcal{U} . The Poincaré metric in $\Delta^* = \{0 < |t| < 1\}$ is given by the (1,1) form

$$\omega_{\text{PM}} = (i/2) \frac{dt \wedge d\bar{t}}{|t|^2 (-\log |t|)^2}.$$

We are writing $-\log |t|$ instead of just $\log |t|$ because we will want to have positive quantities in the computations below. As a check on signs and constants we note the formula

$$(IV.B.1) \quad (i/2) \partial \bar{\partial} (-\log(-\log |t|)) = (1/4) \omega_{\text{PM}}.$$

The inner minus sign is to have $-\log |t| > 0$ so that $\log(-\log |t|)$ is defined. The outer one is to have the expression in parentheses equal to $-\infty$ at $t = 0$ so that we have a psh function. For

$$\varphi = \log(-\log |t|)$$

the curvature form in the trivial bundle over Δ with the singular metric given by $e^{-\varphi}$ has curvature form

$$(IV.B.2) \quad (i/2) \bar{\partial} \partial \log(e^{-\varphi}) = (1/4) \omega_{\text{PM}}.$$

Remark: The functions that appear as coefficients in formally computing (IV.B.1) using the rules of calculus are all in L_{loc}^1 and therefore define distributions. We may then compute ∂ and $\bar{\partial}$ either in the sense of currents or formally using the rules of calculus. An important observation is

$$(IV.B.3) \quad \text{these two methods of computing } \bar{\partial} \partial \varphi \text{ give the same result.}$$

This is in contrast with the situation when we take

$$\varphi = \log |t|$$

in which case we have in the sense of currents the Poincaré-Lelong formula

$$(IV.B.4) \quad (i/\pi) \partial \bar{\partial} \log |t| = \delta_0$$

where δ_0 is the Dirac δ -function at the origin. Anticipating the discussion below, a characteristic feature of the metrics that arise in Hodge theory will be that the principle (IV.B.3) will hold.

Definition: The *Poincaré coframe* has as basis the (1,0) forms

$$\frac{dt_i}{t_i(-\log |t_i|)}, dw_\alpha$$

and their conjugates.

Definition: A metric in the holomorphic vector bundle $E \rightarrow B$ is said to have *logarithmic singularities* along the divisor $Z = \overline{B} \setminus B$ if locally in an open set \mathcal{U} as above and in terms of a holomorphic frame for the bundles and the Poincaré coframe the metric h , the connection matrix $\theta = h^{-1}\partial h$, and the curvature matrix $\Theta_E = \overline{\partial}(h^{-1}\partial h)$ have entries that are Laurent polynomials in the $\log |t_i|$ with coefficients that are real analytic functions in \mathcal{U} .

PROPOSITION IV.B.5: *The Hodge metrics in the Hodge bundles $F^p \rightarrow B$ have logarithmic singularities relative to the canonically extended Hodge bundles $F_e^p \rightarrow \overline{B}$.*

In the geometric case this result may be inferred from the theorem on regular singular points of the Gauss-Manin connection ([Del70]) and (VI.C.1) above. In the general case it is a consequence of the several variable nilpotent orbit theorem ([CKS86]). More subtle is the behavior of the coefficients of the various quantities, especially the Chern polynomials $P(\Theta_{F^p})$, when they are expressed in terms of the Poincaré frame, a topic analyzed in [CKS86] and where the analysis is refined in [Kol87], and to which we now turn.

We recall that a distribution Ψ on a manifold M has a *singular support* $\Psi_{\text{sing}} \subset M$ defined by the property that on any open set $W \subset M \setminus \Psi_{\text{sing}}$ in the complement the restriction $\Psi|_W$ is given by a smooth volume form. A finer invariant of the singularities of Ψ is given by its *wave front set*³⁹

$$WF(\Psi) \subset T^*M.$$

Among other things the wave front set was introduced to help deal with two classical problems concerning distributions:

- (IV.B.6) (a) distributions cannot in general be multiplied;
 (b) in general distributions cannot be restricted to submanifolds
 $N \subset M$.

For (a) the wave front sets should be transverse, and for (b) to define $\Psi|_N$ it suffices to have $TN \subset WF(\Psi)^\perp$.

In the case of currents represented as differential forms with distribution coefficients, multiplication should be expressed in terms of the usual wedge product of forms. For restriction, if N is locally given by $f_1 = \cdots = f_m = 0$, then for a current Ψ we first set $df_i = 0$; i.e., we cross out any terms with a df_i . Then the issue is to

³⁹A good discussion of wave front sets and references to the literature is given in Wikipedia. We will not use them in a technical sense but rather as a suggestion of an important aspect to be analyzed for the Chern polynomials of the Hodge bundles.

restrict the distribution coefficients of the remaining terms to N . Thus the notion of the wave front set for a current Ψ involves both the differential form terms appearing in Ψ as well as the distribution coefficients of those terms.

Definition: The holomorphic bundle $E \rightarrow B$ has *mild logarithmic singularities* in case it has logarithmic singularities and the following conditions are satisfied:

- (i) the Chern polynomials $P(\Theta_E)$ are closed currents given by differential forms with L^1_{loc} coefficients and which represent $P(c_1(E), \dots, c_r(E))$ in $H^*_{\text{DR}}(\overline{B})$;
- (ii) the products $P(\Theta_E) \cdot P'(\Theta_E)$ may be defined by formally multiplying them as L^1_{loc} -valued differential forms, and when this is done we obtain a representative in cohomology of the products of the polynomials in the Chern classes;
- (iii) the restrictions $P(\Theta_E)|_{Z_I^*}$ are defined and represent $P\left(c_1\left(E|_{Z_I^*}\right), \dots, c_r\left(E|_{Z_I^*}\right)\right)$.

We note the opposite aspects of analytic singularities and mild logarithmic singularities: In the former one wants the singularities to create behavior different from that of smooth metrics, either with regard to the functions that are in L^2 with respect to the singular metric, or to create non-zero Lelong numbers in the currents that arise from their curvatures. In the case of mild logarithmic singularities, basically one may work with them as if there were no singularities at all. An important additional point to be explained in more detail below is that the presence of singularities *increases* the positivity of the Chern forms of the Hodge bundles, so that in this sense one uses singularities to positive effect.

The main result, stated below and which will be discussed in the next section, is that the Hodge bundles have mild logarithmic singularities. This would follow if one could show that

(IV.B.7) *When expressed in terms of the Poincaré frame the polynomials $P(\Theta_E)$ have bounded coefficients.*

This is true when Z is a smooth divisor, but when Z is not a smooth divisor this is not the case and the issue is more subtle.

Main result.

THEOREM IV.B.8 ([CKS86], with amplifications in [Kol87], [GGLR17]): *The Hodge bundles have mild singularities.*

The general issues (IV.B.6)(a), (IV.B.6)(b) concerning distributions were raised above. Since currents are differential forms with distribution coefficients, these issues are also present for currents, where as noted above the restriction issue (IV.B.6)(b) involves both the differential form aspect and the distribution aspect of currents. This is part (iii) of the definition and is the property of the Chern polynomials that appears in [GGLR17].

The proof of (i) and part of (ii) in Theorem IV.B.8 is based on the fundamental results in [CKS86], with refinements in [Kol87] concerning a particular multiplicative property (IV.B.6)(a) in the definition of mild logarithmic singularities,⁴⁰ and in [GGLR17] the general multiplicative property and the restriction property of the Chern polynomials is addressed. Both of these involve estimates in the Δ^* -factors in neighborhoods $\mathcal{U} \cong \Delta^{*k} \times \Delta^\ell$ in \overline{B} . In effect these estimates may be intrinsically thought of as occurring in sectors in the co-normal bundle of the singular support of the Chern forms, and in this sense may be thought of as dealing with the wave front sets of these forms.

A complete proof of Theorem IV.B.8 is given in Section 5 of [GGLR17]. In the next section we shall give the argument for the Chern form $\Omega = c_1(\Theta_{\det F})$ of the Hodge line bundle and in the special case when the localized VHS is a nilpotent orbit. The computation will be explicit; the intent is to provide a perspective on some of the background subtleties in the general argument, one of which we now explain.

We restrict to the case when $\mathcal{U} \cong \Delta^{*k} = \{(t_1, \dots, t_k) : 0 < |t_j| < 1\}$, and setting $\ell(t_j) = \log t_j / 2\pi i$ and $x_j = -\log |t_j|$ consider a nilpotent orbit

$$\Phi(t) = \exp\left(\sum_{j=1}^k \ell(t_j) N_j\right) \cdot F_0.$$

Following explicit computations of the Chern form Ω and of the Chern form Ω_I for the restriction of the Hodge line bundle to Z_I^* , the desired result comes down to showing that a limit

$$(IV.B.9) \quad \lim_{x_j \rightarrow \infty} \frac{Q(x)}{P(x)}$$

exists where $Q(x), P(x)$ are particular homogeneous polynomials of the same degree with $P(x) > 0$ if $x_j > 0$. Limits such as (IV.B.9) certainly do not exist in general, and the issue to be understood is how in the case at hand the very special properties of several parameter limiting mixed Hodge structures imply the existence of the limit.

As an application of Theorems III.B.7 and IV.B.8, using the notations from Section I.C we consider a VHS given by a period mapping

$$\Phi : B \rightarrow \Gamma \backslash D.$$

Denoting by

$$F_e \rightarrow \overline{B}$$

the canonically extended Hodge vector bundle we have

THEOREM IV.B.10:

⁴⁰The specific result in [Kol87] is that the integral

$$\int_{\overline{B}} c_1(\Theta_{\det E})^d < \infty, \quad \dim \overline{B} = d$$

of the top power of the Chern form of the Hodge line bundle is finite. This result also follows from the analysis in Section V of [GGLR17].

(i) *The Kodaira-Itaka dimension*

$$\kappa(F_e) \leq 2h^{n,0} - 1.$$

(ii) *Assuming the injectivity of the end piece $\Phi_{*,n}$ of the differential of Φ ,*

$$\kappa(S^{h^{n,0}} F_e) = \dim \mathbb{P} S^{h^{n,0}} F_e;$$

i.e., $S^{h^{n,0}} F_e \rightarrow \overline{B}$ is big.

Proof. It is well known [Gri70], [CMSP17] that the curvature of the Hodge vector bundle has the norm positivity property. In fact, the curvature form is given by

$$(IV.B.11) \quad \Theta_F(e, \xi) = \|\Phi_{*,n}(\xi)(e)\|^2.$$

Concerning the singularities that arise along $\overline{B} \setminus B$, it follows from Theorem IV.B.8 that we may treat the Chern form ω of $\mathcal{O}_{\mathbb{P}F_e}(1) \rightarrow \overline{B}$ as if the singularities were not present.

The linear algebra situation is

$$(IV.B.12) \quad T \otimes F \rightarrow G$$

where $\dim T = \dim B$, $\dim F = h^{n,0}$ and $\dim G = h^{n-1,1}$. By (IV.B.11) condition (III.B.2) is equivalent to the injectivity of $\Phi_{*,n}$, and Theorem IV.B.10 is then a consequence of Theorem III.B.7. \square

This result gives one answer to the question

The Hodge vector bundle is somewhat positive. Just how positive is it?

Since in the geometric case the linear algebra underlying the map (IV.B.12) is expressed cohomologically, in particular cases the result (i) in (IV.B.10) can be considerably sharpened. For example, in the weight $n = 1$ case the method of proof of the theorem gives the

PROPOSITION IV.B.13 ([Bru16a]): *In weight $n = 1$, (i) $\kappa(F_e) \leq 2g - 1$, and (ii) $S^2 F_e \rightarrow \overline{B}$ is big.*

Proof. In this case $D \subset \mathcal{H}_g$ where $g = h^{1,0}$ and \mathcal{H}_g is the Siegel generalized upper-half-plane. We then have

- $T \subset S^2 V^*$;
- $G = V^*$;
- $T \otimes V \rightarrow G$ is induced by the natural contraction map $S^2 V^* \otimes V \rightarrow V^*$.

For any $v \in V$ the last map has image of dimension $\leq g$, and therefore the kernel has dimension $\geq \dim T - g$. This gives (i) in the proposition.

For (ii) we have

$$\begin{array}{ccc} T \otimes S^2 V & \longrightarrow & V^* \otimes V \\ \cap & \nearrow & \downarrow \\ S^2 V^* \otimes S^2 V & & \end{array}$$

For a general $q \in S^2 V$ the contraction mapping \downarrow is injective, and this implies (ii). \square

At the other extreme we have the

PROPOSITION IV.B.14: *Let $\mathcal{M}_{d,n}$ denote the moduli space of smooth hypersurfaces $Y \subset \mathbb{P}^{n+1}$ of degree $d = 2n + 4$, $n \geq 3$. Then the Hodge vector bundle $F \rightarrow \mathcal{M}_{d,n}$ is big.*

Proof. Set $V = \mathbb{C}^{n+2}$ and let $P \in V^{(d)}$ be a homogeneous form of degree d that defines Y . Denote by $J_P \subset \bigoplus_{k \geq d-1} V^{(k)}$ the Jacobian ideal. Then (cf. Section 5 in [CMSP17])

- $T_Y \mathcal{M}_{d,n} \cong V^{(d)} / J_P^{(d)}$;
- $F_Y = H^{n,0}(Y) \cong V^{(d-n-2)}$;
- $G_Y = H^{n-1,0}(Y) \cong V^{(2d-n-2)} / J_P^{(2d-n-2)}$.

It will suffice to show

(IV.B.15) *For general P and general $Q \in V^{(d-n-2)}$ the mapping*

$$V^{(d)} / J_P^{(d)} \xrightarrow{Q} V^{(2d-n-2)} / J_P^{(2d-n-2)}$$

is injective.

Noting that $d - n - 2 = n + 2$ and that it will suffice to prove the statement for one P and Q , we take

$$\begin{aligned} Q &= x_0 \cdots x_{n+1}, \\ P &= x_0^d + \cdots + x_{n+1}^d. \end{aligned}$$

Then $J_P = \{x_0^{2n+3}, \dots, x_{n+1}^{2n+3}\}$ and a combinatorial argument gives (IV.B.15). \square

Remark IV.B.16: The general principle that Proposition IV.B.14 illustrates is this: Let $L \rightarrow X$ be an ample line bundle. Then both for general smooth sections $Y \in |mL|$ and for cyclic coverings $\tilde{X}_Y \rightarrow X$ branched over a smooth Y , as m increases the Hodge vector bundle $F \rightarrow |mL|^0$ over the open set of smooth Y 's becomes increasingly positive in the sense that the k such that $S^k F$ is big decreases, and for $m \gg 0$ F itself is big.

V. PROOF OF THEOREM IV.B.8

V.A. **Reformulation of the result.** We consider a variation of Hodge structure given by a period mapping

$$\Phi : \Delta^{*k} \rightarrow \Gamma_{\text{loc}} \backslash D.$$

Here we assume that the monodromy generators $T_i \in \text{Aut}_Q(V)$ are unipotent with logarithms $N_i \in \text{End}_Q(V)$; Γ_{loc} is the local monodromy group generated by the T_i .

For $I \subset \{1, \dots, k\}$ with complement $I^c = \{1, \dots, k\} \setminus I$ we set

$$\Delta_I^* = \{(t_1, \dots, t_k) : t_i = 0 \text{ for } i \in I \text{ and } t_j \neq 0 \text{ for } j \in I^c\}.$$

From the work of Cattani-Kaplan-Schmid [CKS86] the limit $\lim_{t \rightarrow \Delta_I^*} \Phi(t)$ is defined as a polarized variation of limiting mixed Hodge structures on Δ_I^* . Passing to the primitive parts of the associated graded polarized Hodge structures gives a period mapping

$$\Phi_I : \Delta_I^* \rightarrow \Gamma_{\text{loc}, I} \backslash D_I$$

where D_I is a product of period domains and $\Gamma_{\text{loc}, I}$ is generated by the T_j for $j \in I^c$. This may be suggestively expressed by writing

$$\lim_{t \rightarrow t_I} \Phi(t) = \Phi_I(t_I).$$

However caution must be taken in interpreting the limit, as the “rate of convergence” is not uniform but depends on the sector in which the limit is taken in the manner explained in [CKS86].

We denote by $\Lambda \rightarrow \Delta^{*k}$ and $\Lambda_I \rightarrow \Delta_I^*$ the Hodge line bundles. The Hodge-Riemann bilinear relations give metrics in these bundles and we denote by Ω and Ω_I the respective Chern forms. The result to be proved is

$$(V.A.1) \quad \lim_{t \rightarrow \Delta_I} \Omega = \Omega_I,$$

where again care must be taken in interpreting this equation. In more detail, this means: In Ω set $dt_i = d\bar{t}_i = 0$ for $i \in I$. Then the limit, in the usual sense, as $t \rightarrow \Delta_I$ of the remaining terms exists and is equal to Ω_I . We will write (V.A.1) as

$$(V.A.2) \quad \Omega|_{\Delta_I^*} = \Omega_I.$$

The proof of (V.A.1) that we shall give can easily be adapted to the case when the period mapping depends on parameters.

The limit can also be reduced to the case when Φ is a *nilpotent orbit*. This means that

$$(V.A.3) \quad \Phi(t) = \exp\left(\sum_{i=1}^k \ell(t_i) N_i\right) F$$

where $F \in \check{D}$ and the conditions

- (i) $N : F^p \rightarrow F^{p-1}$,
- (ii) $\Phi(t) \in D$ for $0 < |t| < \epsilon$

are satisfied. This reduction is non-trivial and is given in Section 5 of [GGLR17].

The main points in the proof of Theorem IV.B.8 in the nilpotent orbit case are as follows:

- (a) without changing the associated graded's to Φ and Φ_I we may replace the F in (V.A.3) by an F_0 such that the limiting mixed Hodge structure is \mathbb{R} -split;
- (b) in this case N_I can be completed to an \mathfrak{sl}_2 which we denote by $\{N_I^+, Y_I, N_I\}$;⁴¹
- (c) the Y_I -weight decomposition of N_{I^c} is

$$N_{I^c} = N_{I^c, 0} + N_{I^c, -1} + N_{I^c, -2} + \cdots$$

where $N_{I^c, -m}$ has Y_I -weight $-m$, $m \geq 0$;

- (d) if all the $N_{I^c, -m} = 0$ for $m > 0$, then there is an $\mathfrak{sl}_2^c = \{N_{I^c}^+, Y_{I^c}, N_{I^c}\}$ that commutes with the previous \mathfrak{sl}_2 , and the result (V.A.1) is immediate;

⁴¹There are two ways of doing this—one is the method in [CKS86] and the other one, which is purely linear algebra, is due to Deligne.

(e) in general, by direct computation we have

$$\Omega_I \equiv \Omega + R \pmod{dt_i, d\bar{t}_i} \text{ for } i \in I$$

where the remainder term R consists of expressions $Q(x)/P(x)$ as in (IV.B.9), and then direct computation using the relative filtration property and the fact that for $m > 0$ the $N_{I^c, -m}$ have negative Y_I -weights gives the result.

V.B. Weight filtrations, representations of \mathfrak{sl}_2 and limiting mixed Hodge structures.

The proof of (V.A.1) will be computational, using only that $\Phi(t)$ is a nilpotent orbit (V.A.3) and that the commuting $N_i \in \text{End}_Q(V)$ have the *relative weight filtration property* (RWFP), which will be reviewed below.⁴² The computation will be facilitated by using the representation theory of \mathfrak{sl}_2 adapted to the Hodge theoretic situation at hand. The non-standard but hopefully suggestive notations for doing this will now be explained.

(i) Given a nilpotent transformation $N \in \text{End}_Q(V)$ with $N^{n+1} = 0$ there is a unique increasing weight filtration $W(N)$ given by subspaces

$$(V.B.1) \quad V_k^{W(N)} := W_k(N)V$$

satisfying the conditions

- $N : V_k^{W(N)} \rightarrow V_{k-2}^{W(N)}$,
- $N^k : V_{n+k}^{W(N)} \xrightarrow{\sim} V_{n-k}^{W(N)}$ (Hard Lefschetz property).

Remark that the two standard choices for the ranges of indices in (V.B.1) are

$$\begin{cases} 0 \leq k \leq 2n & \text{(Hodge theoretic)} \\ -n \leq k \leq n & \text{(representation theoretic)}. \end{cases}$$

We will use the first of these.

The weight filtration is self-dual in the sense that using the bilinear form Q

$$(V.B.2) \quad V_k^{W(N)\perp} = V_{2n-k-1}^{W(N)}$$

which gives

$$V_k^{W(N)*} \cong V/V_{2n-k-1}^{W(N)}.$$

The *associated graded* to the weight filtration is the direct sum of the

$$\text{Gr}_\ell^{W(N)} V := V_\ell^{W(N)} / V_{\ell-1}^{W(N)},$$

and the *primitive subspaces* are defined for $\ell \geq n$ by

$$\text{Gr}_{n+k, \text{prim}}^{W(N)} V = \ker \left\{ N^{k+1} : \text{Gr}_{n+k}^{W(N)} V \rightarrow \text{Gr}_{n-k-2}^{W(N)} V \right\}.$$

⁴²The proof in [GGLR17] uses the detailed analysis of limiting mixed Hodge structures from [CKS86], of which the RWFP is one consequence. Part of the point for the argument given here is to isolate the central role played by that property.

(ii) A *grading element* for $W(N)$ is given by a semi-simple $Y \in \text{End}_Q(V)$ with integral eigenvalues $0, 1, \dots, 2n$, weight spaces $V_k \subset V_k^{W(N)}$ for the eigenvalue k , and where the induced maps

$$V_k \xrightarrow{\sim} \text{Gr}_k^{W(N)} V$$

are isomorphisms. Thus

$$V_k^{W(N)} = \bigoplus_{\ell=0}^k V_\ell.$$

Grading elements always exists, and for any one such Y we have

- $[Y, N] = -2N$;
- there is a unique $N^+ \in \text{End}_Q(V)$ such that $\{N^+, Y, N\}$ is an \mathfrak{sl}_2 -triple.

The proof of the second of these uses the first together with the Hard Lefschetz property of $W(N)$.

We denote by \mathcal{U} the standard representation of \mathfrak{sl}_2 with weights $0, 1, 2$. Thinking of \mathcal{U} as degree 2 homogeneous polynomials in x, y we have

- weight $x^a y^b = 2a$, $a + b = 2$;
- $N = \partial_x$ and $N^+ = \partial_y$.

We denote by

$$\mathcal{U}_i = \text{Sym}^i \mathcal{U} \cong \left\{ \begin{array}{l} \text{homogeneous polynomials} \\ \text{in } x, y \text{ of degree } i + 1 \end{array} \right\}$$

the standard $(i + 1)$ -dimensional irreducible representation of \mathfrak{sl}_2 . The N -string associated to \mathcal{U}_i is

$$\{x^{i+1}\} \rightarrow \{x^i y\} \rightarrow \dots \rightarrow \{y^{i+1}\}$$

where $N = \partial_x$. The top of the N -string is the primitive space.

Given (V, Q, N) as above and a choice of a grading element Y , for the \mathfrak{sl}_2 -module

$$V_{\text{gr}} = \bigoplus_{k=0}^{2n} V_k$$

we have a unique identification

$$(V.B.3) \quad V_{\text{gr}} \cong \bigoplus_{i=0}^n H^{n-i} \otimes \mathcal{U}_i$$

for vector spaces H^{n-i} . The notation is chosen for Hodge-theoretic purposes. The N -string associated to $H^{n-i} \otimes \mathcal{U}_i$ will be denoted by

$$(V.B.4) \quad H^{n-i}(-i) \xrightarrow{N} H^{n-i}(-(i-1)) \xrightarrow{N} \dots \xrightarrow{N} H^{n-i}$$

and we define

$$(V.B.5) \quad \text{the Hodge-theoretic weight of } H^{n-i}(-j) \text{ is } n - i + 2j.$$

The representation-theoretic weight of $H^{n-i}(-j)$ is $2j$. It follows that $H^{n-i}(-i)$ is the primitive part of the \mathcal{U}_i -component of V_{gr} .

Relative to Q the decomposition (V.B.3) is orthogonal and the pairing

$$Q_i : H^{n-i}(-i) \otimes H^{n-i}(-i) \rightarrow \mathbb{Q}$$

given by

$$(V.B.6) \quad Q_i(u, v) = Q(N^i u, v)$$

is non-degenerate.

(iii) We recall the

Definition: A *limiting mixed Hodge structure* (LMHS) is a mixed Hodge structure $(V, Q, W(N), F)$ with weight filtration $W(N)$ defined by a nilpotent $N \in \text{End}_Q(V)$ and Hodge filtration F which satisfies the conditions

- (a) $N : F^p \rightarrow F^{p-1}$;
- (b) the form Q_i in (V.B.6) polarizes $\text{Gr}_{n+k, \text{prim}}^{W(N)} V \cong H^{n-k}(-k)$.

The MHS on V induces one on $\text{End}_Q(V)$, and (a) is equivalent to

$$N \in F^{-1} \text{End}_Q(V).$$

We denote by

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

the unique Deligne decomposition of $V_{\mathbb{C}}$ that satisfies

- $W_k(N)V = \bigoplus_{p+q \leq k} I^{p,q}$;
- $F^p V = \bigoplus_{\substack{p' \geq p \\ q}} I^{p',q}$;
- $\bar{I}^{q,p} \equiv I^{p,q} \text{ mod } W_{p+q-2}(N)V$.

The LMHS is \mathbb{R} -split if $\bar{I}^{p,q} = I^{q,p}$. Canonically associated to a LMHS is an \mathbb{R} -split one $(V, Q, W(N), F_0)$ where

$$F_0 = e^{-\delta} F$$

for a canonical $\delta \in I^{-1,-1} \text{End}_Q(V_{\mathbb{R}})$. For this \mathbb{R} -split LMHS there is an evident grading element $Y \in I^{0,0}(\text{End}_Q(V_{\mathbb{R}}))$.

Given a LMHS $(V, Q, W(N), F)$ there is an associated nilpotent orbit

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \Gamma_T \backslash D \\ \Psi & & \Psi \\ t & \longrightarrow & \exp(\ell(t)N)F \end{array}$$

where $\ell(t) = \log t / 2\pi i$ and $\Gamma_T = \{T^{\mathbb{Z}}\}$. Conversely, given a 1-variable nilpotent orbit as described above there is a LMHS. We shall use consistently the bijective correspondence

$$\text{LMHS's} \iff \text{1-parameter nilpotent orbits.}$$

Since

$$\det F_0 = \det F$$

without loss of generality for the purposes of this paper we will assume that our LMHS's are \mathbb{R} -split and therefore have canonical grading elements.

(iv) Let $N_1, N_2 \in \text{End}_Q(V)$ be commuting nilpotent transformations and set $N = N_1 + N_2$. Then there are two generally different filtrations defined on the vector space $\text{Gr}^{W(N_1)} V$:

- (A) the weight filtration $W(N)V$ induces a filtration on any sub-quotient space of V , and hence induces a filtration on $\text{Gr}^{W(N_1)} V$;
- (B) N induces a nilpotent map $\bar{N} : \text{Gr}^{W(N_1)} V \rightarrow \text{Gr}^{W(N_1)} V$, and consequently there is an associated weight filtration $W(\bar{N}) \text{Gr}^{W(N_1)} V$ on $\text{Gr}^{W(N_1)} V$.

Definition: The *relative weight filtration property* (RWFP) is that these two filtrations coincide:

$$(V.B.7) \quad W(N) \cap \text{Gr}^{W(N_1)} V = W(\bar{N}) \text{Gr}^{W(N_1)} V.^{43}$$

We note that \bar{N} is the same as the map induced by N_2 on $\text{Gr}^{W(N_1)} V$, so that (V.B.7) may be perhaps more suggestively written as

$$(V.B.8) \quad W(N) \cap \text{Gr}^{W(N_1)} V = W(N_2) \text{Gr}^{W(N_1)} V.$$

The RWFP is a highly non-generic condition on a pair of commuting nilpotent transformation, one that will be satisfied in our Hodge-theoretic context.

(v) Suppose now that Y_1 is a grading element for N_1 so that the corresponding $\mathfrak{sl}_2 = \{N_1^+, Y_1, N_1\}$ acts on V and hence on $\text{End}_Q(V)$. We observe that the Y_1 -eigenspace decomposition of N_2 is of the form

$$(V.B.9) \quad N_2 = N_{2,0} + N_{2,-1} + \cdots + N_{2,-m}, \quad m > 0$$

where $[Y, N_{2,-m}] = -mN_{2,-m}$. The reason for this is that

$$[N_1, N_2] = 0 \implies \left\{ \begin{array}{l} N_2 \text{ is at the bottom of the } N_1\text{-strings} \\ \text{for } N_1 \text{ acting on } \text{End}_Q(V) \end{array} \right\}.$$

It can be shown that there is an $\mathfrak{sl}'_2 = \{N_2^+, Y_2, N_{2,0}\}$ that commutes with the \mathfrak{sl}_2 above. Thus

- (V.B.10) Given N_1, N_2 as above, there are commuting \mathfrak{sl}_2 's with N_1 and $N_{2,0}$ as nil-negative elements. Moreover, $N_2 = N_{2,0} + (\text{terms of strictly negative weights})$ relative to $\{N_1^+, Y_1, N_1\}$.

It is the ‘‘strictly negative’’ that will be an essential ingredient needed to establish that the limit exists in the main result.

V.C. Calculation of the Chern forms Ω and Ω_I .

Step 1: For a nilpotent orbit (V.A.3) holomorphic sections of the canonically extended VHS over Δ are given by

$$\exp \left(\sum_{j=1}^k \ell(t_j) N_j \right) v, \quad v \in V_{\mathbb{C}}.$$

⁴³There is a shift in indices that will not be needed here (cf. (VI.A.13) below).

Up to non-zero constants the Hodge metric is

$$\begin{aligned} (u, v) &= Q \left(\exp \left(\sum_j \ell(t_j) N_j \right) u, \exp \left(\sum_j \ell(t_j) N \right) \bar{v} \right) \\ &= Q \left(\exp \left(\sum_j \log |t_j|^2 N_j \right) u, \bar{v} \right). \end{aligned}$$

Using the notation (V.B.3) the associated graded to the LMHS as $t \rightarrow 0$ will be written as

$$(V.C.1) \quad V_{\text{gr}} = \bigoplus_{i=0}^n H^{n-i} \otimes \mathcal{U}_i$$

and

$$F^n = \bigoplus_{i=0}^n H^{n-i,0}.$$

For $u \in H^{n-i,0}(-i)$ and $v \in H^{n-i,0}$

$$Q \left(\exp \left(\sum_j \log |t_j|^2 N_j \right) u, \bar{v} \right) = \left(\frac{1}{i!} \right) Q \left(\left(\sum_j \log |t_j|^2 N_j \right)^i u, \bar{v} \right).$$

Setting

$$x_j = -\log |t_j|$$

the metric on the canonically extended line bundle is a non-zero constant times

$$(V.C.2) \quad P(x) = \prod_{i=0}^n \det \left(\left(\sum_j x_j N_j \Big|_{H^{n-i,0}} \right)^i \right).$$

Here to define “det” we set $N = \sum N_j$ and are identifying $H^{n-i}(-i)$ with H^{n-i} using N^i . Note that the homogenous polynomial $P(x)$ is positive in the quadrant $x_j > 0$. The Chern form is

$$(V.C.3) \quad \Omega = \partial \bar{\partial} \log P(x).$$

Step 2: Define

$$\begin{cases} N_I = \sum_{i \in I} x_i N_i, & N_{I^c} = \sum_{j \notin I} x_j N_j \\ N = \sum_{i=1}^k x_i N_i = N_I + N_{I^c} \end{cases}$$

and set

$$P = \prod_{i=0}^n \det \left(N \Big|_{H^{n-i,0}(-i)} \right)^i.$$

Denoting by

$$V_{\text{gr},I} = \bigoplus_{i=0}^n H_I^{n-i,i} \otimes \mathcal{U}_i$$

the associated graded to the LMHS as $t \rightarrow \Delta_I^*$, we define

$$P_I = \prod_{i=0}^n \det \left(N_I \Big|_{H_I^{n-i,0}(-i)} \right)^i.$$

Taking $N_I = N_1$ and $N_{I^c} = N_2$ in (iv) in Section V.B, we have

$$N_{I^c,0} = \text{weight zero component of } N_{I^c}$$

where weights are relative to the grading element Y_I for N_I . Decomposing the RHS of (V.C.1) using the $\mathfrak{sl}_2 \times \mathfrak{sl}'_2$ corresponding to N_I and $N_{I^c,0}$ by (IV.B.10) we obtain

$$(V.C.4) \quad V_{\text{gr}} \cong \bigoplus_{i,j} H_{i,j}^{n-i-j} \otimes \mathcal{U}_i \otimes \mathcal{U}_j$$

where $H_{i,j}^{n-i-j}$ is a polarized Hodge structure of weight $n - i - j$. Note that this decomposition depends on I . On $H_{i,j}^{n-i-j} \otimes \mathcal{U}_i \otimes \mathcal{U}_j$ we have a commutative square

$$\begin{array}{ccc} H^{n-i-j}(-i-j) & \xrightarrow{N_I^i} & H_{i,j}^{n-i-j}(-j) \\ N_{I^c,0}^j \downarrow & & \downarrow N_{I^c,0}^j \\ H_{i,j}^{n-i-j}(-i) & \xrightarrow{N_I^i} & H_{i,j}^{n-i-j}. \end{array}$$

Using (V.C.4) this gives

$$P = \prod_{i,j} \det \left(N_I^i N_{I^c,0}^j \Big|_{H_{i,j}^{n-i-j}(-i-j)} \right) + R$$

where the remainder term R involves the $N_{I^c,-m}$'s for $m > 0$. We may factor the RHS to have

$$P = \prod_{i,j} \det \left(N_I^i \Big|_{H_{i,j}^{n-i-j}(-i-j)} \right) \prod_{i,j} \det \left(N_{I^c,0}^j \Big|_{H_{i,j}^{n-i-j}(-i-j)} \right) + R$$

which we write as

$$(V.C.5) \quad P = P_I \cdot P_{I^c} + R$$

where P_I and P_{I^c} are the two $\prod_{i,j}$ factors. We note that

$$(V.C.6) \quad \text{the remainder term } R = 0 \text{ if we have commuting } \mathfrak{sl}_2 \text{ 's.}^{44}$$

We next have the important observation

LEMMA V.C.7: P_{I^c} is the Hodge metric in the line bundle $\Lambda_I \rightarrow Z_I^*$.

Proof. This is a consequence of the RWFP (V.B.7) applied to the situation at hand when we take $N_1 = N_I$ and $N_2 = N_{I^c}$. \square

⁴⁴To have commuting \mathfrak{sl}_2 's means that $N_{2,-m} = 0$ for $m > 0$.

By (V.C.6), if we have commuting \mathfrak{sl}_2 's, then $R = 0$

$$\begin{aligned}\Omega &= -\partial\bar{\partial}\log P = -\partial\bar{\partial}\log P_I - \partial\bar{\partial}R_{I^c} \\ &\equiv -\partial\bar{\partial}\log P_{I^c} \text{ modulo } dt_i, d\bar{t}_i \text{ for } i \in I \\ &\equiv \Omega_I\end{aligned}$$

and we are done.

In general, we have

$$(V.C.8) \quad \Omega \equiv \Omega_I + S_1 + S_2$$

where

$$(V.C.9) \quad \begin{cases} S_1 = \frac{\partial P_{I^c} \wedge \bar{\partial} R + \partial R \wedge \bar{\partial} P_{I^c} - P_{I^c} \partial \bar{\partial} R}{P_I P_{I^c}} \\ S_2 = \frac{\partial R \wedge \bar{\partial} R}{P_I^2 P_{I^c}^2}. \end{cases}$$

Step 3: We will now use specific calculations to analyze the correction terms S_1, S_2 . The key point will be to use that

$$N = N_I + N_{I^c} = \underbrace{N_I + N_{I^c,0}} + \underbrace{\sum_{m \geq 1} N_{I^c,-m}}$$

where the terms over the first brackets may be thought of as “the commuting \mathfrak{sl}_2 -part of N_I, N_{I^c} ” and the correction term over the second bracket has negative Y_I -weights.

We set

$$h_I^{n-i,0} = \dim H_I^{n-i,0}$$

and for a monomial $\mathcal{P} = x_1^{\ell_1} \cdots x_k^{\ell_k}$ we define

$$\deg_I \mathcal{P} = \sum_{i \in I} \ell_i.$$

LEMMA V.C.10:

(i) *For any monomial \mathcal{P} appearing in P*

$$\deg_I \mathcal{P} \leq n h_I^0 + (n-1) h_I^{1,0} + \cdots + h_I^{n-i,0} = \sum_{i=1}^n i h_I^{n-i,0}.$$

(ii) *If π is any permutation of $1, \dots, k$ and*

$$\ell_{\pi,i} = \sum_{j=1}^n j \left(h_{\{\pi(1), \dots, \pi(i)\}}^{n-j,0} - h_{\{\pi(1), \dots, \pi(i-1)\}}^{n-j,0} \right)$$

then

$$\mathcal{P}_\pi := x_{\pi(1)}^{\ell_{\pi,1}} \cdot x_{\pi(2)}^{\ell_{\pi,2}} \cdots x_{\pi(k)}^{\ell_{\pi,k}} = x_1^{\ell_{\pi,\pi^{-1}(k)}} \cdots x_k^{\ell_{\pi,\pi^{-1}(k)}}$$

appears with a non-zero coefficient in P .

COROLLARY: *The monomials appearing in P are in the convex hull of the monomials \mathcal{P}_π .*

Proof. For $V_{\text{gr},I} = \text{Gr}^{W(N_I)} V$ we have as $\{N_I^+, Y_I, N_I\}$ -modules

$$V_{\text{gr},I} \cong \bigoplus_{i=0}^n H_I^{n-i} \otimes \mathcal{U}_i.$$

Decomposing the RHS as \mathfrak{sl}'_2 -modules we have

$$V_{\text{gr},I} \cong \bigoplus_{i=0}^n H_{a,i-a}^{n-i} \otimes \mathcal{U}_a \otimes \mathcal{U}'_{i-a}$$

where the $H_{a,i-a}^{n-i}$ depend on I . The map

$$N^i : H^{n-i,0}(-i) \rightarrow H^{n-i}$$

gives

$$\bigoplus_{a=0}^i H_{a,i-a}^{n-i,0}(-i) \rightarrow \bigoplus_{a=0}^i H_{a,i-a}^{n-i,0}.$$

The Y_I -weights of vectors in $H_{a,i-a}^{n-i,0}$ are equal to a , and thus $\wedge^{h^{n-i,0}} \left(\bigoplus_{a=0}^i H_{a,i-a}^{n-i,0}(-i) \right)$ has weight $\sum ah_{a,i-a}^{n-i,0}$ and $\wedge^{h^{n-i,0}} \left(\bigoplus_{a=0}^i H_{a,i-a}^{n-i,0} \right)$ has weight $-\sum ah_{a,i-a}^{n-i,0}$. As a consequence

any monomial in $\det \left(N^i \big|_{H^{n-i,0}(-i)} \right)$ drops weights by $2 \sum h_{a,i-a}^{n-i,0}$.

We have

$$(V.C.11) \quad \det \left(\left(N \big|_{H^{n-i,0}(-i)} \right)^i \right) = \det \left(\left((N_I + N_{I^c,0}) \big|_{H^{n-i,0}(-i)} \right)^i \right) + T$$

where $T =$ terms involving $N_{I^c,\text{neg}}$. For any monomial \mathcal{P} in a minor involving the $N_{I^c,\text{neg}}$ of total weight $-d$,

$$2 \deg_I \mathcal{P} + d = 2 \sum_{a=0}^i ih_{a,i-a}^{n-i,0}, \quad d > 0$$

and so

$$\deg_I \mathcal{P} < \sum_{a=0}^i ah_{a,i-a}^{n-i,0}.$$

Putting everything together, we have (V.C.10) where

$$(V.C.12) \quad T = \left\{ \begin{array}{l} \text{linear combinations of monomials } \mathcal{P} \\ \text{satisfying } \deg_I \mathcal{P} < \sum_{i=0}^n \sum_{a=0}^i ah_{a,i-a}^{n-i,0} \end{array} \right\}.$$

Using the bookkeeping formula $h_I^{n-i,0} = \sum_{i=a}^n h_{a,i-a}^{n-i,0}$ we obtain

$$\sum_{i=0}^n \sum_{a=0}^i ah_{a,i-a}^{n-i,0} = \sum_{a=0}^i \sum_{i=a}^n h_{a,i-a}^{n-i,0} = \sum_{a=0}^n ah_I^{n-a,0}$$

which gives

$$P = P_I P_{I^c} + \left(\text{correction term with } \deg_I < \sum_{a=0}^n ah_I^{n-a,0} \right)$$

where $\deg_I P_I = \sum_{a=0}^n ah_I^{n-a,0}$ and $\deg_I P_{I^c} = 0$, giving (i) in (V.C.10).

A parallel argument shows that for $I \cap J = \emptyset$

$$\begin{aligned} D_{I \cup J} &:= \prod_{i=0}^n \det \left(\left(N_I + N_{I,0} \Big|_{H_{I \cup J}^{n-i,0}(-i)} \right)^i \right) \\ &\quad + \text{a correction with } \deg_I < \sum_{a=0}^m ah_I^{n-a,0}. \end{aligned}$$

By the definition of $H_{I \cup J}^{n-i}$,

$$\det \left(\left(N_I + N_{J,0} \Big|_{H_{I \cup J}^{n-i,0}(-i)} \right)^i \right) \neq 0$$

and

$$\deg_I \left(\det \left(\left(N_I + N_{J,0} \Big|_{H_{I \cup J}^{n-i,0}(-i)} \right)^i \right) \right) = \sum_{a=0}^n ah_I^{n-a,0}$$

while automatically

$$\deg_{I \cup J}(\text{all terms of } D_{I \cup J}) = \sum_{a=0}^n ah_{I \cup J}^{n-a,0}.$$

Thus

$$\begin{aligned} \deg_J \det \left(\left(N_I + N_{J,0} \Big|_{H_{I \cup J}^{n-i,0}(-i)} \right)^i \right) &= \deg_{I \cup J} \det \left(\left(N_I + N_{J,0} \Big|_{H_{I \cup J}^{n-i,0}(-i)} \right)^i \right) \\ &\quad - \deg_I \det \left(\left(N_I + N_{J,0} \Big|_{H_{I \cup J}^{n-i,0}(-i)} \right)^i \right) \\ &= \sum_{a=0}^n a (h_{I \cup J}^{n-a,0} - h_I^{n-a,0}). \end{aligned}$$

Proceeding inductively on $\{\pi(1)\} \subset \{\pi(1), \pi(2)\} \subset \dots \subset \{\pi(1), \dots, \pi(k)\}$ we obtain, if $N_{\{\pi(1), \dots, \pi(\ell)\}, 0} =$ weight 0 piece of $N_{\{\pi(1), \dots, \pi(\ell)\}}$ with respect to $\text{Gr}^{W(N)}_{\{\pi(1), \dots, \pi(k)\}}$ then

$$\prod_{i=0}^n \det \left(\left(N_{\{\pi(1)\}} + N_{\{\pi(1), \pi(2)\}, 0} + \dots + N_{\{\pi(1), \dots, \pi(k)\}, 0} \Big|_{H^{n-i,0}} \right)^i \right)$$

is a *non-zero* multiple of $x_{\pi(1)}^{\ell_1} x_{\pi(2)}^{\ell_2} \dots x_{\pi(k)}^{\ell_k}$. This is our \mathcal{P}_π . Tracking the correction terms we have

$$P = \sum_{\pi} C_{\pi} \mathcal{P}_{\pi} + \text{terms strictly in the convex hull of the } \mathcal{P}_{\pi}$$

where $C_{\pi} \neq 0$ for all π . This proves (ii) in V.C.10. □

Step 4: Referring to (V.C.8) and (V.C.9), from Lemma V.C.10 we have:

- (a) R_1 has $\deg_I R_1 < \deg_I P_I$, and all monomials satisfy (i) in V.C.10.
- (b) R_2 is a sum of products of monomials $\mathcal{P}_1 \mathcal{P}_2$ where each $\deg_I \mathcal{P} < \deg_I P_I$ and \mathcal{P}_i satisfies (i) in V.C.10.

To complete the proof we have

LEMMA V.C.13: *Given a monomial \mathcal{P} in the I -variables satisfying $\deg_I \mathcal{P} < \deg_I P_I$ and (ii) in Lemma V.C.10,*

$$\lim_{t \rightarrow \Delta_I^*} \mathcal{P}/P_I = 0.$$

Proof. Implicit in the lemma is that the limit exists. We note that $t \rightarrow \Delta_I^*$ is the same as $x_i \rightarrow \infty$ for $i \in I$. We also observe that the assumptions in the lemma imply that there is a positive degree monomial \mathcal{P}' with $\deg_I(\mathcal{P}'\mathcal{P}) = \deg_I P_I$ and where $\mathcal{P}'\mathcal{P}$ lies in the convex hull of the \mathcal{P}_π 's for P_I . Using this convex hull property we will show that

$$(V.C.14) \quad \mathcal{P}'\mathcal{P}/P_I \text{ is bounded as } x_i \rightarrow \infty \text{ for } i \in I.$$

Since $\lim_{x_i \rightarrow \infty} \mathcal{P}'(x) = \infty$, this will establish the lemma.

We now turn to the proof of (V.C.14). Because the numerator and denominator are homogeneous of the same degree, the ratio is the same for (x_1, \dots, x_k) and $(\lambda x_1, \dots, \lambda x_k)$, $\lambda > 0$.

For simplicity, reindex so that $I = \{1, \dots, d\}$. Suppose that $x_\nu = (x_{\nu 1}, \dots, x_{\nu d})$ is a sequence of points in $(x_i > 0, i \in I)$ such that

$$\lim_{\nu \rightarrow \infty} \frac{\mathcal{P}'\mathcal{P}(x_\nu)}{D_I(x_\nu)} = \infty.$$

Consider a successive set of subsequences such that for all i, j , we have one of three possibilities:

- (i) $\lim_{\nu \rightarrow \infty} x_{\nu i}/x_{\nu j} = \infty$;
- (ii) $x_{\nu i}/x_{\nu j}$ is bounded above and below, which we write as $x_{\nu i} \equiv x_{\nu j}$;
- (iii) $\lim_{\nu \rightarrow \infty} x_{\nu i}/x_{\nu j} = 0$.⁴⁵

Now replace our sequence by this subsequence. Let I_i, \dots, I_r be the partition of I such that

$$i \equiv j \iff \text{(ii) holds for } i, j$$

and order them so that (i) holds for $i, j \iff i \in I_{m_1}, j \in I_{m_2}$ and $m_1 < m_2$. We may thus find a $C > 0$ such that $\frac{1}{C} \leq x_{\nu i}/x_{\nu j} \leq C$ if i, j in same I_m , and for any $B > 0$

$$x_{\nu i}/x_{\nu j} > B^{m_2 - m_1} \text{ if } i \in I_{m_1}, j \in I_{m_2}, \nu \text{ sufficiently large.}$$

By compactness, we may pick a subsequence so that $\lim_{\nu \rightarrow \infty} (x_{\nu i}/x_{\nu j}) = C_{ij}$ if $i, j \in$ same I_m .

Now introduce variables y_1, \dots, y_d and let

$$x_i = a_i y_m \text{ if } i \in I_m, \quad a_i/a_j = C_{ij}, a_i > 0.$$

⁴⁵In effect we are doing a sectoral analysis in the co-normal bundle to the stratum $\Delta^* - I$, which explains the wave front set analogy mentioned above.

We may restrict our cone by taking

$$\tilde{N}_m = \sum_{i \in I_m} a_i N_i.$$

This reduces us to the case $|I_m| = 1$ for all m , i.e.,

$$\lim_{\nu \rightarrow \infty} x_{\nu i} / x_{\nu j} = \infty \text{ if } i < j.^{46}$$

Thus for any B ,

$$x_{\nu i} / x_{\nu j} > B^{j-i} \text{ for } \nu \gg 0.$$

Now

$$\frac{x_{\nu_1}^{m_1} x_{\nu_2}^{m_2} \cdots x_{\nu_d}^{m_d}}{x_{\nu_1}^{\ell_1} x_{\nu_2}^{\ell_2} \cdots x_{\nu_d}^{\ell_d}} \rightarrow 0$$

if $m_1 + m_2 + \cdots + m_d = \ell_1 + \ell_2 + \cdots + \ell_d$ and $m_1 < \ell_1$, or $m_1 = \ell_1$ and $m_2 < \ell_2, \dots$. Thus

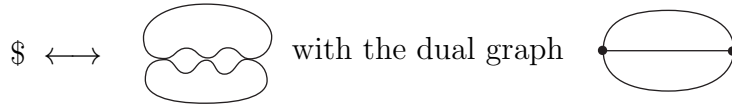
$$P_I = cM_{\{1,2,\dots,d\}} + \text{terms of slower growth as } \nu \rightarrow \infty, c > 0,$$

i.e.,

$$(\mathcal{P}_{\{1,2,\dots,d\}} / \text{other terms})(x_\nu) > B.$$

Since $\mathcal{P}'\mathcal{P}$ belongs to the convex hull of the \mathcal{P}_π , $(\mathcal{P}'\mathcal{P} / \mathcal{P}_{\{1,2,\dots,d\}})(x_\nu)$ is bounded as $\nu \rightarrow \infty$. This proves the claim. \square

Example V.C.15: An example that illustrates most of the essential points in the argument is provided by a neighborhood Δ^3 of the dollar bill curve



in $\overline{\mathcal{M}}_2$. The family may be pictured as follows:

⁴⁶In effect we are making a generalized base change $\Delta^{*d} \rightarrow \Delta^{*k}$ such that for the pullback to Δ^{*d} the coordinates y_m go to infinity at different rates.

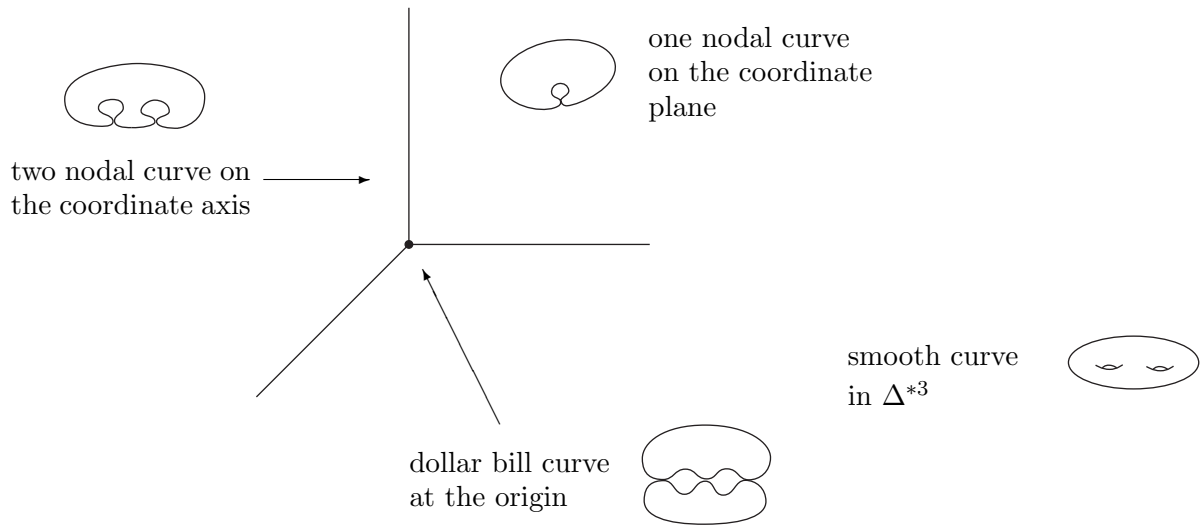
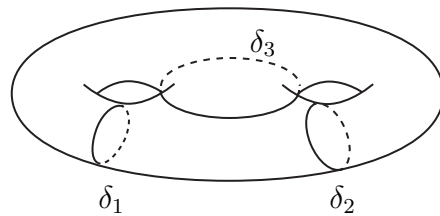


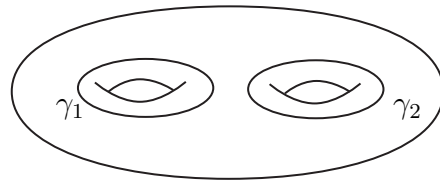
FIGURE 1

With the picture



each of the coordinate planes outside the axes is a family of nodal curves where one of the vanishing cycles δ_i has shrunk to a point. Along each of the coordinate axes two of the three cycles have shrunk to a second node, and at the origin we have the dollar bill curve.

We complete the δ_i to a symplectic basis by adding cycles γ_i .⁴⁷



The corresponding monodromies around the coordinate axes are Picard-Lefschetz transformations with logarithms

$$N_i(\mu) = (\mu, \delta_i)\delta_i$$

⁴⁷Here γ_3 is not drawn in.

and with matrices

$$N_1 = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \quad N_2 = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \quad N_3 = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right).$$

Setting as usual $\ell(t) = \log t/2\pi i$, the normalized period matrix is

$$\Omega(t) = \begin{pmatrix} \ell(t_1) + \ell(t_3) & \ell(t_3) \\ \ell(t_3) & \ell(t_2) + \ell(t_3) \end{pmatrix} + \begin{pmatrix} \text{holomorphic} \\ \text{term} \end{pmatrix}.$$

The corresponding nilpotent orbit is obtained by taking the value at the $t_i = 0$ of the holomorphic term, and by rescaling this term may be eliminated.

Setting $L(t) = (-\log |t|)/4\pi^2$ and $\text{PM}(t) = (i/2)\bar{\partial}\partial \log L(t)$, the metric in the canonically framed Hodge vector bundle is the 2×2 Hermitian matrix

$$H(t) = \begin{pmatrix} L(t_1) + L(t_3) & L(t_3) \\ L(t_3) & L(t_2) + L(t_3) \end{pmatrix};$$

in the Hodge line bundle the metric is

$$\begin{aligned} h(t) &= L(t_1)L(t_2) + (L(t_1) + L(t_2))L(t_3) \\ &= L(t_1)L(t_2) + L(t_1t_2)L(t_3). \end{aligned}$$

Setting

$$\begin{aligned} \omega &= \partial\bar{\partial} \log h(t) = \partial\bar{\partial}(\log(L(t_1)L(t_2) + L(t_1t_2)L(t_3))) \\ \omega_3 &= \partial\bar{\partial} \log L(t_1t_2) \end{aligned}$$

we will show that

$$(V.C.16) \quad \omega|_{t_3=0} \text{ is defined and is equal to } \omega_3.$$

Proof. Setting $\psi = \partial h/h$ and $\eta = \partial\bar{\partial} h/h$ we have

$$\omega = -\psi \wedge \bar{\psi} + \eta.$$

Now

$$\psi = \frac{\partial(L(t_1)L(t_2) + L(t_2t_2)L(t_3))}{L(t_1)L(t_2) + L(t_1t_2)L(t_3)}.$$

Setting $dt_3 = 0$ the dominant term of what is left is the left-hand term in

$$\frac{\partial L(t_1t_2)}{\frac{L(t_1)L(t_2)}{L(t_3)} + L(t_1t_2)} \longrightarrow \frac{\partial L(t_1t_1)}{L(t_1t_2)},$$

and the arrow means that the limit as $t_3 \rightarrow 0$ exists and is equal to the term on the right.

For η , letting \equiv denote modulo dt_3 and $\overline{dt_3}$ and taking the limit as above

$$\eta \equiv \frac{\partial\bar{\partial} L(t_1t_2)}{\frac{L(t_1)L(t_2)}{L(t_3)} + L(t_1t_2)} \longrightarrow \frac{\partial\bar{\partial} L(t_1t_2)}{L(t_1t_2)},$$

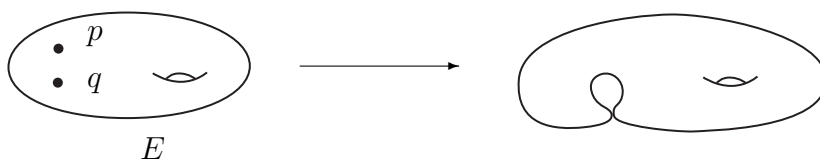
which gives the result. □

We next observe that

$$(V.C.17) \quad \omega_3|_{t_2=0} \text{ is defined and is equal to zero.}$$

Proof. The computation is similar to but simpler than that in the proof of (V.C.16). \square

Interpretations: The curves pictured in Figure 1 map to an open set $\Delta^3 \subset \overline{\mathcal{M}}_2$. The PHS's of the smooth curves in Δ^{*3} vary with three parameters. Those on the codimension 1 strata such as $\Delta^{*2} \times \Delta$ vary in moduli with two parameters. Their normalizations are



and their LMHS's vary with two parameters with

$$\begin{cases} \text{Gr}_1(\text{LMHS}) \cong H^1(E) \\ \text{Gr}_0(\text{LMHS}) \cong \mathbb{Q} \end{cases}$$

and where the extension data in the LMHS is locally given by $\text{AJ}_E(p - q)$. Thus $\text{Gr}(\text{LMHS})$ varies with one parameter and for the approximating nilpotent orbit is constant along the curves $t_1 t_2 = c$. This local fibre of the map $\overline{\mathcal{M}}_2 \rightarrow \overline{\mathcal{P}}$ is part of the closed fibre parametrized by E .

Along the codimension 2 strata such as $\Delta^* \times \Delta^2$ the curves vary in moduli with 1-parameter. Their normalizations are



and the moduli parameter is locally the cross-ratio of $\{p, q; p', q'\}$. The LMHS's are purely Hodge-Tate and thus $\text{Gr}(\text{LMHS})$ has no continuous parameters. In summary

- Φ_e is locally 1-1 on Δ^{*3} ;
- $\Phi_{e,*}$ has rank 1 on $\Delta^{*2} \times \Delta$;
- Φ_e is locally constant on $\Delta^* \times \Delta^2$.

As $c \rightarrow 0$ the fibres of Φ_e on the $\Delta^{*2} \times \Delta$ tend to the coordinate axis $\Delta^* \times \Delta^2$ along which Φ_e is locally constant.

Proof of Theorem I.A.14 in the $\dim B = 2$ case. We are given $\Phi : B \rightarrow \Gamma \setminus D$ where $B = \overline{B} \setminus Z$ with $Z = \sum Z_i$ a normal crossing divisor whose components are irreducible, smooth curves around which the monodromy T_i is unipotent and non-trivial. We may also assume that Φ_* is generically 1-1. The mapping Φ is then proper and the images $\mathcal{P} = \Phi(B) \subset \Gamma \setminus D$ is a closed complex surface. We want to show that \mathcal{P} has a canonical completion to a complex analytic surface $\overline{\mathcal{P}}$ over which the Hodge line bundle over \mathcal{P} extends to an ample line bundle $\Lambda_e \rightarrow \overline{\mathcal{P}}$. We choose our labelling so that the LMHS is locally constant along the Z_i for $i = 1, \dots, k$ and is not locally constant along the remaining Z_j . Then the curves Z_i for $i = 1, \dots, m$ are the ones to be contracted to give $\overline{\mathcal{P}}$. It is classical that the criterion to be able to contract the Z_i to normal singularities is that the intersection matrix

$$(V.C.18) \quad \|Z_i \cdot Z_j\|_{1 \leq i, j \leq k} < 0$$

be negative definite.

Since the monodromy of LMHS along Z_i is finite, the canonically extended Hodge line bundle

$$\Lambda_{e,i} \rightarrow Z_i$$

is of finite order, and moreover the canonically extended Chern ω_e may be restricted to Z_i and

$$\omega_i = \omega_{e,i}|_{Z_i} = 0$$

where ω_i is the Chern form of the Hodge line bundle $\Lambda_i = \Lambda_e|_{Z_i}$. Since $\omega_e^2 > 0$ our desired result (V.C.18) follows from the Hodge index theorem. Since $\Lambda_e|_{Z_i}$ is of finite order a power of it is trivial and therefore for some m the line bundle $\Lambda_e^m \rightarrow \overline{B}$ descends to a line bundle $\Lambda_{e,\overline{\mathcal{P}}}^m \rightarrow \overline{\mathcal{P}}$. Although $\overline{\mathcal{P}}$ is singular and ω_e only descends to a singular differential form on $\overline{\mathcal{P}}$, the classical Kodaira theorem may be extended to this case to give that $\Lambda_{e,\overline{\mathcal{P}}}^m \rightarrow \overline{\mathcal{P}}$ is ample (cf. Section VI in [GGLR17] for the details).

VI. APPLICATIONS, FURTHER RESULTS AND SOME OPEN QUESTIONS

VI.A. The Satake-Baily-Borel completion of period mappings.

We will discuss the proof of Theorem (1.A.15)* as stated in the introduction. There are three steps in the argument:

- (a) construction of the completion $\overline{\mathcal{P}}$ of \mathcal{P} ;
- (b) analysis of the extended Chern form ω_e on \overline{B} ; and
- (c) extension of the classical Kodaira theorem to the case of the singular variety $\overline{\mathcal{P}}$ where the Chern form up on \overline{B} has mild singularities.

Step (b) was discussed in Section V above and using it step (c) can be done by extending essentially standard arguments. For this we refer to Section 6 in [GGLR17] and here shall take up step (a). What follows is not a formal proof; the intent is to illustrate some of the key ideas behind the argument given in [GGLR17]. There are four parts to the argument:

(a1) localize the period mapping to

$$\Phi : \Delta^{*k} \times \Delta^\ell \rightarrow \Gamma_{\text{loc}} \backslash D$$

where $\Gamma_{\text{loc}} = \{T_1, \dots, T_k\}$ is the local monodromy group and determine the structure of local image

$$\Phi(\Delta^k \times \Delta^\ell) \subset \bar{\mathcal{P}};$$

(a2) using the first step and the proper mapping theorem, show that the fibres \mathcal{F} of the set-theoretic map

$$\Phi_e : \bar{B} \rightarrow \bar{\mathcal{P}}$$

are compact analytic subvarieties of \bar{B} , and from this infer that $\bar{\mathcal{P}}$ has the structure of a compact Hausdorff topological space and Φ_e is a proper mapping;

(a3) define the sheaf $\mathcal{O}_{\bar{\mathcal{P}}}$ whose sections over an open set $\mathcal{U} \subset \bar{\mathcal{P}}$ are the continuous functions f such that $f \circ \Phi_e$ is holomorphic in $\Phi_e^{-1}(\mathcal{U})$, and show that $\mathcal{O}_{\bar{\mathcal{P}}}$ endows $\bar{\mathcal{P}}$ with the structure of an analytic variety; and

(a4) show that certain identifications of connected components of the fibres that result from the global action of monodromy give a finite equivalence relation (this uses [CDK95]).

In this paper we shall mainly discuss the first step isolating the essential point of how the relative weight filtration property enters in the analysis of how the local period mapping in (a1) extends across the boundary strata of $\Delta^{*k} \times \Delta^\ell$ in $\Delta^k \times \Delta^\ell$.

Step (a2) is based on the following (cf. [Som78]). Let

$$\Phi : B \rightarrow \Gamma \backslash D$$

be a period mapping where $B = \bar{B} \backslash Z$ with $Z = \sum_i Z_i$ a normal crossing divisor with unipotent monodromy T_i around Z_i . The Φ may be extended across all the $Z_i^* := Z_i \backslash (\cap_{j \neq i} Z_j \cap Z_j)$ around which $T_i = \text{Id}$ and the resulting mapping

$$\Phi_e : B_e \rightarrow \Gamma \backslash D$$

is proper.⁴⁸ Thus $\Phi_e(B_e) \subset \Gamma \backslash D$ is a closed analytic variety. This result will be used for B , and also for the period mappings

$$\Phi_I : Z_I^* \rightarrow \Gamma_I \backslash D_I$$

given by taking the associated gradeds to the limiting mixed Hodge structures along the strata $Z_I^* := Z_I \backslash (\cap_{J \supset I} Z_J \cap Z_I)$.

Step (a3), which is work still in progress, is based on analysis of the global structure of the fibres \mathcal{F} , specifically positivity properties of the co-normal sheaf $I_{\mathcal{F}}/I_{\mathcal{F}}^2$ (which is the co-normal bundle $N_{\mathcal{F}/\bar{B}}^*$ in case \mathcal{F} is smooth).

⁴⁸This result uses that an integral element of the IPR the holomorphic sectional curvatures are $\leq -c$ for some $c > 0$ and the Ahlfors-Schwarz lemma. For the former we refer to Theorem VI.B.1 below. The latter is a by now standard deep fact in complex function theory.

Turning to the discussion of step (a1) we shall only consider the key special case when $\ell = 0$ (there are no parameters) and Φ is a nilpotent orbit (V.B.3). Then Φ is given by taking the orbit of the image of the homomorphism of complex Lie groups

$$\rho : \mathbb{C}^{*k} \rightarrow G_{\mathbb{C}}$$

whose differential is

$$\rho_*(t_i \partial / \partial t_i) = N_i.$$

Referring to the discussion following the statement of Theorem (I.A.15)* in the introduction and recalled below we will prove the following

PROPOSITION VI.A.1: *There exists a mapping*

$$\mu : \Delta^{*k} \rightarrow \mathbb{C}^N$$

*whose fibres are exactly the fibres of the set-theoretic mapping Φ_e localized to Δ^{*k} and with monodromy group Γ_{loc} (cf. VI.A.3 below).*

The mapping μ in the proposition will be given by monomials, and for this reason it will be called a *monomial mapping*.

The mapping Φ_e localized to Δ^{*k} arises from a period mapping

$$(VI.A.2) \quad \Phi : \Delta^{*k} \rightarrow \Gamma_{\text{loc}} \backslash D$$

given by a nilpotent orbit. The corresponding variation of Hodge structure over Δ^{*k} induces variations of polarized limiting mixed Hodge structures along the open boundary strata $\Delta_I^* = \{(t_1, \dots, t_k) \in \Delta^k : t_i = 0 \text{ for } i \in I \text{ and } t_j \neq 0 \text{ for } j \in I^c\}$. Passing to the primitive parts of the associated graded gives period mappings

$$\Phi_I : \Delta_I^* \rightarrow \Gamma_I \backslash D_I,$$

where Γ_I is generated by the T_i for $i \in I$. The set-theoretic mapping Φ_e is given by

$$(VI.A.3) \quad \Phi_e : \Delta^k \rightarrow \Gamma_{\text{loc}} \backslash D \amalg \left(\coprod_I \Gamma_{I, \text{loc}} \backslash D_I \right)$$

where $\Phi_e|_{\Delta_I^*}$ and $\Phi_{I, \text{loc}}$ is the monodromy group given by the VHS over Δ_I^* .

The proof of Proposition VI.A.1 will be given in several steps, as follows.

1. determine the connected components of the nilpotent orbit

$$(VI.A.4) \quad \exp \left(\sum_j \ell(t_j) N_j \right) \cdot F : \Delta^{*k} \rightarrow \Gamma_{\text{loc}} \backslash D;$$

2. extend that argument to determining the connected components of the mappings Φ_I , which will also be given by nilpotent orbits

$$(VI.A.5) \quad \Phi_I(t_I) = \exp \left(\sum_{j \in I^c} \ell(t_j) N_j \right) \cdot F_I : \Delta_I^* \rightarrow \Gamma_I \backslash D_I;$$

3. show that for $I \subset J$ the closure in $\overline{\Delta_I^*} \cap \Delta_J^*$ of a fibre of Φ_I is equal to a connected component of a fibre of Φ_J ; and
4. show that the limit of a sequence of fibres of Φ_I is contained in a fibre of Φ_J .

As for removing the “connected component” qualifiers go, part of this deals with the identifications induced by global monodromy that was mentioned above.

The basic idea in the construction appears already in step 1; the most interesting part of the argument is step 3 where the relative weight filtration property provides the key.

Step 1: We consider the question: *What are the conditions that a monomial*

$$(VI.A.6) \quad t^B = t_1^{b_1} \cdots t_k^{b_k}, \quad b_i \in \mathbb{Z}$$

be constant on the fibres of (VI.A.2)? For this we let

$$R = \left\{ A = (a_1, \dots, a_k) : \sum_i a_i N_i = 0 \right\} \subset \mathbb{R}^k$$

be the set of relations on the $N_i \in \mathfrak{g}$.⁴⁹ We note that R is defined over \mathbb{Q} .

PROPOSITION VI.A.7: *The conditions that the monomial (VI.A.6) be constant on the fibres of (VI.A.2) are*

$$A \cdot B = \sum_j a_j b_j = 0, \quad A \in R.$$

Proof. The vector field induced by $N = \sum a_j N_j \in \mathfrak{g}_{\mathbb{R}}$ is nowhere vanishing on D .⁵⁰ Thus on the one hand

$$\rho_* \left(\sum_j a_j t_j \partial / \partial t_j \right) \text{ is tangent to a fiber of } \Phi \iff \sum_j a_j N_j = 0.$$

On the other hand, the condition that the monomial (VI.A.6) be constant on the orbits of the vector field $\sum_j a_j t_j \partial / \partial t_j$ on Δ^{*k} is

$$\left(\sum_j a_j t_j \partial / \partial t_j \right) t^B = (A \cdot B) t^B = 0. \quad \square$$

This simple computation contains one of the key ideas in the construction of the monomial mapping

$$\mu : \Delta^k \rightarrow \mathbb{C}^N.$$

We next consider the question:

Are there enough monomials (VI.A.6) satisfying $A \cdot B = 0$ for all $A \in R$ and where $b_j \in \mathbb{Z}^{\geq 0}$ to separate the connected components of the fibres of (VI.A.4)?

This is the existence result that is needed to give local charts that will define the fibres of (VI.A.4) up to connected components. It is a consequence of the following

PROPOSITION VI.A.8: *The subspace R^\perp is spanned by vectors B where all $b_i \in \mathbb{Q}^{\geq 0}$.*

This is a consequence of a result in linear programming, known as *Farkas’ alternative theorem*. We refer to Section 3 in [GGLR17] for details and a reference.

⁴⁹Here we use the [GGLR17] notations $\mathfrak{g} = \text{End}_{\mathbb{Q}}(V)$ and $\mathfrak{g}_{\mathbb{R}} = \text{End}_{\mathbb{Q}}(V_{\mathbb{R}})$.

⁵⁰This is because $D \cong G_{\mathbb{R}}/H$ where H is a compact subgroup of $G_{\mathbb{R}}$. The Lie algebra \mathfrak{h} then contains no non-zero nilpotent elements in $\text{End}_{\mathbb{Q}}(V_{\mathbb{R}})$.

Step 2: We consider the period mappings (VI.A.5)

$$\Phi_I : Z_I^* \rightarrow \Gamma_{\text{loc}, I} \setminus D_I$$

given by the variation of polarized limiting mixed Hodge structures on the open smooth strata Z_I^* . Again restricting to the case of a nilpotent orbit (VI.A.5) we may ask for the analogue of the question in Step 1 for this nilpotent orbit.

The key observations here are that here *both the weight and the Hodge filtrations enter*, and *since Φ_I maps to the associated graded relative to the weight filtration $W(N_I)$ any operation that decreases $W(N_I)$ has no effect*. Recalling from Section V.B our notations

- $N_I = \sum_{i \in I} N_i$;
- $Y_I =$ grading element for N_I and $\{N_I, Y_I, N_I^+\}$ is the resulting \mathfrak{sl}_2 ;
- for $j \in I^c$ we have $N_j = N_{j,0} + N_{j,-1} + \cdots$ where $N_{j,-m}$ is the $-m$ weight space for Y_I ;

it follows that the nilpotent orbit (VI.A.5) is the same as the nilpotent orbit using $N_{j,0}$ in place of N_j , and in place of Proposition VI.A.7 we have

PROPOSITION VI.A.9: *The connected components of the fibres of (VI.A.5) are the level sets of monomials t^B where $b_j \in \mathbb{Z}^{\geq 0}$ and*

$$\sum_{j \in I^c} b_j N_j \in W_{-1}(N_I) \mathfrak{g}.$$

Moreover, recalling that the Chern form of the Hodge line bundle is given by $\omega_I = \omega_e|_{Z_I^*}$ these connected components are exactly the connected integral varieties of the exterior differential system

$$\omega_I = 0.$$

We recall our notation $\Delta_I^* = \{t \in \Delta^{*k} : t_i = 0 \text{ for } i \in I \text{ and } t_j \neq 0 \text{ for } j \in I^c\}$ and denote by

$$(VI.A.10) \quad \mu_I : \Delta_I^* \rightarrow \mathbb{C}^{N_I}$$

the monomial map constructed in the same way as the monomial map constructed from Propositions VI.A.7 and VI.A.8 using the t^B for a generating set of vectors $B \in R^\perp$ with the $b_i \in \mathbb{Z}^{\geq 0}$. The to be constructed compact analytic variety is $\bar{\mathcal{P}}$ is set theoretically the disjoint union

$$(VI.A.11) \quad \bar{\mathcal{P}} = \mathcal{P} \amalg \left(\coprod_I \mathcal{P}_I \right)$$

where \mathcal{P}_I is a finite quotient of the union of the images

$$\Phi_I(Z_I^*) \subset \Gamma_{\text{loc}, I} \setminus D_I.$$

To complete the proof of the construction of $\bar{\mathcal{P}}$ as compact analytic variety two issues need to be addressed:

- (i) set-theoretically, the inverse image of $\Phi_I(\Delta_I^*) \subset \mathcal{P}_I$ is a finite cover of $\mu_I(\Delta_I^*)$ and we need to describe analytic functions that will separate the sheets of this covering,⁵¹ and
- (ii) we need to show that the analytic varieties \mathcal{P}_I fit together to give the structure of an analytic variety on $\overline{\mathcal{P}}$.

Step 3: For the second of the two issues above we observe that the restriction

$$t^B|_{\Delta_I^*} = 0 \text{ if } b_i > 0 \text{ for some } i \in I.$$

To establish (ii) we have the

PROPOSITION VI.A.12: *For $I \subsetneq J$ so that $\Delta_J^* \subset \overline{\Delta_I^*}$, the closure of a level set of μ_I is contained in a level set of μ_J . Moreover the limit in Δ_J^* of level sets in Δ_I^* is contained in a level set of μ_J .*

Proof. This will be a consequence of the relative weight filtration property (RWFP) (V.B.7) that we now recall in a form adapted to the proof of (VI.A.12).

Given $A, B \subset \{1, \dots, k\}$ with $A \cap B = \emptyset$, we denote by

$$N_B = N_{B,0} + N_{B,-1} + N_{B,-2} + \dots$$

the Y_A -eigenspace decomposition of N_B relative to the \mathfrak{sl}_2 $\{N_A^+, Y_A, N_A\}$. Then the nilpotent operator

$$N_{B,0}|_{\mathrm{Gr}_m^{W(N_A)} V} : \mathrm{Gr}_m^{W(N_A)} V \rightarrow \mathrm{Gr}_m^{W(N_A)} V$$

induces a weight filtration on $\mathrm{Gr}_\bullet^{W(N_A)} V$. Another weight filtration on this vector space is defined by

$$\frac{W_\bullet(N_A + N_B) \cap W_m(N_A)}{W_\bullet(N_A + N_B) \cap W_{m-1}(N_A)}.$$

The RWFP is that these two filtrations coincide; i.e.,

$$(VI.A.13) \quad \frac{W_{m+m'}(N_A + N_B) \cap W_m(N_A)}{W_{m+m'}(N_A + N_B) \cap W_{m-1}(N_A)} = W_{m'} \left(N_{B,0}|_{\mathrm{Gr}_m^{W(N_A)} V} \right).$$

Returning to the proof of (VI.A.12), what must be proved is that for $I \subsetneq J$

$$(VI.A.14) \quad \sum_{j \in I^c} a_j N_j \in W_{-1}(N_I) \implies \sum_{j \in J^c} a_j N_j \in W_{-1}(N_J).$$

On the RHS we have used that $N_j \in W_{-2}(N_J)$ for $j \in J$, so the sum is really over $j \in J^c$. What (VI.A.14) translates into is that if t^A is a monomial that is constant on the fibres of Φ_I , then the restriction $t^A|_{\Delta_J^*}$ is constant on the level sets of Φ_J .

We let $X = \sum_{j \in I^c} a_j N_j \in W_{-1}(N_I)\mathfrak{g}$, where the “ \in ” is because

$X \in \mathcal{Z}(N_I) \implies X$ has only negative weights in the N_I -string decomposition of \mathfrak{g} .

⁵¹Here there is both a local issue dealing with the fibres of $\Phi_I : \Delta_I^* \rightarrow \Gamma_I \backslash D_I$, and a global issue arising from the possibility that two connected components of the fibre of $\Phi_I : \Delta_I^* \rightarrow \Gamma_I \backslash D_I$ may be subsets of a single fibre of Φ_I on Z_I^* due to the global action of monodromy.

Write

$$X = X_0 + X_{-1} + X_{-2} + \cdots$$

in terms of the eigenspace decomposition of Y_J . Since N_{J-I} is in the -2 eigenspace of Y_I , we have

$$[N_{J-I}, X_0] = 0.$$

Now decompose X_0 into Y_I eigenspace components

$$X = X_{0,-m} + X_{0,-(m+1)} + \cdots, \quad m \geq 1.$$

Then

$$[N_{J-I,0}, X_{0,-m}] = 0 \implies X_{0,-m} \in \text{Gr}_{-m}^{W(N_I)} \mathfrak{g} \text{ lies in } \mathcal{Z} \left(N_{J-I,0} \Big|_{\text{Gr}_{-m}^{W(N_I)} \mathfrak{g}} \right)$$

and consequently

$$X_{0,-m} \in W_0 \left(N_{J-I,0} \Big|_{\text{Gr}_{-m}^{W(N_I)} \mathfrak{g}} \right).$$

Applying (VI.A.13) with $A = I$, $B = J - I$ gives (VI.A.14), which proves VI.A.12. \square

Step 4: At this point we have constructed a monomial mapping

$$\mu : \mathbb{C}^k \rightarrow \mathbb{C}^N$$

whose fibres are unions of the fibres of the nilpotent orbit (V.A.3). The final local step is to refine the construction to have

$$(VI.A.15) \quad \begin{array}{ccc} \mathbb{C}^k & \xrightarrow{\eta} & \mathbb{C}^k \\ & \searrow \mu & \swarrow \tilde{\mu} \\ & \mathbb{C}^N & \end{array}$$

where $\tilde{\mu}$ is a monomial mapping with connected fibres. The basic idea already occurs when $k = N = 1$ and $\mu(t) = t^m$; then “ $t^{1/m}$ ” separates the fibres of μ .

In general we suppose that

$$\mu(t) = (t^{I_1}, \dots, t^{I_N})$$

where

$$t^{I_j} = t_1^{i_{j1}} \cdots t_k^{i_{jk}}.$$

Define a map $\mathbb{Z}^k \rightarrow \mathbb{Z}^N$ by

$$e_j \rightarrow (i_{1j}, \dots, i_{Nj}).$$

Then identifying \mathbb{Z}^N with $\text{Hom}(\mathbb{Z}^N, \mathbb{Z})$, up to a finite group the image $\Lambda \subset \mathbb{Z}^N$ is defined by $\Lambda^\perp \subset \mathbb{Z}^N$. Setting $\tilde{\Lambda} = (\Lambda^\perp)^\perp$, for the finite abelian group $\tilde{\Lambda}/\Lambda$ we have

$$(VI.A.16) \quad \tilde{\Lambda}/\Lambda \cong \bigoplus \mathbb{Z}/d_i \mathbb{Z}.$$

The mapping η in (VI.A.15) will be a $|\tilde{\Lambda}/\Lambda|$ -to-1 monomial map. To construct it, at the level of exponents of monomials we will have

$$(VI.A.17) \quad \begin{array}{ccc} \mathbb{Z}^k & \xrightarrow{B} & \mathbb{Z}^k \\ & \searrow A & \swarrow \tilde{A} \\ & \mathbb{Z}^N & \end{array}$$

where $\text{Im}(A) = \Lambda$, $\text{Im}(\tilde{A}) = \tilde{\Lambda}$ and $\mathbb{Z}^k/\text{Im}(B) \cong \tilde{\Lambda}/\Lambda$. Such a map always exists and may be constructed using (VI.A.16). We then use (VI.A.17) to define (VI.A.15) where $\tilde{\mu}$ is a monomial map with connected fibres. \square

For the issue arising from the global action of monodromy we refer to Section 3 in [GGLR17].

Example VI.A.18: This is a continuation of Example V.C.15 above. Since N_1, N_2, N_3 are linearly independent, the mapping $\Delta^3 \rightarrow \Gamma_{\text{loc}} \backslash D$ given by the corresponding nilpotent orbit is 1-1.⁵²

The interesting situation is on the face $\{0\} \times \Delta^{*2}$ given by $t_1 = 0$. There we observe from the computations in Example V.C.15 that the induced maps

$$\overline{N}_2, \overline{N}_3 : \text{Gr}_1^{W(N_1)} V \rightarrow \text{Gr}_1^{W(N_1)} V$$

are equal. This gives the relation $\overline{N}_2 - \overline{N}_3 = 0$ which leads to the monomial $t_2 t_3$ that is constant on the fibres of the period mapping

$$\Phi_1 : \{0\} \times \Delta^{*2} \rightarrow \Gamma_{1,\text{loc}} \backslash D_1.⁵³$$

On the codimension 2 strata the associated graded to the LMHS's are Hodge-Tate, so the corresponding period mappings are constant.⁵⁴

We conclude this section by discussing the following question:

What are the Zariski tangent spaces to $\overline{\mathcal{P}}$?

More precisely,

$$(VI.A.19) \quad \textit{What is the kernel of the mapping } T_b \overline{B} \rightarrow T_{\Phi_e(b)} \overline{\mathcal{P}}?$$

Recalling the notation ω_e for the Chern form of the canonically extended Hodge line bundle $\Lambda_e \rightarrow \overline{B}$, we want to define in each tangent space to \overline{B} the meaning of the equations

$$(VI.A.20) \quad \omega_e(\xi) = 0, \quad \xi \in T_b \overline{B}.$$

The issue is that ω_e is not smooth, continuous, or even bounded. If $b \in B$, then (VI.A.20) has the usual meaning $\Phi_*(\xi) = 0$. If $b \in Z_I^*$ and $\xi \in T_b Z_I^*$ is tangent to Z_I^* , then since $\omega_e|_{Z_I^*} = \omega_I$ (VI.A.20) means that $\Phi_{I,*}(\xi) = 0$.

⁵²In this case $D = \mathcal{P}_2$.

⁵³In this case $D_1 = \mathcal{P}$.

⁵⁴We note that the limit of the parabolas $t_2 t_3 = c$ as $c \rightarrow 0$ will be the coordinate axes, confirming the limit part of the statement in VI.A.12 in this case.

Thus the interesting case is when $b \in Z_I^*$ and ξ is a normal vector to Z_I^* in \overline{B} . This amounts to the situation of a 1-parameter VHS

$$\Phi : \Delta^* \rightarrow \Gamma_T \backslash D, \quad \Gamma_T = T^{\mathbb{Z}}$$

and (VI.A.20) becomes the condition

$$(VI.A.21) \quad \Phi_{e,*}(\partial/\partial t)|_{t=0} = 0.$$

In this case we have the

PROPOSITION VI.A.22: *If $T \neq \text{Id}$, then $\Phi_{e,*}(\partial/\partial t)|_{t=0} \neq 0$.*

Proof. If $T = \exp N$ is unipotent, then the methods used in Section V.A above and Section 3 in [GGLR17] give for the Chern form on Δ^*

$$\omega_e \geq C \frac{dt \wedge d\bar{t}}{|t|^2(-\log|t|)^2}, \quad C > 0.$$

In general T is quasi-unipotent and after a base change $t' = t^m$ we will have

$$\begin{array}{ccc} \Delta'^* & \xrightarrow{\Phi'} & \Gamma_{T'} \backslash D \\ \pi \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\Phi} & \Gamma_T \backslash D \end{array}$$

where T' is unipotent. Then from

$$\pi^* \left(\frac{dt \wedge d\bar{t}}{|t|^2(-\log|t|)^2} \right) = C' \frac{dt' \wedge d\bar{t}'}{|t'|^2(-\log|t'|)^2}, \quad C' > 0$$

we may infer the proposition.⁵⁵ □

From VI.A.22 we may draw the

CONCLUSIONS VI.A.23: (i) *If the N_j , $j \in I^c$, are linearly independent modulo $W_{-1}(N_1)$, then $\omega_e > 0$ in the normal spaces to Z_I^* .*

(ii) *For nilpotent orbits, $\omega_e(\xi) = 0 \iff \mu_*(\xi) = 0$ where $\mu : \Delta^{*k} \rightarrow \mathbb{C}^N$ is the monomial map.*

We comment that (ii) holds in the general situation in Section 3 of [GGLR17] where local quasi-charts are constructed for arbitrary VHS over $\Delta^{*k} \times \Delta^\ell$.

Finally we use the results of Section V above and Section 3 in [GGLR17] to summarize the properties of the EDS (VI.A.20):

- (VI.A.24) (a) (VI.A.20) defines a coherent, integrable sub-sheaf $\mathcal{J} \subset \mathcal{O}_{\overline{B}}(T\overline{B})$;
 (b) the maximal leaves of \mathcal{J} are closed, complex analytic subvarieties of \overline{B} ;
 (c) as a set, $\overline{\mathcal{P}}$ is the quotient of the fibration of \overline{B} given by the leaves of \mathcal{J} .

⁵⁵It is interesting to note that the pullback of smooth forms under a branched covering map vanish along the branch locus. For the Poincaré metric this is not the case, illustrating again the general principle that in Hodge theory singularities increase positivity.

Singular integrable foliations and their quotients have been introduced and studied in [Dem12b].

VI.B. Norm positivity and the cotangent bundle to the image of a period mapping.

(i) *Statement of results.* Let $\Phi : B \rightarrow \Gamma \backslash D$ be a period mapping with image a quasi-projective variety $\mathcal{P} \subset \Gamma \backslash D$. The $G_{\mathbb{R}}$ -invariant metric on D constructed from the Cartan-Killing form on $\mathfrak{g}_{\mathbb{R}}$ induces a Kähler metric on the Zariski open set \mathcal{P}^0 of smooth points of \mathcal{P} . We denote by $R(\eta, \xi)$ and $R(\xi)$ the holomorphic bi-sectional and holomorphic sectional curvatures respectively.

THEOREM VI.B.1: *There exists a constant $c > 0$ such that*

- (i) $R(\xi) \leq -c$ for all $\xi \in T\mathcal{P}^0$;
- (ii) $R(\eta, \xi) \leq 0$ for all $\eta, \xi \in T\mathcal{P}^0 \times_{\mathcal{P}^0} T\mathcal{P}^0$;
- (iii) For any $b \in \mathcal{P}_o$ there exists a $\xi \in T_b\mathcal{P}^0$ such that $R(\eta, \xi) \leq -c/2$ for all $\eta \in T_b\mathcal{P}^0$.

Observe that using (II.G.3) from [BKT13] (iii) follows from (i) and (ii). As a corollary to (ii) we have

- (iv) $R(\eta, \xi) \leq -c/2$ is an open set in $T\mathcal{P}^0 \times_{\mathcal{P}^0} T\mathcal{P}^0$.

We note that (iii) implies that this open set projects onto each factor in $\mathcal{P}^0 \times \mathcal{P}^0$.

As applications of the proof of Theorem VI.B.1 and consideration of the singularity issues that arise we have the following extensions of some of the results of Zuo [Zuo00] and others (cf. Chapter 13 in [CMSP17]):

(VI.B.2) \mathcal{P} is of log-general type,

(VI.B.3) $\mathrm{Sym}^m \Omega_{\mathcal{P}}^1(\log)$ is big for $m \geq m_0$.

The result in (VI.B.2) means that for any desingularization $\tilde{\mathcal{P}}$ of $\overline{\mathcal{P}}$ with $\tilde{\mathcal{P}}$ lying over \mathcal{P} and $\tilde{Z} = \tilde{\mathcal{P}} \setminus \tilde{\mathcal{P}}$, the Kodaira dimension

$$\kappa \left(K_{\tilde{\mathcal{P}}}(\tilde{Z}) \right) = \dim \mathcal{P}.$$

The result in (VI.B.3) means that

$$\mathrm{Sym}^m \Omega_{\tilde{M}}^1(\log \tilde{Z}) \text{ is big for } m \geq m_0.$$

The proof will show that we may choose m_0 to depend only on the Hodge numbers for the original VHS.

The proof will also show that

(VI.B.2)_s \mathcal{P} is of stratified-log-general type,

(VI.B.3)_s $\mathrm{Sym}^m \Omega_{\mathcal{P}}^1(\log)$ is stratified-big for $m \geq \bar{m}_0$.

Here stratified-log-general type means that there is a canonical stratification $\{\mathcal{P}_I^*\}$ of $\overline{\mathcal{P}}$ such that each stratum \mathcal{P}_I^* is of log-general type. There is the analogous definition for stratified big.

Without loss of generality, using the notations above we may take $\widetilde{M} = \overline{B}$, $\widetilde{\mathcal{P}} = B$ and $\widetilde{Z} = Z$; we shall assume this to be the case.

Remark: The results of Zuo, Brunearbe and others are essentially that $K_{\overline{\mathcal{P}}}(\log)$ and $\Omega_{\overline{\mathcal{P}}}^1(\log)$ are weakly positive in the sense of Viehweg. This is implied by (VI.B.3).

The proof of Theorem VI.B.1 will be done first in the case

$$(VI.B.4) \quad B = \overline{B} \text{ and } \Phi_* \text{ is everywhere injective.}$$

It is here that the main ideas and calculations occur.

The singularities that arise are of the types

$$(VI.B.5) \quad \begin{cases} (a) & \text{where } \Phi_* \text{ fails to be injective (e.g., on } \mathcal{P}_{\text{sing}}), \\ (b) & \text{on } Z = \overline{B} \setminus B \text{ where the VHS has singularities,} \\ (c) & \text{the combination of (a) and (b).} \end{cases}$$

As will be seen below, there will be a coherent sheaf I with

$$\Phi_*(TB) \subset I \subset \Phi^*(T(\Gamma \setminus D)).^{56}$$

Denoting by I^o the open set where I is locally free, there is an induced metric and corresponding curvature form for I^o , and with the properties (i), (ii) in the theorem for I^o Theorem VI.B.1 will follow from the curvature decreasing property of holomorphic sub-bundles, which gives

$$R(\eta, \xi) = \Theta_{T^{p^o}}(\eta, \xi) \leq \Theta_{I^o}(\eta, \xi).$$

As for the singularities, if we show that

$$(VI.B.6) \quad \kappa(\det I^o(\log)) = \dim B$$

$$(VI.B.7) \quad \text{Sym}^m I^o(\log) \text{ is big}$$

then (VI.B.2) and (VI.B.3) will follow from the general result: If over a projective variety Y we have line bundles L, L' and a morphism $L \rightarrow L'$ that is an inclusion over an open set, then

$$(VI.B.8) \quad L \rightarrow Y \text{ big} \implies L' \rightarrow Y \text{ is big.}$$

We will explain how (VI.B.6) and (VI.B.7) will follow from (VI.B.8) for suitable choices of Y, L and L' .

(ii) *Basic calculation.* It is convenient to use Simpson's system of Higgs bundles framework (cf. [Sim92] and Chapter 13 in [CMSP17]) whereby a VHS is given by a system of holomorphic vector bundles E^p , and maps

$$E^{p+1} \xrightarrow{\theta^{p+1}} E^p \otimes \Omega_B^1 \xrightarrow{\theta^p} E^{p-1} \otimes \wedge^2 \Omega_X^2$$

⁵⁶Here we are identifying a coherent sub-sheaf of a vector bundle with the corresponding family of linear subspaces in the fibres of the vector bundle. The coherent sheaf I will be a subsheaf of the pull-back $\Phi^*T(D \setminus \Gamma)_h$ of the horizontal tangent spaces to $\Gamma \setminus D$. The critical step in the calculation will be that it is integrable as a subsheaf $\Phi^*T(\Gamma \setminus D)_h$.

that satisfy

$$(VI.B.9) \quad \theta^p \wedge \theta^{p+1} = 0.$$

Thus there is induced

$$E^{p+1} \xrightarrow{\theta^{p+1}} E^p \otimes \Omega_B^1 \xrightarrow{\theta^p} E^{p-1} \otimes \text{Sym}^2 \Omega_B^1,$$

and the data $(\bigoplus_p E^p \otimes \text{Sym}^{k-p} \Omega_B^1, \bigoplus_p \theta^p)$ for any k with $k \geq p$ is related to the notion of an infinitesimal variation of Hodge structure (IVHS) (cf. 5.5 ff. in [CMSP17]).

In our situation the vector bundles E^p will have Hermitian metrics with Chern connections D^p . The metrics define adjoints

$$\theta^{p*} : E^p \rightarrow E^{p+1} \otimes \overline{\Omega_B^1},$$

and in the cases we shall consider if we take the direct sum over p we obtain

$$(E, \nabla = \theta^* + D + \theta), \quad \text{with (VI.B.9) equivalent to } \nabla^2 = 0.$$

The properties uniquely characterizing the Chern connection together with $\nabla^2 = 0$ give for the curvature matrix of E^p the expression

$$(VI.B.10) \quad \Theta_{E^p} = \theta^{p+1} \wedge \theta^{p+1*} + \theta^{p*} \wedge \theta^p,$$

which is a difference of non-negative terms each of which has the norm positivity property (III.A.3) (cf. [Zuo00] and Chapter 13 in [CMSP17]).

For a PVHS (V, Q, ∇, F) we now set

$$E^p = \text{Gr}^p \text{Hom}_Q(V, V), \quad -n \leq p \leq n$$

where Gr^p is relative to the filtration induced by F on $\text{Hom}_Q(V, V)$. At each point b of B there is a weight zero PHS induced on $\text{Hom}_Q(V, V) = \mathfrak{g}$ and

$$E_b^p = \mathfrak{g}^{p, -p}$$

with the bracket

$$[\ , \] : E^p \otimes E^q \rightarrow E^{p+q}.$$

Thinking of θ as an element in $\mathfrak{g} \otimes \Omega_B^1$, the integrability condition VI.B.9 translates into

$$(VI.B.11) \quad [\theta, \theta] = 0.$$

We shall use the notation

$$\text{Gr}^p = \text{Gr}^p \text{Hom}_Q(V, V)$$

rather than E^p for this example.

The differential of Φ gives a map

$$\Phi_* : TB \rightarrow \text{Gr}^{-1}.$$

Definition: $I \subset \text{Gr}^{-1}$ is the coherent subsheaf generated by the sections of Gr^{-1} that are locally in the image of Φ_* over the Zariski open set where Φ_* is injective.

For ξ a section of I we denote by ad_ξ the corresponding section of Gr^{-1} . The integrability condition (VI.B.11) then translates into the first part of the

PROPOSITION VI.B.12: I is a sheaf of abelian Lie sub-algebras of $\bigoplus_p \text{Gr}^p$. For η, ξ sections of I

$$\Theta_{\text{Gr}^{-1}}(\eta, \xi) = -\|\text{ad}_\xi^*(\eta)\|^2.$$

Proof. For $\eta, \xi \in \text{Gr}^{-1}$ the curvature formula (VI.B.10) is

$$\Theta_{\text{Gr}^{-1}}(\eta, \xi) = \|\text{ad}_\xi(\eta)\|^2 - \|\text{ad}_\xi^*(\eta)\|^2.$$

The result then follows from $\text{ad}_\xi(\eta) = [\xi, \eta] = 0$ for $\eta, \xi \in I$. \square

On the open set where I^o is a vector bundle with metric induced from that on Gr^{-1} we have

$$\Theta_{I^o}(\eta, \xi) \leq \Theta_{\text{Gr}^{-1}}(\eta, \xi) \leq 0.$$

The first term is the holomorphic bi-sectional curvature for the induced metric on $\Phi(B)$.

To complete the proof of Theorem VI.B.10 we need to show the existence of $c > 0$ such that for all ξ of unit length

$$(VI.B.13) \quad \|\text{ad}_\xi^*(\xi)\| \geq c.$$

The linear algebra situation is this: At a point of B we have

$$V = \bigoplus_{p+q=n} V^{p,q}$$

and ξ is given by maps

$$A_p : V^{p,q} \rightarrow V^{p-1,q+1}, \quad \left\lfloor \frac{n+1}{2} \right\rfloor \leq p \leq n.$$

In general a linear map

$$A : E \rightarrow F$$

between unitary vector spaces has *principal values* λ_i defined by

$$Ae_i = \lambda_i f_i, \quad \lambda_i \text{ real and non-zero}$$

where e_i is a unitary basis for $(\ker A)^\perp$ and f_i is a unitary basis for $\text{Im } A$. The square norm is

$$\|A\|^2 = \text{Tr } A^*A = \sum_i \lambda_i^2.$$

We denote by $\lambda_{p,i}$ the principal values of A_p . The $\lambda_{p,i}$ depend on ξ , and the square norm of ξ as a vector in $T_p B \subset T_{\Phi(p)}(\Gamma \backslash D)$ is

$$\|\xi\|^2 = \sum_p \sum_i \lambda_{p,i}^2.$$

In the above we now replace V by $\text{Hom}_Q(V, V)$ and use linear algebra to determine the principal values of ad_ξ^* . These will be quadratic in the $\lambda_{p,i}$'s, and then

$$\|\text{ad}_{\xi^*}(\xi)\|^2$$

will be quartic in the $\lambda_{p,i}$. A calculation gives

$$(VI.B.14) \quad \|\text{ad}_{\xi^*}(\xi)\|^2 = \sum_p \left(\frac{\sum_i a_p \lambda_{p,i}^4}{(\sum_i \lambda_{p,i}^2)^2} \right)$$

where the a_p are non-negative integers that are positive if $A_p \neq 0$, and from this by an elementary algebra argument we may infer the existence of the $c > 0$ in Theorem VI.B.1. \square

At this point we have proved the theorem. The basic idea is very simple:

For a VHS the curvature (VI.B.10) of the Hodge bundles is a difference of non-negative terms, each of which is of norm positivity type where the “A” in Definition III.A.3 is a Kodaira-Spencer map or its adjoint. For the $\text{Hom}_{\mathbb{Q}}(V, V)$ variation of Hodge structure, $A(\xi)(\eta) = [\xi, \eta] = 0$ by integrability. Consequently the curvature form has a sign, and a linear algebra calculation gives the strict negativity $\Theta_I(\xi, \xi) \leq -c\|\xi\|^4$ for some $c > 0$.⁵⁷

(iii) *Singularities.* The singularity issues were identified in (VI.B.5), and we shall state a result that addresses them. The proof of this result follows from the results in [CKS86] as extended in [GGLR17], [Kol87] and the arguments in [Zuo00].⁵⁸

Using the notations introduced in (ii) above, a key observation is that the differential

$$\Phi_* : TB \rightarrow \text{Gr}^{-1}$$

extends to

$$\Phi_* : T\bar{B} \langle -Z \rangle \rightarrow \text{Gr}_e^{-1}$$

where $T\bar{B} \langle -Z \rangle = \Omega_{\bar{B}}^1(\log Z)^*$ and Gr_e^{-1} is the canonical extension to \bar{B} of $\text{Gr}^{-1} \rightarrow B$. This is just a reformulation of the general result (cf. [CMSP17]) that for all p , $\theta^p : E^p \rightarrow E^p \otimes \Omega_{\bar{B}}^1$ extends to

$$(VI.B.15) \quad \theta_e^p : E_e^p \rightarrow E_e^{p-1} \otimes \Omega_{\bar{B}}^1(\log Z).$$

As noted above, the image $\Phi_*TB \subset \text{Gr}^{-1}$ generates a coherent subsheaf $I \subset \text{Gr}^{-1}$ and from (VI.B.15) we may infer that I extends to a coherent subsheaf $I_e \subset \text{Gr}_e^{-1}$. As in [Zuo00] we now blow up \bar{B} to obtain a vector sub-bundle of the pullback of Gr^{-1} and note that $I_e \subset \text{Gr}_e^{-1}$ will be an integrable sub-bundle.

The metric on Gr^{-1} induces a metric in I and we use the notations

- $\varphi =$ Chern form of $\det I^{o*}$;
- $\omega =$ Chern form of $\mathcal{O}_{\mathbb{P}I^{o*}}(1)$.

THEOREM VI.B.16: *Both φ and ω extend to closed, $(1, 1)$ currents φ_e and ω_e on \bar{B} and $\mathbb{P}I_e^*$ that respectively represent $c_1(\det I_e^*)$ and $c_1(\mathcal{O}_{\mathbb{P}I_e^*})(1)$. They have mild singularities and satisfy*

⁵⁷This first proof of the result that appeared in the literature was Lie-theoretic where the metric on \mathfrak{g} was given by the Cartan-Killing form. As will be illustrated below the above direct algebra argument is perhaps more amenable to the computation in examples.

⁵⁸These arguments have been amplified at a number of places in the literature; cf. [VZ03] and [Pă16].

- $\varphi_e \geq 0$ and $\varphi_e > 0$ on an open set;
- $\omega_e \geq 0$ and $\omega_e > 0$ on an open set.

With one extra step this result follows from singularity considerations similar to those in Section IV above. The extra step is that

$$I_e \text{ is not a Hodge bundle, but rather it is the kernel of the map } \theta^{-1} : \text{Gr}_e^{-1} \rightarrow \text{Gr}_e^{-2} \otimes \Omega_e^1 \otimes \Omega_{\bar{B}}^1(\log Z).$$

As was noted in [Zuo00], either directly or using (5.20) in [Kol87] we may infer the stated properties of φ_e and ω_e . \square

Remark: It is almost certainly not the case that any sub-bundle $G \subset \text{Gr}_e^{-1}$ will have Chern forms with mild singularities. The bundle I_e is special in that it is the kernel of the map $\text{Gr}_e^{-1} \rightarrow \text{Gr}_e^{-2} \otimes \Omega_{\bar{B}}^1(\log Z)$. Although we have not computed the 2nd fundamental form of $I_e \subset \text{Gr}_e^{-1}$, for reasons to be discussed below it is reasonable to expect it to also have good properties.

The issue of the curvature form of the induced metric on the image $\mathcal{P} = \Phi(B) \subset \Gamma \backslash D$ seems likely to be interesting. Since the metric on the smooth points $\mathcal{P}^0 \subset \mathcal{P}$ is the Kähler metric given by the Chern form of the augmented Hodge line bundle, the curvature matrix of $T\mathcal{P}^0$ is computed from a positive (1,1) form that is itself the curvature of a singular metric. In the 1-parameter case the dominant term in ω is the Poincaré metric $\text{PM} = dt \otimes d\bar{t}/|t|^2(-\log|t|)^2$, and the curvature of the PM is a positive constant times $-\text{PM}$. One may again suspect that the contributions of the lower order terms in ω are less singular than PM. This issue may well be relevant to Question I.A.11.

(iv) *Examples.* On the smooth points of \mathcal{P}^0 of the image of a period mapping the holomorphic bi-sectional curvature satisfies

$$(VI.B.17) \quad R(\eta, \xi) \leq 0,$$

and for η, ξ in an open set in $T\mathcal{P}^0 \times_{\mathcal{P}^0} T\mathcal{P}^0$ it is strictly negative. This raises the interesting question of the degree of flatness of $T^*\mathcal{P}^0$. In the classical case when D is a Hermitian symmetric domain and $B = \Gamma \backslash D$ is compact this question has been studied by Mok [Mok87] and others. In case B is a Shimura variety the related question of the degree of flatness of the extended Hodge bundle F_e over a toriodal compactification of $\Gamma \backslash D$ is one of current interest (cf. [Bru16a], [Bru16b] and the references cited there). This issue will be further discussed in Section VI.E.

Here we shall discuss the equation

$$\Theta_{I^0}(\eta, \xi) = 0$$

over the smooth locus \mathcal{P}^0 of \mathcal{P} . In view of (VI.B.10) this equation is equivalent to

$$\text{ad}_{\xi^*}(\eta) = 0, \quad \eta \in I.$$

To compute the dimension of the solution space to this equation, we use the duality

$$\ker(\text{ad}_{\xi}^*) = (\text{Im}(\text{ad}_{\xi}))^\perp$$

to have

$$(VI.B.18) \quad \dim \ker(\text{ad}_\xi^*) = \dim (\text{coker} (\text{Im}\{\text{ad}_\xi : \text{Gr}^0 \rightarrow \text{Gr}^{-1}\})).$$

Since I depends on the particular VHS, at least as a first step it is easier to study the equation

$$(VI.B.19) \quad \text{Ad}_{\xi^*}(\eta) = 0, \quad \eta \in \text{Gr}^{-1}.$$

Because the curvature form decreases on the sub-bundle $I \subset \text{Gr}^{-1}$, over \mathcal{P}^0 we have

$$(VI.B.18) \implies (VI.B.19)$$

but in general not conversely.

Example 1: For weight $n = 1$ with $h^{1,0} = g$, with a suitable choice of coordinates the tangent vector ξ is given by $g \times g$ symmetric matrix A , and on Gr^{-1} we have

$$(VI.B.20) \quad \dim \ker(\text{ad}_\xi^*) = \binom{g - \text{rank } A + 1}{2}$$

Proof. At a point we may choose a basis for that $Q = \begin{pmatrix} 0 & Ig \\ -Ig & 0 \end{pmatrix}$ and

$$\begin{aligned} F^1 \text{ is given by } & \begin{pmatrix} \Omega \\ Ig \end{pmatrix}, \quad \text{Im } \Omega > 0 \\ \xi \in \text{Gr}^{-1} \text{ is given by } & \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad A = {}^t A \\ \eta \in \text{Gr}^0 \text{ is given by } & \begin{pmatrix} C & 0 \\ 0 & -{}^t C \end{pmatrix}. \end{aligned}$$

Then

$$[\xi, \eta] = \begin{pmatrix} 0 & AC + {}^t CA \\ 0 & 0 \end{pmatrix}.$$

Diagonalizing A and using (VI.B.18) we obtain (VI.B.20).

Example 2: For weight $n = 2$, ξ is given by

$$A = h^{2,0} \times h^{1,1} \text{ matrix.}$$

We will show that on Gr^{-1}

$$(VI.B.21) \quad \dim \ker(\text{ad}_\xi^*) = (h^{2,0}\text{-rank } A)(h^{1,1}\text{-rank } A).$$

Proof. We may choose bases so that $Q = \text{diag}(I_{h^{2,0}}, -I_{h^{1,1}}, I_{h^{2,0}})$ and

$$F^2 \text{ is given by } \begin{pmatrix} \Omega \\ 0 \\ i\Omega \end{pmatrix}, \quad \Omega \text{ non-singular,}$$

$$\xi \text{ is given by } \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & {}^tA \\ 0 & 0 & 0 \end{pmatrix},$$

$$\eta \text{ is given by } \begin{pmatrix} C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & -{}^tC \end{pmatrix}.$$

Then

$$[\xi, \eta] = \begin{pmatrix} 0 & AC - DA & 0 \\ 0 & 0 & {}^tAD + {}^t(AC) \\ 0 & 0 & 0 \end{pmatrix}.$$

Choosing bases so that $A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ and $D = \begin{pmatrix} D_{11} & D_{12} \\ -{}^tD_{12} & D_{22} \end{pmatrix}$, we have

$$AC - DA = \begin{pmatrix} C_{11} & -D_{11} & D_{12} \\ 0 & -{}^tD_{12} & 0 \end{pmatrix}.$$

Setting $\text{rk}(E) = \text{rank } E$ for a matrix E , this gives

$$\begin{matrix} & \text{rk } A & h^{2,0}\text{-rk } A \\ \text{rk } A & \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \\ h^{1,1}\text{-rk } A & \end{matrix}$$

where the $*$'s are arbitrary. □

As in the $n = 1$ case we note that

$$(VI.B.22) \quad A \text{ of maximal rank} \iff \ker(\text{ad}_\xi^*) = 0.$$

Example 3: Associated to a several parameter nilpotent orbit

$$\exp\left(\sum_i \ell(t_i)N_i\right) \cdot F$$

is a nilpotent cone $\sigma = \{N_\lambda = \sum \lambda_i N_i, \lambda_i > 0\}$ and the weight filtration $W(N)$ is independent of $N \in \sigma$. As discussed in Section 2 of [GGLR17], without loss of generality in what follows here we may assume that the LMHS associate to $N \in \sigma$ is \mathbb{R} -split. Thus there is a single $Y \in \text{Gr}^0 \text{Hom}_Q(V, V)$ such that for any $N \in \sigma$

$$[Y, N] = -2N,$$

and using the Hard Leftschetz Property $N^k : \mathrm{Gr}_{n+k}^{W(N)}(V) \xrightarrow{\sim} \mathrm{Gr}_{n-k}^{W(N)}(V)$ we may uniquely complete Y, N to an $\mathfrak{sl}_2\{N, Y, N^+\}$. Let $\mathfrak{g}_\sigma \subset \mathrm{End}(\mathrm{Gr}_\bullet^{W(N)} V)$ be the Lie algebra generated by the N_i and Y . The properties of this important Lie algebra will be discussed elsewhere; here we only note that \mathfrak{g}_σ is semi-simple and that the nilpotent orbit gives a period mapping

$$\Delta^{*k} \xrightarrow{\Phi_\sigma} \Gamma_{\mathrm{loc}} \backslash D_\sigma$$

where $D_\sigma = G_{\sigma, \mathbb{R}}/H_\sigma$ is a Mumford-Tate sub-domain of D . Of interest are the holomorphic bi-sectional curvatures of $\Phi_\sigma(\Delta^{*k})$. We shall not completely answer this, but shall give a proof of the

PROPOSITION VI.B.23: $\Theta_I(\eta, N) = 0$ for all $N \in \sigma$, if and only if, $\eta \in \mathcal{Z}(\mathfrak{g}_\sigma)$.

Proof. We denote by $\mathfrak{g}_\mathbb{C} = \bigoplus_p \mathfrak{g}^{p, -p}$ the Hodge decomposition on the associated graded to the limiting mixed Hodge structure defined by σ . The Hodge metric is given on $\mathfrak{g}_\mathbb{C}$ by the Cartan-Killing form, and its restriction to $\mathfrak{g}^{-1, -1}$ is non-degenerate.⁵⁹ The decomposition of $\mathfrak{g}_\mathbb{C}$ into N -strings for the \mathfrak{sl}_2 given by $\{N, Y, N^+\}$ is orthogonal with respect to the Hodge metric, from which we may infer that the adjoint ad_{N^*} acts separately on each N -string. The picture is something like

$$\eta \circ \begin{array}{c} \xrightarrow{\eta} \\ \xleftarrow{N^*} \end{array} \circ \begin{array}{c} \xrightarrow{\eta} \\ \xleftarrow{N^*} \end{array} \circ \begin{array}{c} \xrightarrow{\eta} \\ \xleftarrow{N^*} \end{array} \circ.$$

Because N is an isomorphism the same is true of N^* ; consequently

$$\Theta_I(\eta, N) = 0 \iff \eta \text{ belongs to an } N\text{-string of length 1,}$$

and this implies that $[\eta, Y] = [\eta, N^+] = 0$. By varying N over σ we may conclude the proposition. \square

Example 4: One of the earliest examples of the positivity of the Hodge line bundle arose in the work of Arakelev ([Ara71]). For 1-parameter families it gives an *upper* bound on the degree of the Hodge line bundle in terms of the degree of the logarithmic canonical bundle of the parameter spaces.⁶⁰ This result has been extended in a number of directions; we refer to [CMSP17], Section 13.4 for further general discussion and references to the literature.

One such extension is due to [Zuo00], [VZ03] and [VZ06]. This proof of that result centers around the above observation that the curvature of Hodge bundles has a sign on the kernels of Kodaira-Spencer mappings. There is a new ingredient in the argument that will be useful in other contexts and we shall now explain this. As above there are singularity issues that arise where the differential of Φ fails to be injective. These may be treated in a similar manner to what was done above, and for simplicity of exposition and to get at the essential new point we shall assume

⁵⁹The decomposition of $\mathfrak{g}_\mathbb{C}$ into the primitive sub-spaces and their images under powers of N depends on the particular N . The Hodge metric on $\mathfrak{g}^{-1, -1}$ is only definite on the subspaces arising from the primitive decomposition for such an N .

⁶⁰Using the above notations, the logarithmic canonical bundle of the parameter space is $K_{\overline{B}}(Z)$.

that Φ_* is everywhere injective and that the relevant Kodaira-Spencer mappings have constant rank.

The basic Arakelov-type inequality then exists at the curvature level. For a variation of Hodge structure (V, Q, ∇, F) over B with a completion to \bar{B} with $Z = \bar{B} \setminus B$ a reduced normal crossing divisor, the inequality is

$$(VI.B.24) \quad \left(\begin{array}{c} \text{curvature of} \\ \det \text{Gr}^p V \end{array} \right) \leq C_p \left(\begin{array}{c} \text{curvature of} \\ \det \Omega_{\bar{B}}^1(\log Z) \end{array} \right)$$

where C_p is a positive constant that depends on the ranks of the Kodaira-Spencer mappings. Here we will continue using the notations

$$(VI.B.25) \quad \begin{cases} \text{Gr}^p V = F^p V / F^{p+1} V, \\ \text{Gr}^p V \xrightarrow{\theta} \text{Gr}^{p-1} V \otimes \Omega_{\bar{B}}^1(\log Z). \end{cases}$$

The second of these was denoted by θ^p above; we shall drop the “ p ” here but note that below we shall use θ^ℓ to denote the ℓ^{th} iterate of θ .

Proof of (VI.B.24). Using the integrability condition (VI.B.9) the iterates of (VI.B.25) give

$$\text{Gr}^p V \xrightarrow{\theta^\ell} \text{Gr}^{p-\ell} V \otimes \text{Sym}^\ell \Omega_{\bar{B}}^1(\log Z)$$

We use the natural inclusion $\text{Sym}^\ell \Omega_{\bar{B}}^1(\log Z) \subset \otimes^\ell \Omega_{\bar{B}}^1(\log Z)$ and consider this map as giving

$$(VI.B.26) \quad \text{Gr}^p V \xrightarrow{\theta^\ell} \text{Gr}^{p-\ell} V \otimes \left(\otimes^\ell \Omega_{\bar{B}}^1(\log Z) \right).$$

There is a filtration

$$\ker \theta \subset \ker(\theta^2) \subseteq \dots \subseteq \ker \theta^{p+1} = \text{Gr}^p V$$

and $\text{Gr}^p V$ has graded quotients

$$\ker \theta, \frac{\ker \theta^2}{\ker \theta}, \dots, \frac{\text{Gr}^p V}{\ker \theta^p}.$$

The crucial observation (and what motivates the above use of \otimes^ℓ rather than Sym^ℓ) is

$$(VI.B.27) \quad \begin{aligned} & \frac{\ker \theta^\ell}{\ker \theta^{\ell+1}} \hookrightarrow \text{Gr}^{p-\ell+1} V \otimes \left(\otimes^\ell \Omega_{\bar{B}}^1(\log Z) \right) \\ & \text{lies in } K^{p-\ell+1} \otimes \left(\otimes^{\ell-1} \Omega_{\bar{B}}^1(\log Z) \right) \text{ where} \\ & K^{p-\ell+1} = \ker \left\{ \text{Gr}^{p-\ell+1} V \xrightarrow{\theta} \text{Gr}^{p-\ell} V \otimes \Omega_{\bar{B}}^1(\log Z) \right\}. \end{aligned}$$

From this we infer that

- (i) K^p, K^{p-1}, \dots, K^o all have negative semi-definite curvature forms;

$$(ii) \frac{\ker \theta^\ell}{\ker \theta^{\ell-1}} \hookrightarrow K^{p-\ell+1} \otimes \left(\bigotimes^{\ell-1} \Omega_B^1(\log Z) \right)$$

which gives

$$(iii) \det \left(\frac{\ker \theta^\ell}{\ker \theta^{\ell-1}} \right) \hookrightarrow \wedge^{d_{p,\ell}} \left(K^{p-\ell+1} \otimes \left(\bigotimes^{\ell-1} \Omega_B^1(\log Z) \right) \right).$$

Using

$$(iv) \det \operatorname{Gr}^p V \cong \bigotimes_{\ell=1}^{p+1} \det \left(\frac{\ker \theta^\ell}{\ker \theta^{\ell-1}} \right)$$

and combining (iv), (iii) and (ii) at the level of curvatures gives (VI.B.24). \square

Note: In [GGK08] there are results that in the 1-parameter case express the “error term” in the Arakelov inequality by quantities involving the ranks of the Kodaira-Spencer maps and structure of the monodromy at the singular points.

VI.C. The Iitaka conjecture.

One of the main steps in the general classification theory of algebraic varieties was provided by a proof of the Iitaka conjecture. An important special case of this conjecture is the

THEOREM VI.C.1: *Let $f : X \rightarrow Y$ be a morphism between smooth projective varieties and assume that*

- (i) $\operatorname{Var} f = \dim Y$ (i.e., the Kodaira-Spencer maps are generically 1-1);
- (ii) the general fibre $X_y = f^{-1}(y)$ is of general type.

Then the Kodaira-Iitaka dimensions satisfy

$$(VI.C.2) \quad \kappa(X) \geq \kappa(X_y) + \kappa(Y).$$

As noted in the introduction, this result was proved with one assumption (later seen to not be necessary) by Viehweg ([Vie83a], [Vie83b]), and in general by Kollár [Kol87]. The role of positivity of the Hodge vector bundle had earlier been identified in [Fuj78], [Kaw81], [Kaw83], [Kaw85] and Ueno [Uen74], [Uen78]. Over the years there has been a number of interesting results concerning the positivity of the Hodge vector bundle and, in the geometric case, the positivity of the direct images of the higher pluricanonical series; cf. [PT14] and also [Pă16] and [Sch15] for recent results and a survey of some of what is known together with further references.

To establish (VI.C.2) one needs to find global sections of ω_X^m . From

$$h^0(\omega_X^m) = h^0(f_*\omega_X^m)$$

and

$$f_*\omega_X^m = f_*\omega_{X/Y}^m \otimes \omega_Y^m,$$

the issue is to find sections of $f_*\omega_{X/Y}^m$. If for example $f_*\omega_{X/Y}^m$ is generically globally generated, then we have at least approximately $h^0(\omega_{X_y}^m) \cdot h^0(\omega_Y^m)$ sections of $f_*\omega_X^m$ which leads to the result.

To find sections of $f_*\omega_{X/Y}^m$, if for example $\mathrm{Sym}^m f_*\omega_{X/Y}$ has sections, then since the multiplication mapping

$$(VI.C.3) \quad \mathrm{Sym}^m f_*\omega_{X/Y} \rightarrow f_*\omega_{X/Y}^m$$

is injective on decomposable tensors, we get sections of the image. In general the issue is to find sections of $\mathrm{Sym}^{m''} f_*\omega_{X/Y}^{m'}$ and use an analogue of (VI.C.3).

The arguments in [Vie83b], [Kol87] have two main aspects:

- (i) the use of Hodge theory;
- (ii) algebro-geometric arguments using Viehweg's notion of weak positivity.

In the works cited above, Hodge theory is used to show that under the assumptions in (VI.C.1) for $m \gg 0$ we have

$$\kappa(\det f_*\omega_{X/Y}^m) = \dim Y.$$

From this one wants to infer that $\mathrm{Sym}^{m''} f_*\omega_{X/Y}^{m'}$ has sections for $m', m'' \gg 0$. This is where (ii) comes in.

The objectives of this section are twofold. One is to show that in case local Torelli holds for $f : X \rightarrow Y$ step (ii) may be directly circumvented by using the special form $\Theta_F = -{}^t\bar{A} \wedge A$ of the curvature in the Hodge vector bundle where A has algebro-geometric meaning that leads to positivity properties.⁶¹ The other is to discuss Viehweg's branched covering construction, which provides a mechanism to apply the positivity properties of the Hodge vector bundle to the pluri-canonical series.

One issue has been that the assumptions (i), (ii) plus Viehweg's branched covering method give

$$(VI.C.4) \quad \kappa(\det(f_*\omega_{X/Y}^m)) = \dim Y.$$

From this one wants to show that

$$(VI.C.5) \quad \left\{ \begin{array}{l} \left(\det f_*\omega_{X/Y}^m \right)^{m'} \text{ has} \\ \text{lots of sections} \end{array} \right\} \implies \left\{ \begin{array}{l} \mathrm{Sym}^{m''} \left(f_*\omega_{X/Y}^m \right) \text{ has} \\ \text{lots of sections} \end{array} \right\}.$$

The Viehweg-Kollár method proves (VI.C.4) using the positivity properties of the Hodge *line* bundle, and from this goes on to infer (VI.C.5) by an algebro-geometric argument involving Viehweg's concept of weak positivity for a coherent sheaf.⁶² The positivity of the Hodge *vector* bundle does not enter directly.

We will give a four-step sketch of the proof of Theorem VI.C.1, one that avoids the use of weak positivity. The first three steps follow from the discussions above. The fourth step uses a variant of Viehweg's argument to derive Hodge theoretic information from the pluri-canonical series together with (III.B.7) and (III.B.8).

⁶¹As noted above, A is the end piece of the differential of the period mapping.

⁶²The definition for a vector bundle was recalled above.

Step one (already noted above): Suppose that the fibres of $X \xrightarrow{f} Y$ are smooth and that $\Phi_{*,n}$ is injective at a general point. Then by Theorem III.B.1

$$\kappa \left(\text{Sym}^{h^{n,0}}(f_*\omega_{X/Y}) \right) = \dim \mathbb{P}_{f_*\omega_{X/Y}}(1) > \dim Y.$$

Step two: This is the same as step one but where we allow singular fibres. What is needed to handle these follows from the discussion in Section IV.B.

Reformulation of step two: We set $S^k = \text{Sym}^k$ and

- $F_e = f_*\omega_{X/Y}$;
- $\omega_k =$ curvature form of $\mathcal{O}_{\mathbb{P}S^k F_e}(1)$

where ω_k is the (1,1) form computed using the induced Hodge metric in $S^k F$. Then $\omega_k > 0$ in the vertical tangent space to a general fibre of $\mathbb{P}S^k F_e \rightarrow Y$. Using the injectivity of the Kodaira-Spencer map

$$T_y Y \rightarrow \text{Hom}(F_y^n, F_y^{n-1}/F_y^n)$$

at a general point $y \in Y$, from Theorem III.B.1 it follows that for all $k \geq h^{n,0}$

$$(VI.C.6) \quad \omega_k > 0 \text{ in the horizontal space at a general point in } (\mathbb{P}S^k F_e)_y.$$

This gives

$$\omega > 0 \text{ in an open set in } \mathbb{P}S^k F_e,$$

which implies that $\mathcal{O}_{\mathbb{P}S^k F_e}(1)$ is big. This in turn implies the same for $S^k F_e$.

Step three: Since $S^k f_*\omega_{X/Y}$ is big, $S^\ell(S^k f_*\omega_{X/Y})$ is generically globally generated for $\ell \gg 0$. It follows that the direct summand $S^{k\ell} f_*\omega_{X/Y}$ is generically globally generated for $\ell \gg 0$. Since

$$S^{k\ell} f_*\omega_{X/Y} \rightarrow f_*\omega_{X/Y}^{k\ell}$$

is injective on decomposable tensors, we obtain at least approximately

$$h^0 \left(\omega_{X_y}^{k\ell} \right) h^0 \left(\omega_Y^{k\ell} \right)$$

sections of $\omega_X^{k\ell}$.

Remark VI.C.7: In the geometric case an alternative geometric argument that $S^k f_*\omega_{X/Y}$ is big may be given as follows:

First, for a family $W \xrightarrow{g} Y$ with smooth general fibre $W_y = g^{-1}(y)$, the condition (III.B.3) for bigness of $g_*\omega_{W/Y}$ is:

- (i) for general $y \in Y$ the Kodaira-Spencer map $\rho_y : T_y Y \rightarrow H^1(TW_y)$ should be injective;
- (ii) for general $\psi \in (g_*\omega_{W/Y})_y = H^0(\Omega_{W_y}^n)$ the map

$$H^1(TW_y) \xrightarrow{\psi} H^1(\Omega_{W_y}^{n-1})$$

should be injective on the image $\rho_y(T_y Y) \subset H^1(TW_y)$.

Next, for a family $X \rightarrow Y$ with generically injective Kodaira-Spencer mappings, we set

$$W = X \overbrace{\times_Y \times \cdots \times_Y}^k X.$$

Then $H^0(\Omega_W^n)$ contains $\otimes^k H^0(\Omega_X^n)$ as a direct summand and the same argument as in the proof of Theorem (III.B.7) gives that $\otimes^k f_*\omega_{X/Y}$ is big.

Finally, we may apply a similar argument to a desingularization of the quotient of $W \rightarrow Y$ by the action of the symmetric group. Then the general fibre is

$$X_y^{(k)} = \text{desingularization of } \text{Sym}^k X_y$$

with

$$S^k H^0(\Omega_{X_y}^n) = \text{direct summand of } H^0(\Omega_{X_y^{(k)}}^n)$$

and again the argument in the proof of Theorem III.B.7 will apply.

Step four: The idea is to apply the reformulation of step two with $f_*\omega_{X/Y}^m$ replacing $f_*\omega_{X/Y}$. This will be discussed below in which the pluricanonical series of a smooth variety will be seen to have Hodge theoretic interpretations.⁶³

An alternate more direct approach would be to have a metric in

$$\mathcal{O}_{\mathbb{P}S^k f_*\omega_{X/Y}^m}(1)$$

whose Chern form ω is positive in the space at a general point of $\mathbb{P}S^k f_*\omega_{X/Y}^m$. For this approach to work one would need to have a metric in

$$f_*\omega_{X/Y}^m$$

that has a property analogous to that obtained using the Kodaira-Spencer map in the case $m = 1$ considered above. Here one possibility might be to use the relative Bergman kernel metrics that have appeared from the recent work of a number of people; cf. [PT14] and the references cited therein. This possibility will be discussed further at the end of Section VI.C.

Before giving the detailed discussion we will give some general comments.

(a) A guiding heuristic principle is

(VI.C.8) *For families $f : X \rightarrow Y$ of varieties of general type the*

$$f_*\omega_{X/Y}^k$$

become more positive as k increases.

The reasons are

⁶³An alternative approach to the Iitaka conjecture which replaces the use of Hodge theory by vanishing theorems and also uses the cyclic covering trick has been used by Kollár (*Ann. of Math.* **123** (1986), 11–42). The relation between vanishing theorems and Hodge theory is classical dating to Kodaira-Spencer and Akizuki-Nakano in the 1950's. As noted by a number of people, Hodge theory in some form and the curvature properties of the Hodge bundles seem to generally be lurking behind the positivity of direct images of pluricanonical sheaves.

- $f_*\omega_{X/Y} \geq 0$ by Hodge theory, and $S^k f_*\omega_{X/Y}$ becomes more positive with k by the argument in the proof of Theorem III.B.7;
- the map

$$S^\ell f_*\omega_{X/Y}^k \rightarrow f_*\omega_{X/Y}^{k:\ell}$$

is non-trivial since it is injective on decomposable tensors (similar to the proof of Clifford's theorem);

- passing to the quotient increases positivity (curvatures increase on quotient bundles).

(b) In what follows we will without comment make simplifying modifications by replacing $f : X \rightarrow Y$ by $f' : X' \rightarrow Y'$ where all the varieties are smooth and where

- $Y' \rightarrow Y$ is an isomorphism outside a codimension 2 subvariety of Y ;
- X' is a desingularization of $X \times_{Y'} Y$ and f' is flat;
- the fibres $X'_{y'} = f'^{-1}(y')$ are smooth outside a normal crossing divisor in Y' around which the local monodromies are unipotent ([Vie83b], page 577).

(c) We now recall the two constructions from the introduction that associate to $f_*\omega_{X/Y}^m$ a variation of Hodge structure; these constructions will be done first for a fixed smooth W and then for a family $f : X \rightarrow Y$ where W is a typical general fibre X_y . The objectives are to illustrate how pluricanonical series can give rise to Hodge structures.

(i) The case of fixed W , Hodge structures associated to $H^0(K_W^m)$.

We summarize and establish notation for the standard construction of a cyclic covering

$$\widetilde{W}_\psi \rightarrow W$$

associated to $\psi \in H^0(K_W^m)$ whose divisor $(\psi) \in |mK_X|$ is smooth. The construction is

$$(VI.C.9) \quad \begin{array}{ccc} \widetilde{W}_\psi & \subset & K_W = \text{total space of the line bundle} \\ \pi \downarrow & & \downarrow \\ W & = & W \end{array}$$

where

$$\widetilde{W}_\psi = \{(w, \eta) : \eta \in K_{W,w}, \eta^m = \psi(w)\}.$$

Then the direct image

$$(VI.C.10) \quad \psi_* K_{\widetilde{W}_\psi} \cong \bigoplus_{i=0}^{m-1} K_W^{m-i}$$

which gives

$$(VI.C.11) \quad H^0(K_{\widetilde{W}_\psi}) \cong \bigoplus_{i=0}^{m-1} H^0(K_W^{m-i}).$$

In this way pluri-differentials on W become ordinary differentials on \widetilde{W}_ψ ; this is the initial step in the relation between the pluricanonical series and Hodge theory.

We observe that the cyclic group \mathbb{G}_m acts on $\widetilde{W}_\psi \rightarrow W$ with the action by $\zeta = e^{2\pi i/m}$ on $H^0(K_{\widetilde{W}_\psi})$ and by ζ^{i+1} on $H^0(K_X^{m-i})$ in (VI.C.11).

We also observe the diagram

$$(VI.C.12) \quad \begin{array}{ccc} \widetilde{W}_\psi & \xrightarrow{\sim} & \widetilde{W}_{\lambda\psi} \\ \downarrow & & \downarrow \\ W & \xlongequal{\quad} & W \end{array}$$

induced by scaling the action of $\lambda \in \mathbb{C}^*$ on $K_W \rightarrow W$ in (VI.C.9). *The isomorphism $\xrightarrow{\sim}$ in (VI.C.12) depends on the choice of $\lambda^{1/m}$.*

We denote by $H^0(K_W^m)^0 \subset H^0(K_W^m)$ the ψ 's with smooth divisor (ψ), and we set

$$\mathbb{P}^0 = \mathbb{P}H^0(K_W^m)^0 \subset \mathbb{P} = H^0(K_W^m).$$

Then $H^0(\mathcal{O}_{\mathbb{P}}(1)) \cong H^0(K_W^m)^*$, and the identity gives a canonical section

$$(VI.C.13) \quad \Psi \in H^0(W \times \mathbb{P}, K_W^m \boxtimes \mathcal{O}_{\mathbb{P}}(1)).$$

Definition: (Ψ) is the *universal divisor of the (ψ)'s for $\psi \in H^0(K_W^m)$.*

To construct a universal family of cyclic coverings $\widetilde{W}_\psi \rightarrow W$ it is necessary to choose an auxiliary cyclic covering

$$\widetilde{\mathbb{P}} \xrightarrow{q} \mathbb{P},$$

with $q^*\mathcal{O}_{\mathbb{P}}(1) = L^m$ for an ample line bundle $L \rightarrow \widetilde{\mathbb{P}}$. From

$$W \times \widetilde{\mathbb{P}} \xrightarrow{\text{id} \times q} W \times \mathbb{P}$$

we obtain

$$H^0(W \times \widetilde{\mathbb{P}}, K_W^m \boxtimes L^m) \cong H^0(W \times \widetilde{\mathbb{P}}, (K_W \boxtimes L)^m)$$

and using the pullback to $W \times \widetilde{\mathbb{P}}$ of Ψ in (VI.C.13) there is a cyclic covering

$$\widetilde{W}_\Psi := \widetilde{W} \times \widetilde{\mathbb{P}} \xrightarrow{h} W \times \mathbb{P}$$

branched over the universal divisor (Ψ) and depending on the choice of $\widetilde{\mathbb{P}} \rightarrow \mathbb{P}$. In this way the choice of isomorphisms in (VI.C.12) necessitated by choosing an m^{th} root of λ may be made uniform. We observe that

$$(VI.C.14) \quad \begin{cases} \widetilde{\Psi} \in H^0(K_{\widetilde{W}_\Psi/\widetilde{\mathbb{P}}}) \\ \widetilde{\Psi}^m = p^*\Psi. \end{cases}$$

Setting $\widetilde{W}_\Psi^0 = h^{-1}(W \times \mathbb{P}^0)$ the total space and the fibres of

$$\widetilde{W}_\Psi^0 \rightarrow \mathbb{P}^0$$

are irreducible and smooth. This gives a period mapping, or equivalently a VHS,

$$(VI.C.15) \quad \Phi : \mathbb{P}^0 \rightarrow \Gamma \backslash D.$$

PROPOSITION VI.C.16: *The Hodge vector bundle*

$$F_\Psi \cong \bigoplus_{i=0}^{m-1} H^0(K_W^{m-i}) \otimes L^{i+1}.$$

Arguments as in sections (1.4)–(1.7) of [Vie83b] show that for $m \gg 0$ local Torelli holds for the period mapping (VI.C.15). In fact, we have the

PROPOSITION VI.C.17: *For $m \gg 0$ the part*

$$\Phi_{*,n} : T\tilde{\mathbb{P}}^0 \rightarrow \text{Hom}(H^0(K_W^m) \otimes L, F_\Psi^{n-1}/F_\Psi^n)$$

of the end piece of $\Phi_{,n}$ is injective.*

This is proved by showing that Φ_* is injective on certain of the eigenspaces for the \mathbb{G} -equivariant action in the picture, and that the ζ -eigenspace is among those that are included.

Discussion of singularities: For $\psi \in H^0(K_W^m)$ the Finsler-type norm

$$\begin{aligned} \|\psi\| &= \int_W (\psi \wedge \bar{\psi})^{1/m} \\ &= \int_{\widetilde{W}_\psi} \tilde{\psi} \wedge \bar{\tilde{\psi}} \end{aligned}$$

is equal to the square of the Hodge length of $\tilde{\psi} = \psi^{1/m} \in H^0(K_{\widetilde{W}_\psi})$. Even when the divisor (ψ) acquires singularities so that \widetilde{W}_ψ becomes singular, the Hodge length of the canonical section $\tilde{\psi}$ will remain finite. However, although $\|\psi\|$ is continuous in ψ , it is not smooth as its derivatives detect singularities of the degenerating Hodge structures.⁶⁴ In terms of limiting mixed Hodge structures, $\tilde{\Psi}$ lies in the lowest possible weight part.

The second method of associating Hodge theoretic data to the pluricanonical series $H^0(K_W^m)$ is the following: Let \widetilde{W}_Ψ be a desingularization of a completion of \widetilde{W}_Ψ^0 and

$$\widetilde{W}_\Psi \xrightarrow{\tilde{\pi}} \mathbb{P}$$

the resulting fibration. If $\dim \widetilde{W}_\Psi = \tilde{n}$, then $H^{\tilde{n}}(\widetilde{W}_\Psi)$ has a polarized Hodge structure. The general Hodge theory of maps such as $\tilde{\pi}$ is contained in the *decomposition theorem* ([dCM09]). In the case at hand a special feature arises in that from Proposition VI.C.16 we may infer that

(VI.C.18) $H^{\tilde{n},0}(\widetilde{W}_\Psi)$ contains as a direct summand

$$\bigoplus_{i=0}^{m-1} H^0(K_W^{m-i}) \otimes H^0(\mathcal{O}_{\mathbb{P}}(n_i)).$$

Here the $n_i > 0$ are determined by the $\nu_i > 0$ in

$$q_* L^{i+1} \cong \bigoplus_{\mathbb{P}} \mathcal{O}_{\mathbb{P}}(\nu_i).$$

⁶⁴Norms of this type appear in [NS68] and have been used extensively in the recent literature (cf. [Pă16] for a summary and survey).

(ii) The case of a family $f : X \rightarrow Y$; Hodge structures associated to $f_*\omega_{X/Y}^m$.

The following is a sketch of the proof of Theorem VI.C.1. It is intended to point out some of the geometric aspects of the arguments in [Vie83b], in particular the way in which the Hodge-theoretic interpretations enter into those arguments, referring to that paper for the details.

Since the publication of [Vie83b] several important general results concerning the pluricanonical series have been established (cf. [Dem12a]), and we shall assume the following:

- for general $y \in Y$, X_y is smooth and there is an ℓ such that for all $m = k\ell$, $k \geq 1$, the linear system $|mK_{X_y}|$ is ample; whenever an m appears below it will be of this form; and
- the assumptions to have local Torelli in the form used in [Vie83b] are satisfied for $f_*\omega_{X/Y}^m$.

The *basic diagram* is

$$(VI.C.19) \quad \begin{array}{ccccccc} \tilde{\mathcal{X}} & \longrightarrow & \mathcal{X}_2 & \longrightarrow & \mathcal{X}_1 & \longrightarrow & X \\ \tilde{f} \downarrow & & \downarrow \tilde{h} & & \downarrow h & & \downarrow f \\ \tilde{\mathbb{P}} & \xrightarrow{g} & \tilde{\mathbb{P}} & \xrightarrow{q} & \mathbb{P} & \xrightarrow{p} & Y \end{array}$$

where

- $\mathbb{P} = \mathbb{P}f_*\omega_{X/Y}^m$, so that $p_*\mathcal{O}_{\mathbb{P}}(1) = f_*\omega_{X/Y}^{m*}$ is the *dual* of $f_*\omega_{X/Y}^m$;
- $\mathcal{X}_1 = X \times_Y \mathbb{P}$;
- $\tilde{\mathbb{P}} \xrightarrow{q} \mathbb{P}$ is a cyclic branched covering where there is an ample line bundle $L \rightarrow \tilde{\mathbb{P}}$ with

$$q^*\mathcal{O}_{\mathbb{P}}(1) = L^m;$$

- $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathbb{P}} \tilde{\mathbb{P}}$; and
- $\tilde{\mathcal{X}} \rightarrow \mathcal{X}_2$ is the cyclic covering obtained by globalizing the construction of $\tilde{W}_{\Psi} \rightarrow W \times \tilde{\mathbb{P}}$ given by the completion of \tilde{W}_{Ψ}^0 in (VI.C.14) above.

The players in the basic diagram are

- $f^{-1}(y) = X_y$;
- $p^{-1}(y) = \mathbb{P}H^0(\omega_{X_y}^m)$, whose points are $[\psi]$ where $\psi \in H^0(\omega_{X_y}^m)$;
- $q^{-1}([\psi]) = \{[\psi_i]\}$ where $[\psi_i] \xrightarrow{q} [\psi]$ under the cyclic covering;
- Ψ is the tautological section of $\omega_{\mathcal{X}_1/\mathbb{P}}^m \otimes h^*\mathcal{O}_{\mathbb{P}}(1)$;
- $\tilde{\Psi}$ is the tautological section of

$$\omega_{\mathcal{X}_2/\tilde{\mathbb{P}}}^m \otimes \tilde{h}^*L^m = \left(\omega_{\mathcal{X}_2/\tilde{\mathbb{P}}} \otimes \tilde{h}^*L \right)^m;$$

- $\tilde{\mathcal{X}} \rightarrow \mathcal{X}_2$ is the m -sheeted cyclic covering obtained by extracting an m^{th} root of $\tilde{\Psi}$.

We will denote the fibre over y of $\tilde{\mathcal{X}} \rightarrow Y$ by

$$(VI.C.20) \quad \tilde{\mathcal{X}}_y = \bigcup_{[\psi] \in \mathbb{P}H^0(\omega_{X_y}^m)} \tilde{X}_{y,\psi}$$

where X_y corresponds to W and $\tilde{X}_{y,\psi}$ to \tilde{W}_ψ above.

There are two families of varieties constructed from the basic diagram. Denoting the composition $q \circ g \circ \tilde{f}$ by G for the first we have

$$(VI.C.21) \quad G : \tilde{\mathcal{X}} \rightarrow \mathbb{P}$$

whose fibre over $(y, [\psi]) \in \mathbb{P}$ is $\tilde{X}_{y,\psi}$. The second is

$$(VI.C.22) \quad F : \tilde{\mathcal{X}} \rightarrow Y$$

whose fibre over $y \in Y$ is the variety (VI.C.20).

There are two important observations concerning these families:

(VI.C.23) *Generic local Torelli holds for both families; and*

(VI.C.24) *The basic diagram (VI.C.19) is commutative.*

The fibres of $G_*\omega_{\tilde{\mathcal{X}}/\mathbb{P}}$ are given by

$$(G_*\omega_{\tilde{\mathcal{X}}/\mathbb{P}})_{(y, [\psi])} = H^0(\omega_{\tilde{X}_{y,\psi}}).$$

From (VI.C.23) we have

$$(VI.C.25) \quad G_*\omega_{\tilde{\mathcal{X}}/\mathbb{P}} \geq 0 \text{ and } \det G_*\omega_{\tilde{\mathcal{X}}/\mathbb{P}} > 0 \text{ on an open set.}$$

From (VI.C.25) we have

$$(VI.C.26) \quad \text{For } k \gg 0, \text{ both } S^k G_*\omega_{\tilde{\mathcal{X}}/\mathbb{P}} \text{ and } S^k F_*\omega_{\tilde{\mathcal{X}}/Y} \text{ are big.}$$

To complete the proof of Theorem VI.C.1 the argument one first might try to make is this:

$H^0(\omega_{X_y}^m)$ is a direct factor of $H^0(\omega_{\tilde{X}_{y,\psi}})$, and from the commutativity of the basic diagram (VI.C.19) it follows that $f_\omega_{X/Y}^m$ is a direct factor of the Hodge vector bundle associated to the family $\tilde{\mathcal{X}} \rightarrow Y$. By the local Torelli property (VI.C.23) with the implication (VI.C.26), it follows that $S^k f_*\omega_{X/Y}^m$ is big.*

However, this argument is not correct; the issue is more subtle. The problem is that under the mapping

$$\tilde{\mathcal{X}} \xrightarrow{H} \mathcal{X}_1$$

in (VI.C.19) we do *not* have

$$H_*\omega_{\tilde{\mathcal{X}}/\mathbb{P}} \cong \bigoplus_{i=0}^{m-1} \omega_{\mathcal{X}_1/\mathbb{P}}^{m-i},$$

but rather

$$H_*\omega_{\tilde{\mathcal{X}}/\mathbb{P}} \cong \bigoplus_{i=0}^{m-1} \omega_{\mathcal{X}_1/\mathbb{P}}^{m-i} \otimes h^*\mathcal{O}_{\mathbb{P}}(i+1)$$

where the $\mathcal{O}_{\mathbb{P}}(i+1)$'s reflect the global twisting of the identification (VI.C.10). This would not be an issue if $\mathcal{O}_{\mathbb{P}}(1)$ were positive. This positivity trivially holds along the fibres of $\mathbb{P} \rightarrow Y$, but since $p_*\mathcal{O}_{\mathbb{P}}(1) = f_*\omega_{X/Y}^m$ any positivity of $f_*\omega_{X/Y}^m$ becomes negativity of $\mathcal{O}_{\mathbb{P}}(1)$ in directions normal to fibres of $\mathbb{P} \rightarrow Y$. An additional step is required.

The key observation is that (cf. [Vie83b], page 587)

$$\text{For } a > 0, \omega_{X_1/\mathbb{P}}^{ma+a+1} \otimes h^*\mathcal{O}_{\mathbb{P}}(a) \text{ is a direct summand of } G_*\omega_{\tilde{X}/\mathbb{P}}^{a+1}.$$

It is the additional factor of $a+1$ in $\omega_{X_1/\mathbb{P}}^{ma+a+1}$ that offsets the negativity of $\mathcal{O}_{\mathbb{P}}(a)$ in the normal direction of the fibres of $\mathbb{P} \rightarrow Y$.

Thus what needs to be shown is

$$(VI.C.27) \quad S^k G_*\omega_{\tilde{X}/\mathbb{P}}^{a+1} \text{ is big for } k \gg 0.$$

This follows from (VI.C.27) if we have

$$S^{a+1} G_*\omega_{\tilde{X}/\mathbb{P}} \rightarrow G_*\omega_{\tilde{X}/\mathbb{P}}^{a+1} \rightarrow 0,$$

and this may be accomplished generically in Y by choosing $m \gg 0$.

Remark VI.C.28: We conclude with a comment and a question. The comment is that perhaps the most direct way to prove the Iitaka Conjecture VI.C.2 would be to use the curvature properties of the Finsler-type metric in $f_*\omega_{X/Y}^m$. Specifically, referring to [Ber09], [BPǎ12] and [PT14] for details we set

$$\mathbb{P}^* = \mathbb{P}(f_*\omega_{X/Y}^m)^*.$$

Then $\mathcal{O}_{\mathbb{P}^*}(1) \xrightarrow{\hat{\pi}}$ Y has fibres

$$(VI.C.29) \quad \mathcal{O}_{\mathbb{P}^*}(1)_{(y, [\eta])} = H^0(\omega_{X_y}^m)/\eta^\perp, \quad \eta \in H^0(\omega_{X_y}^m)^*$$

and

$$\hat{\pi}_*\mathcal{O}_{\mathbb{P}^*}(1) = f_*\omega_{X/Y}^m.$$

Following [Kaw82] and [PT14], we define a metric in $\mathcal{O}_{\mathbb{P}^*}(1)$ by taking the infimum of the $\|\psi\|$'s where $\psi \in H^0(\omega_{X_y}^m)$ projects to a fixed vector in the quotient (VI.C.29). Leaving aside the question of singularities, for this metric the form ω_m satisfies

$$\omega_m \geq 0.$$

We do not expect to have $\omega_m > 0$ in an open set in \mathbb{P}^* : this is already not the case when $m = 1$. Instead we carry out a similar construction replacing $f_*\omega_{X/Y}^m$ by $S^{m'} f_*\omega_{X/Y}^m$ and denote by $\omega_{m',m}$ the Chern form of the corresponding $\mathcal{O}(1)$ -bundle.

Question: Assuming $\text{Var } f = \dim Y$, for $m' \gg 0$ do we have $\omega_{m',m} > 0$ in an open set?

This is true when $m = 1$, and if it holds for general m and the issue of singularities can be handled one would have a direct ‘‘curvature’’ proof of VI.C.2. In fact, heuristic reasoning suggests the following

CONJECTURE VI.C.30: *Under the assumptions in Theorem VI.C.1, let $m \geq 1$ be such that for general $y \in Y$ the bundle $K_{X_y}^m$ is globally generated. Then for $P_m(X_y) = h^0(K_{X_y}^m)$*

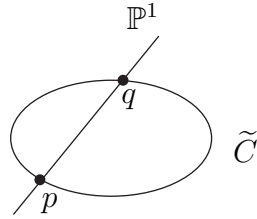
$$\mathrm{Sym}^{m'} f_* \omega_{X/Y}^m \text{ is big for } m' \geq P_m(X_y).$$

VI.D. The Hodge vector bundle may detect extension data.

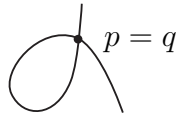
PROPOSITION VI.D.1: *On a subvariety $Y \subset Z_I^*$ along which the period mapping Φ_I is locally constant, the extended Hodge bundle $F_{e,I} \rightarrow Y$ is flat. It may however have non-trivial monodromy.*

Proof. $F_{e,I}$ is filtered by $W_\bullet(N_I) \cap F_{e,I}$. For \mathbb{V}_I the local system corresponding to $\Phi_I : Z_I^* \rightarrow \Gamma_I \backslash D_I$, the Gauss-Manin connection ∇ acting on $\mathcal{O}_{Z_I^*}(\mathbb{V}_I)$ preserves $W_\bullet(N_I)$, and on the associated graded to $W_\bullet(N_I)\mathbb{V}_I$ it preserves the associated graded to $W_\bullet(N_I) \cap F_{e,I}$. It follows that ∇ preserves $F_{e,I} \subset \mathcal{O}_Y(\mathbb{V})$. \square

Example VI.D.2: On an algebraic surface S suppose we have a smooth curve \tilde{C} of genus $g(\tilde{C}) \geq 2$, and that through each pair of distinct points $p, q \in \tilde{C}$ there is a unique rational curve meeting \tilde{C} at two points.



Suppose moreover that along the diagonal $D \subset \tilde{C} \times \tilde{C}$ the \mathbb{P}^1 becomes simply tangent. Then $\tilde{C} + \mathbb{P}^1$ is a nodal curve, and for $m \gg 0$ the pluricanonical map given by $|m\omega_{\tilde{C} + \mathbb{P}^1}|$ contracts the \mathbb{P}^1 and we obtain an irreducible stable curve $C_{p,q}$ with arithmetic genus $p_a(C_{p,q}) = 3$.



As \mathbb{P}^1 becomes tangent we obtain a cusp.

The extension data for the MHS is given (cf. [Car80]) by

$$\mathrm{AJ}_{\tilde{C}}(p - q) \in J(\tilde{C}).$$

When we turn around $p = q$ we interchange p, q ; after base change $t \rightarrow t^2$ the extension class is well defined locally. Globally, it is another story.

The vector space $H^0(\omega_{C_{p,q}})$, which is the fibre of the canonically extended Hodge bundle, has the 2-dimensional fixed subspace $H^0(\Omega_{\tilde{C}}^1)$ and variable 1-dimensional quotient represented by a differential of the third kind $\varphi_{p,q} \in H^0(\Omega_{\tilde{C}}^1(p + q))$ having non-zero residues at p, q . As $p \rightarrow q$ the differential $\varphi_{p,q}$ tends to a differential of

the second kind $\varphi_{2p} \in H^0(\Omega_{\tilde{C}}(2p))$ with non-zero polar part at p . For the global monodromy over $Y = \tilde{C} \times \tilde{C} \setminus D$, the extension class is given for $\psi \in H^0(\Omega_{\tilde{C}}^1)$ by

$$\psi \rightarrow \int_p^q \psi \text{ (modulo periods).}$$

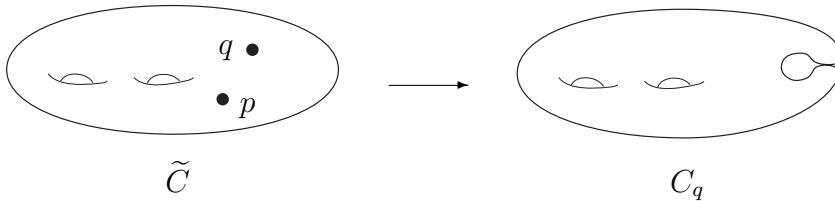
The action of $\pi_1(Y)$ on $H_1(\tilde{C}, \{p, q\})$ may then be shown to give an infinite subgroup of $H_1(\tilde{C}, \mathbb{Z})$, which implies the assertion.

CLAIM VI.D.3: *The Hodge vector bundle may detect continuous extension data.*

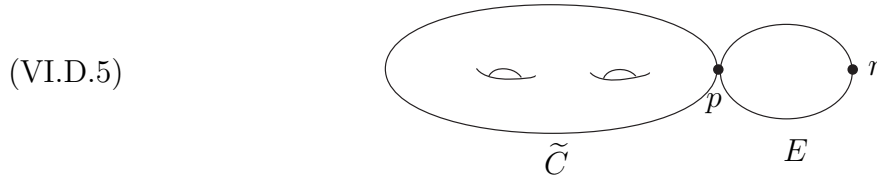
Here continuous extension data means the extension classes that arise in the induced filtration of $F_e \rightarrow Y$ where there is a variation of limiting mixed Hodge structure over Y with F_e the Hodge vector bundle.

Example VI.D.4: We let \tilde{C} be a smooth curve with $g(\tilde{C}) \geq 2$ and $p \in \tilde{C}$ a fixed point. We construct a family $C_q, p \in \tilde{C}$, of stable curves as follows.

- For $q \neq p$ we identify p, q



- For $p = q$, we obtain a curve



In this way we obtain a VMHS parametrized by \tilde{C} and with trivial monodromy.

For the filtration on the canonically extended Hodge bundle there is a fixed part

$$W_1(N) \cap F_e^1 \cong H^0(\Omega_{\tilde{C}}^1),$$

and a variable part whose quotient is

$$W_2(N) \cap F_e^1 / W_1(N) \cap F_e^1 = \begin{cases} \mathbb{C}\varphi_{p,q} & \text{for } q \neq p \\ \mathbb{C}\varphi_{E,r} & \text{for } q = p. \end{cases}$$

The notation means that $\varphi_{E,r}$ is the differential of the third kind on the normalization of E and with residues at ± 1 at the two points over the node. Since there is no monodromy we may normalize the $\varphi_{p,q}$ and $\varphi_{E,r}$.

Combining the above we have over \tilde{C} an exact sequence

$$0 \rightarrow H^0(\Omega_{\tilde{C}}^1) \otimes \mathcal{O}_{\tilde{C}} \rightarrow F_e \rightarrow \mathcal{O}_{\tilde{C}} \rightarrow 0$$

with non-trivial extension class

$$e = \text{“Identity”} \in H^1(\mathcal{O}_{\tilde{C}}) \otimes H^0(\Omega_{\tilde{C}}^1).$$

VI.E. The exterior differential system defined by a Chern form. In this section we will discuss the exterior differential system

$$(VI.E.1) \quad \omega = 0$$

defined by the Chern form of the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ where $E \rightarrow X$ is a Hermitian vector bundle whose curvature has the norm positivity property (III.A.1). Without assuming the norm positivity property, this type of EDS has been previously studied in [BK77] and [Som59] and also appeared in [Kol87].

Here our motivation is the following question:

(VI.E.2) *Under what conditions can one say that the Kodaira-Iitaka dimension of $E \rightarrow X$ is equal to its numerical dimension?*

PROPOSITION VI.E.3: *The exterior differential system (VI.E.1) defines a foliation of $\mathbb{P}E$ by complex analytic subvarieties $W \subset \mathbb{P}E$ with the properties*

(i) *W meets the fibres of $\mathbb{P}E \xrightarrow{\pi} X$ transversely; thus $W \rightarrow \pi(W)$ is an étalé map;*

(ii) *the restriction $E|_{\pi(W)}$ is flat.*

Proof. Since $\omega > 0$ on the fibres of $\mathbb{P}E \rightarrow X$, the vectors $\xi \in T_{(x,[e])}\mathbb{P}E$ that satisfy $\omega(\xi) = 0$ project isomorphically to TX . The image of these vectors is the subspace (here identifying ξ with $\pi_*(\xi)$)

$$(VI.E.4) \quad \{\xi \in T_x X : A(e \otimes \xi) = 0\}.$$

This is the same as the subspace of $T_x X$ defined by

$$\Theta_E(e \otimes \xi) = 0,$$

which implies that $E|_{\pi(W)}$ is flat. □

Remark: Given any holomorphic bundle map

$$(VI.E.5) \quad A : TX \otimes E \rightarrow G,$$

if we have a metric in $E \rightarrow X$ we may use it to identify $E \cong E^*$ and then define the horizontal sub-bundle $H \subset T\mathcal{O}_{\mathbb{P}E}(1)$. It follows that (VI.E.4) defines a C^∞ distribution (with jumping fibre dimensions) in $T\mathcal{O}_{\mathbb{P}E}(1)$, and when the map (VI.E.5) arises from the curvature of the metric connection as in (III.A.3) this distribution is integrable and the maximal leaves of the corresponding foliation of $\mathbb{P}E$ by complex analytic subvarieties are described by Proposition VI.E.3.

The restrictions $E|_{\pi(W)}$ being flat, the monodromy is discrete. Heuristic arguments suggest that the maximal leaves $W \subset \mathbb{P}E$ are *closed* analytic subvarieties.

CONJECTURE VI.E.6: *Finite monodromy provides the necessary and sufficient condition to have the equality*

$$\kappa(E) = n(E)$$

of Kodaira-Iitaka and numerical dimensions of a holomorphic vector bundle having a Hermitian metric whose curvature satisfies the norm positivity condition.

The idea is that the quotient $\mathbb{P}E/\sim$, where \sim is the equivalence relation given by the connected components of the foliation defined by (VI.E.1), exists as a complex analytic variety of dimension equal to $n(E)$, and there is a meromorphic mapping

$$\mathbb{P}E \dashrightarrow \mathbb{P}E/\sim$$

together with an ample line bundle on $\mathbb{P}E/\sim$ that pulls back to $\mathcal{O}_{\mathbb{P}E}(1)$. The rather simple guiding model here is the dual of the universal sub-bundle over the Grassmannian that was discussed above. In fact, the conjecture holds if $E \rightarrow X$ is globally generated with metrics induced from the corresponding mapping to a Grassmannian.

We note that the foliation defined by the null space of the holomorphic bi-sectional curvature on quotients of bounded symmetric domains has been studied in [Mok87]. In this case the leaves are generally not closed.

Finally we point out the very interesting papers [CD17a] and [CD17b]. In these papers the authors construct examples of smooth fibrations

$$f : X \rightarrow B$$

of a surface over a curve such that for $E = f_*\omega_{X/B}$ one has

$$E = A \oplus Q$$

where A is an ample vector bundle and Q is a flat $\mathcal{U}(m, \mathbb{C})$ -bundle with infinite monodromy group.⁶⁵ In this case the leaves of the EDS (VI.E.1) may be described as follows: For each $b \in B$ we have

$$\mathbb{P}Q_b^* \subset \mathbb{P}E_b^*$$

and using the flat connection on Q^* the parallel translate of any point in $\mathbb{P}Q_b^*$ defines an integral curve of the EDS.

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⁶⁵We note that $Q \subset f_*\omega_{X/B} \subset R_f^1\mathbb{C}_X$ is *not* flat relative to the Gauss-Manin convention on $R_f^1\mathbb{C}_X$.

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