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January 18, 1960

Dr. Ivar Stakgold, Head
Mathematics Branch
Office of Naval Research
Department of the Navy
Washington 25, D. C.

Ref: ONR:432:IS:lgb, NR 043-201

Dear Dr. Stakgold:

Following up my letter of November 10, 1959 to you, I have today sent you under separate cover ten (10) copies of the Final Report on Contract Nonr 1358(06). I trust that you will receive them in good order.

Sincerely yours,

A. H. Taub

cc: Dr. Oppenheimer ✓

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THE SCIENTIFIC FILES OF JOHN VON NEUMANN

Report on Contract Nonr 1358(06)

November 1959

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1. The funds of the subject contract have been used to support work on the scientific material in the files left by John von Neumann. This material is quite varied in nature, ranging from rough notes to almost completed, to completed unpublished manuscripts.

2. Experts in the subject matter of the various manuscripts were asked to review them with a view toward their publication in their entirety. If they were deemed unsuitable for such publication, they were to be judged as to their suitability for having a short review published. A review consists of a description of the manuscript given in sufficient detail so that a reader could determine whether he was interested enough to take steps to see the full manuscript, which is deposited in the library of the Institute for Advanced Study. The publication of the reviews would enable the scientific public at large to benefit from the ideas propounded by, and the techniques used by von Neumann in papers he did not finish or publish.

3. The manuscripts which were reviewed and material related to manuscripts which are to be published for the first time in the collected works have been gathered into File A. Attachment A lists the items in this file and contains a description of each item in the file. The description consists either of a copy of the review of the manuscript, if one has been written, or a short listing of the contents of the folder containing the item in question.

4. It should be pointed out that before funds from the subject contract were made available, various mathematicians devoted quite a bit of time to studying the von Neumann files. One result of this work has been the publication of a posthumous paper by von Neumann:

> Non-isomorphism of Certain Continuous Rings (with an introduction by I. Kaplansky). Ann. Math. 67, No. 3 (1958), pp. 485-496.

5. Two papers in which von Neumann was a co-author have appeared since his death. These are:

> The Jacobi Method for Real Symmetric Matrices (with H. H. Goldstine and F. J. Murray). J. Assoc. for Computing Machinery 6, No. 1 (1959), pp. 59-96.

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

A Study of a Numerical Solution to a Two-Dimensional Hydrodynamical Problem (with A. Blair, N. Metropolis, A. H. Taub and M. Tsingou). MTAC XIII (1959), pp. 145-164.

6. The files contained many manuscripts related to lecture courses at Princeton University, The Institute for Advanced Study and to lectures given before the American Mathematical Society. Some of these lectures have been published in mimeograph form. The lecture notes have been gathered into File B. Attachment B lists the lecture notes and material related to them in File B.

7. A number of unpublishable, partially completed manuscripts were found and gathered into File C. These are listed and described in Attachment C.

8. Attachment D contains a listing of rough notes not in manuscript form, written by von Neumann on various scientific subjects. The notes themselves are in a file called File D.

9. Attachment E consists of a bibliography of books, papers and reports written by J. von Neumann. Government reports which are still classified are omitted from this bibliography.

A. H. Taub
Principal Investigator

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

ATTACHMENT A

FILE A: Manuscripts Reviewed and Papers First Published in the Collected Works

		Page
A. 1	Comparison of Cells. Reviewed by G. Hunt	5
A. 2	Characterization of Factors of Type II_1. Reviewed by I. Kaplansky.	6
A. 3	Independence of F_∞ From the Sequence ν. Reviewed by I. Halperin.	10
A. 4	Continuous Geometries With a Transition Probability. Reviewed by I. Halperin. Earlier drafts of this manuscript are in this folder also.	13
A. 5	Quantum Logics. (Strict- and Probability-Logics.) Reviewed by A. H. Taub. An earlier draft is also included in this folder.	18
A. 6	Lattice Abelian Groups. Reviewed by G. Birkhoff.	23
A. 7	Measure in Functional Spaces. Reviewed by I. Halperin. Halperin's corrections to the manuscript are in folder.	25
A. 8	Representation of Certain Linear Groups by Unitary Operators in Hilbert Space. Reviewed by G. W. Mackey.	33
A. 9	On the Principles of Large Scale Computing Machines.	37
A. 10	Nonlinear Capacitance or Inductance Switching, Amplifying and Memory Devices.	38
A. 11	Notes on the Photo-Disequilibrium-Amplification Scheme. Reviewed by John Bardeen.	39
A. 12	First and Second Reports on the Numerical Calculation of Flow Problems. The folder contains drafts of the reports, and copies of these.	40
A. 13	Discussion of a Maximum Problem.	42
A. 14	Numerical Method for Determination of Value and Best Strategies of 0-Sum, 2-Person Game With Large Number of Strategies. Reviewed by H. W. Kuhn and A. W. Tucker.	43
A. 15	Symmetric Solution of Some General n Person Games. Reviewed by O. Morgenstern.	45
A. 16	Static Solution of Einstein Field Equations for Perfect Fluid With $T^{\rho}_{\rho} = 0$. Reviewed by A. H. Taub.	46
A. 17	On Relativistic Gas-Degeneracy and the Collapsed Configuration of Stars. Reviewed by A. H. Taub.	48
A. 18	The Point Source Model. Reviewed by A. H. Taub	50

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

		Page
A. 19	The Point Source Solution, Assuming a Degeneracy of the Semi-Relativistic Type $p = K\rho^{4/3}$ Over the Entire Star. Reviewed by A. H. Taub.	51
A. 20	Discussion of DeSitter's Space and of Dirac's Equation in it. Reviewed by A. H. Taub.	52
A. 21	The Taylor Instability Problem. Reviewed by H. H. Goldstine.	54
A. 22	Recent Theories of Turbulence. This folder has copies of this report and correspondence relating to it.	56
A. 23	Description of the Conformal Mapping Method for the Integration of Partial Differential Equation Systems with 1+2 Independent Variables. Reviewed by A. H. Taub.	57

ATTACHMENT B

Listing of Contents of Folders in FILE B: Lecture Notes, Drafts of These, and Material Absorbed in These. 61

ATTACHMENT C

Listing of Contents of Folders in FILE C: Unpublishable Manuscripts. 63

ATTACHMENT D

Listing of Contents of Folders in FILE D: Notes Not in Manuscript Form. 72

ATTACHMENT E

Bibliography. 76

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

COMPARISON OF CELLS

Reviewed by G. Hunt

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von Neumann dealt with the equivalence of measure spaces in several contexts. The present manuscript proves in detail the following theorem, which is well known for linear intervals:

Theorem: Let P be an n-cell (that is, a compact n-dimensional parallelepiped) and let ρ, σ be Radon measures on P which have the same finite total mass, which attribute no mass to the boundary of P or to any set reduced to a single point, and which attribute strictly positive mass to every non-empty open set. Then, there is a homeomorphism ϕ of P on itself which leaves fixed each point of the boundary and which carries ρ into σ (in the sense that $\rho(\phi(M)) = \sigma(M)$ for every Borel set M).

For $n > 1$, a set whose dimension lies between 0 and n may carry part of the mass of ρ or σ. This fact renders the theorem implausible, at first glance, and causes all the complications of the proof.

Notes to the reader: 1) In the proof of Lemma 1, the assertion concerning functional determinants is incorrect. It is easily proved, however, that the function displayed in the proof does all that is required.

2) On page 20, it is mistakenly asserted twice that the uniform limit of homeomorphisms is itself a homeomorphism. That the maps U_1^o and U_2^o are indeed homeomorphisms follows quickly from the properties of the construction.

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CHARACTERIZATION OF FACTORS OF TYPE II_1

Reviewed by I. Kaplansky

In [2], the first of the series on rings of operators, there occurs on pages 321-322 the following passage:

"A more general problem which will be discussed by the second named author in a forthcoming paper arises now from the following motive: It would be desirable to characterize those abstract algebras, in which one and only one function T(A) with the properties of the trace (as discussed above) exists."

A draft of this manuscript was found after von Neumann's death. It is hand-written, 21 pages long, and untitled. At the top appear the words, "Rewrite this with some changes!"

Five axioms are given for a certain class of rings, and then 47 lemmas develop the theory, ending abruptly in the middle of a sequence of ideas. No theorem is stated. Manifestly the work was intended to culminate in the theorem that a ring satisfies the axioms if and only if it is a factor of Type II_1 or I_n.

With the aid of later developments, it is easy to complete the proof; one way of doing so will be indicated below. How von Neumann himself planned to continue must be left to the conjecture of the informed reader.

Axioms (A) and (B) postulate an algebra R with unit over the complex numbers with a conjugate linear involution $*$. Define H to be the set of self-adjoint elements, D to be the set of sums of squares of self-adjoint elements. Axiom (C) asserts the existence of a unique trace T: a real-linear function on H satisfying $T(1) = 1$, $T(A^2) > 0$ for $A \neq 0$, and $T(U^*AU) = T(A)$ for U unitary, i.e. $UU^* = U^*U = 1$. Axiom (D) asserts that for any A in H there exists a real α such that $A + \alpha \in D$ (we are identifying scalars with scalar multiples of 1). The final axiom will be stated shortly.

Later investigations of this kind almost always began by assuming

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

a C^*-algebra as the fundamental object, adding suitable axioms to this foundation. We shall indicate below how the two points of view can be blended. It is noteworthy that the first four of von Neumann's axioms are purely algebraic, and that the fifth merely asserts the completeness of a certain metric.

We shall now state the 47 lemmas in condensed form, with parenthetical observations where appropriate.

L 1.-3. Elementary facts about D.

L 4. Facts about $\mathfrak{B}_A$ = the set of all α such that $\alpha - A^2 \in D$.

Definition. $\|A\|^2$ = greatest lower bound of $\mathfrak{B}_A$.

L 5. $\|A\|^2 \geqq T(A^2)$.

L 6. H is a (real) normed linear space under $\| \ \|$. [Proof of the triangle inequality requres a device of the Schwartz inequality type.]

L 7. For $A, B \in H$, $\|AB + BA\|$ and $\|i(AB - BA)\|$ are at most $\|A\| \ \|B\|$. [Another non-trivial argument.]

Definition. $|A|^2 = T(A^2)$.

L 8. H is a normed linear space under $| \ |$.

[Indeed it is a pre-Hilbert space with inner product $T(AB + BA)/2$, and the next lemma is the Schwartz inequality.]

L $8\frac{1}{2}$. $T(AB + BA) \leqq 2 \ \|A\| \ \|B\|$.

Now axiom (E) is stated: the unit sphere in the $\| \ \|$-norm is $| \ |$-complete.

L 9. Continuity properties relative to the two norms.

L 10. Passage to any $\| \ \|$-sphere.

L 11. H is $\| \ \|$-complete. [At this point H is a (real) Banach space under one norm and a pre-Hilbert space in a smaller norm. One cannot yet say anything about a Banach algebra, let alone C^*-algebra.]

L 12.-15. Existence of cos A, sin A for $A \in H$, and their properties.

L 16. For $A, B \in H$, $T(i(AB - BA)) = 0$. [The proof uses the unitary elements $\cos tB + i \sin tB$. At this point the trace can easily be extended

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to all of R and enjoys the property $T(AB) = T(BA)$ for all A and B.]

L 17. $T(A^*A) > 0$ for $A \neq 0$.

L 18. For $A \in R$, $B \in H$, $T(A^*B^*BA) \leq \|B\|^2 T(A^*A)$.

L 19.-25. These lemmas develop the procedure for taking continuous functions of self-adjoint elements, using approximating polynomials. [At this point we can easily argue that for $A \in H$, $\|A\|$ is the sup of the spectrum of A. Now we define $\|B\|$ for any element of R by $\|B\|^2 = \|B^*B\|$ and argue briefly that R is a C^*-algebra. It is possible that this goal could be reached by a shorter route.]

L 26, 27, $27\frac{1}{2}$, 28. Connections between positivity, trace, and norm.

L 29. If $A_i \in D$, and every finite sum ΣA_i is bounded by a fixed constant, then ΣA_i converges in $|\ |$.

L 30. $|AX + XA| \leq \|A\|\ |X|$.

L 31. The right annihilator of any element of R is generated by a projection. [The argument is lengthy and uses Lemmas 29 and 30. At this point we see that R is an AW^*-algebra, in the sense of [2]. (Note that the trace rules out the possible existence of an uncountable set of orthogonal projections.) The assumed uniqueness of the trace (not used at all in the von Neumann manuscript) implies that the center of R is just the complex numbers. In other words, R is an AW^*-factor with trace. From the paper of Feldman [1], or of Widom [4], one deduces that R is a ring of operators; indeed R is weakly closed when it acts on the Hilbert space obtained by completing R in the $|\ |$-norm. Of course, R must be of type I_n or II_1.

This, then, is how von Neumann's project can be completed with the aid of the work done 15-20 years later. The remainder of the von Neumann manuscript is evidence that these later writers in part reproduced things he knew.]

L 32. A and A^*A have the same right projection.

L 33. The right and left projections of an element have the same trace. [This, it goes without saying, requires a delicate argument.]

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

L 34.-35. Routine matters on the partial ordering of projections.

L 36.-37. The existence of the least upper bound of projections E, F and the fact that it is the projection of E + F.

L 38. $T(E \cap F) \leqq T(E) + T(F)$. [Deduced, as usual, by examining $F(1 - E)$.]

L 39.-40. Similar discussions for an infinite set of projections.

L 41.-45. This is a sequence of lemmas, leading up to the following: if A_i is a Cauchy sequence in the $|\ |$-norm, and there is a constant γ such that $T(X^* A_i^* A_i X) \leqq \gamma T(X^* X)$ for every X in R, then A_i is $|\ |$-convergent.

---

[1] J. Feldman, Embedding of AW^*-algebras, Duke Math. J. 23 (1956), 303-307.

[2] F. J. Murray and J. von Neumann, On rings of operators, Ann. of Math. 37 (1936), 116-229.

[3] I. Kaplansky, Projections in Banach algebras, Ann. of Math. 53 (1951), 235-249.

[4] H. Widom, Embedding in algebras of type I, Duke Math. J. 23 (1956), 309-324.

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INDEPENDENCE OF F_∞ FROM THE SEQUENCE ν

Reviewed by I. Halperin

1. Let $L_n = L_n(F)$ denote the lattice of linear subspaces of the n-1 dimensional projective geometry $P_{n-1}(F)$, where F is a fixed division ring. Then L_n can be imbedded in L_{nm} for every integer m.

Suppose ν is any increasing sequence of integers $1 = n_0 < n_1 < n_2 \ldots$ with each $\frac{n_\ell}{n_{\ell-1}}$ an integer ≥ 1 (from now on, sequence shall mean: a sequence with these properties). We may suppose $L_{n_0} \subset L_{n_1} \subset \ldots$; then the set theoretic sum of the L_{n_ℓ} forms a metric space and the metric closure is a continuous geometry $L_\infty{}^\nu(F)$, as shown by von Neumann (see: Examples of continuous geometries, P.N.A.S., vol. 22 (1936), pp. 101-108).

In this manuscript von Neumann proves the Theorem: $L_\infty{}^\nu(F) = L_\infty(F)$ does not depend on the particular sequence ν; moreover, for every element $a \neq 0$ in $L_\infty(F)$, the elements x with $0 \leq x \leq a$ form a continuous geometry isomorphic to $L_\infty(F)$.

2. The argument is carried out in terms of the "coordinating" rings of the L_n, $L_\infty{}^\nu$, namely F_n, $F_\infty{}^\nu$. Here F_n is the ring of n x n matrices with elements in F; F_n is imbedded in F_{nm} and $F_\infty{}^\nu$ is obtained from the F_{n_ℓ}, $\ell \geq 0$, as explained in Section 3 of von Neumann's paper: The non-isomorphism of certain continuous rings, Annals of Math., vol. 67 (1958), pp. 485-496.

3. It is easy to prove that $F_\infty{}^\nu$ and $F_\infty{}^\mu$ are ring isomorphic if μ is a subsequence of ν, hence whenever μ, ν have a common subsequence.

A sequence $\nu = (n_\ell;\ \ell = 0, 1, 2, \ldots)$ is called complete if every integer

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n is a divisor of some n_ℓ; $F_\infty^{\ \nu}$ and $F_\infty^{\ \mu}$ are then isomorphic if ν, μ are both complete.

For any θ with $0 < \theta < 1$ and arbitrary sequence ν there exists a subsequence $\nu' = (n_\ell;\ \ell \geq 0)$ and a complete sequence $\pi = (p_\ell;\ \ell \geq 0)$ such that $\frac{p_\ell}{p_{\ell-1}} \leq \frac{n_\ell}{n_{\ell-1}}$ for every ℓ, and $\lim_{\ell\to\infty} \frac{p_\ell}{n_\ell} = \theta$. From this von Neumann proves that $F_\infty^{\ \pi}$ is ring isomorphic to the ring $eF_\infty^{\ \nu}e$ where e is any idempotent in $F_\infty^{\ \nu}$ with rank $e = \theta$; and hence $F_\infty^{\ \nu}$, $F_\infty^{\ \mu}$ are isomorphic for any two sequences μ, ν.

The Theorem of Section 1 above now follows readily.

4. The last paragraph of the manuscript begins the proof that F_∞ and Z_∞ are ring isomorphic if Z is the centre of F and F has finite (linear) order over Z; but the manuscript breaks off at this point.

In his paper: Algebraic Theory of Continuous Geometries, P.N.A.S., vol. 23 (1937), pp. 16-22, especially page 21, von Neumann states that he proved this by algebraic considerations, mainly with the help of a theorem of T. Skolem and E. Noether.

In his third colloquium lecture (American Mathematical Society Colloquium Lectures, Continuous Geometry, 4 lectures by John von Neumann, delivered at Pennsylvania State College, State College, Pennsylvania, U.S.A., Sept. 7-10, 1937), von Neumann sketched the following proof of this theorem that F_∞ and Z_∞ are ring isomorphic.

Let $\mathcal{R} \times \mathcal{S}$ denote the direct product of two rings $\mathcal{R}$, $\mathcal{S}$, each of which contains Z in its centre. This means: $\mathcal{R} \times \mathcal{S}$ consists of all finite, formal sums $\Sigma(\alpha_i \times \xi_i)$ with α_i in $\mathcal{R}$, ξ_i in $\mathcal{S}$, and the usual identifications relative

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

to Z; in particular, $(\omega\alpha) \times \xi = \alpha \times (\omega\xi)$ if ω is in Z.

Suppose $\mathcal{R} = F_n$ where F is a division ring with centre Z. Then for every integer m, F_{nm} can be identified with $F_n \times Z_m$.

Suppose F is of linear order k over its centre Z with k finite and > 1. Then

$$F \times F' = F' \times F = Z_k$$

where F' denotes the division ring composed of the elements of F with addition as in F but multiplication the transpose of multiplication in F. Now, formally, F_∞ is the closure of

$$\begin{aligned} F \times Z_k \times Z_k \ldots &= F \times (F' \times F) \times (F' \times F) \times \ldots \\ &= (F \times F') \times (F \times F') \times \ldots \\ &= Z_k \times Z_k \times \ldots \quad . \end{aligned}$$

Since Z_∞ is the closure of the last expression, F_∞ and Z_∞ are ring isomorphic if F is of finite order k over its centre and k > 1. The statement holds trivially if k = 1.

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CONTINUOUS GEOMETRIES WITH A TRANSITION PROBABILITY

Reviewed by I. Halperin

This manuscript gives in detail the material summarized in von Neumann's fourth colloquium lecture (American Mathematical Society Colloquium Lectures, Continuous Geometry, 4 lectures by John von Neumann, delivered at Pennsylvania State College, State College, Pennsylvania, U.S.A., Sept. 7-10, 1937).

Von Neumann had previously invented continuous geometries as a generalization of finite dimensional projective geometries. Here he investigates those continuous geometries which can be used to describe quantum mechanical phenomena; specifically, he postulates that there is defined in the continuous geometry a suitable transition probability. The final result, after 200 pages of deep reasoning is (essentially): every such geometry with transition probability can be identified with the projection geometry of a finite (II_1 or I_m) factor in some finite or infinite dimensional Hilbert space.

This result indicates that continuous geometries do not provide new useful mathematical descriptions of quantum mechanical phenomena beyond those already available from rings of operators. But this investigation, like von Neumann's previous study of certain Jordan algebras (On an algebraic generalization of the quantum mechanical formalism (Part I). Rec. Math. (Mat. Sbornik) N.S. vol. 1 (1936), pp. 415-484) deepens our understanding of the theory of rings of operators in Hilbert space. Many arguments carried out in that theory by using the vectors of Hilbert space, are here given "algebraically," that is, without using vectors.

This manuscript postulates an abstract system L of elements a, b, c, ... satisfying 13 axioms. Some of these axioms force L to be a projective or continuous geometry which is orthocomplemented and possesses sufficiently many "unitary" automorphisms (see condition (T) of Chapter III, below). Chapter IV of the manuscript applies to all such geometries and is thus a continuation of Part II of von Neumann's 1935-37 Princeton lectures (to be published by Princeton University Press).

The manuscript consists of 9 chapters.

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Chapter I. This enumerates and gives the phenomenological interpretation of the following 13 axioms on L:

Axioms I, II, III, IV, V are equivalent to: L is an orthocomplemented modular lattice; (-a) denotes the orthogonal complement of a.

Axioms VI, VII, VIII: For all a, b in L with $a \neq 0$, a "probability of transition from a to b," P(a, b), is defined with: $0 \leq P(a, b) \leq 1$, $P(a, b) = 1$ if and only if $a \leq b$, and $P(a, b + c) = P(a, b) + P(a, c)$ whenever b and c are orthogonal.

Axiom IX (completeness): If a_n is a sequence in L with $P(c, a_n)$ convergent for every $c \neq 0$, then there exists an element a in L such that $P(c, a_n)$ converges to P(c, a) for every $c \neq 0$.

Axiom X (continuity): for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that,

$$P(1, a + b) \leq P(1, a) + \epsilon \quad \text{whenever } P(1, b) \leq \delta, \text{ and } a \neq 0.$$

Axiom XI (irreducibility): if for some fixed a,

$$c = ca + c\,(-a)$$

for every c, then $a = 0$ or $a = 1$.

Axiom XII (invariance of P(a, b)): if $c \to c'$ is any (<, -) automorphism of L then $P(a, b) = P(a', b')$.

Axiom XIII (existence of (<, -) automorphisms of L, $c \to c'$): this consists of two parts:

XIII_1: If $P(1, a) < P(1, b)$ there exists such $c \to c'$ with $a' = b$.

XIII_2: If $P(1, a) = P(1, b)$ there exists such $c \to c'$ with $a' = b$ and $c' = c$ for all c orthogonal to both a and b.

Chapters II, III. Axiom IX, along with axioms VI, VII, VIII, forces the orthocomplemented modular lattice L to be complete. Axiom X then forces L to be "continuous" (in the sense that von Neumann's axioms on continuity of lattice operations do hold). Axiom XI forces L to be irreducible. Thus L is a projective or continuous geometry with a dimension function D(a).

Then Axiom XIII forces P(1, a) to be identical with D(a).

(Actually, Axiom X is a consequence of Axioms I-IX, XIII since Kaplansky has now proved that every complete, orthocomplemented modular lattice is necessarily "continuous" (Annals of Mathematics, vol. 61 (1955), pp. 524-541). Also, Axiom IX is only required in the weaker form:

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Axiom IX': If a_n is an increasing sequence in L then for some a in L, $P(c, a_n)$ converges to $P(c, a)$ for every $c \neq 0$.

Finally, omission of Axiom XI would lead to a more general discussion of the (possibly) reducible geometries with transition probability.)

Suppose L is an orthocomplemented projective or continuous geometry but exclude the projective geometries of dimension 0, 1 or 2. Then L can be identified with the lattice of all principal right ideals of a discrete or continuous *-rank ring $\mathfrak{R}$, that is, $\mathfrak{R}$ is a discrete or continuous rank ring and possesses a conjugation $x \rightarrow x^*$; every a in L is of the form $(e)_r$ with hermitian idempotent e and $(1 - e)_r$ is the orthogonal complement of a.

Call an element u in $\mathfrak{R}$ unitary if $u^*u = 1$, equivalently $uu^* = 1$. Then Axiom XIII (along with Axioms VI, VII, VIII, IX) implies:

Condition $(T_{1/5})$ (existence of sufficiently many unitary automorphisms): whenever $D(a) = D(b) \leq 1/5$ there exists a unitary u such that the inner automorphism of $\mathfrak{R}$: $x \rightarrow u x u^{-1}$ generates a $(<, -)$ automorphism of L which maps a on b.

Now without assuming Axiom XIII or the existence of the function P(a, b), if we exclude also the projective geometries of dimension 3, condition $(T_{1/5})$ implies condition (T) (in which the value of D(a) = D(b) is not restricted to be $\leq 1/5$).

Chapter IV. This requires only that $\mathfrak{R}$ be a *-rank ring satisfying condition (T) and that its L is not a projective geometry consisting of one point. Call x in $\mathfrak{R}$ definite if $x = z^*z$ for some z in $\mathfrak{R}$. Then x + y is definite whenever x, y are definite and x, -x are both definite only if x = 0. Hence the centre of $\mathfrak{R}$, which is a division ring Z, has characteristic 0; thus Z can be taken to include the rational real numbers.

Write $x \geq y$ if x, y are hermitian and x - y is definite. Write $||x|| \leq k$ if k is a rational real number and $k - x^*x \geq 0$. Let $\mathfrak{B}$ be the set of all x with $||x|| \leq k$ for some k. The usual properties of $||\ ||$ are deduced, except for the fact: $||x|| \leq k$ for all $k > 0$ implies $x = 0$. Spectral-type theorems about p(x) are established "algebraically" where $p(\xi)$ is a polynomial in ξ with rational real coefficients and x is a hermitian element in $\mathfrak{B}$.

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Chapter V. A non-negative, real-valued function $t(x)$ is called a trace in $\mathfrak{R}$ if $t(x)$ is defined for every x in $\mathfrak{B}$ and: $t(x + y) = t(x) + t(y)$, $t(x) = t(x^*)$, $t(x) = 0$ only if $x = 0$, $t(u \times u^*) = t(x)$ for every unitary element u, and $t(1) = 1$. For such a trace, define $|x| = \sqrt{t(x^*x)}$. Then $|x|$ is an absolute value on $\mathfrak{B}$ and $|x-y|$ defines a metric on $\mathfrak{B}$. If such a trace exists then $||x|| \leq k$ for all $k > 0$ does force $x = 0$.

Chapters VI, VII. From now on assume that L satisfies all axioms I-XIII and is not a projective geometry of dimension 0, 1, 2 or 3. Then $P(a, b) = D(b)/D(a)$ whenever $a \geq b$ with $a \neq 0$, and a close analogue of the spectral theory of bounded operators in Hilbert space exists. A trace on $\mathfrak{R}$ does exist and it is unique.

$\mathfrak{B}$ has the following completeness property: if x_n is a sequence with all $||x_n|| \leq k$ for some fixed k and $|x_n - x_m| \to 0$ then there exists an x in $\mathfrak{B}$ with $|x_n - x| \to 0$. The centre Z of $\mathfrak{R}$ consists of all real of all complex numbers.

Chapter VIII. $\mathfrak{B}$ is a (not necessarily complete) inner product space if inner product is defined by: $(x, y) = t(xy^*)$ in case the centre Z consists of all real numbers, and by: $(x, y) = t(xy^*) - it(ixy^*)$ if Z consists of all complex numbers. Moreover, $(zx, y) = (x, z^*y)$, $(x^*, y^*) = (x, y)^*$, $(x, x)^{\frac{1}{2}} = |x|$ (as defined in Chapter V). $P(a, b)$ can be expressed as $(f, g)/|f|^2 = \frac{|fg|^2}{|f|^2}$ where f, g are hermitian idempotents with $a = (f)_r$, $b = (g)_r$.

Let Q denote the inner product completion of $\mathfrak{B}$, so Q is a real or complex Hilbert space of some finite or infinite dimension. Then the ring operations and the conjugation operation (*) in $\mathfrak{B}$ can be extended to Q.

For each u in $\mathfrak{R}$ define operators R_u, L_u on part of $\mathfrak{B}$ as follows: $R_u x$ is defined and $= xu$ if xu is in B, $L_u x$ is defined and $= ux$ if ux is in $\mathfrak{B}$. Then L_u, R_u are linear (possibly unbounded) operators with domains dense in Q; their closures $\tilde{L}_u$, $\tilde{R}_u$ permit non-pathological ring and * operations. If u is in B then R_u, L_u are bounded linear operators with closures defined everywhere in Q: $R_u x = xu$, $L_u x = ux$ for all x in Q.

Let $\mathcal{M}$, $\mathcal{N}$ denote the set of all $\tilde{R}_u$, $\tilde{L}_u$ respectively with u in $\mathfrak{B}$. Then $\mathcal{M}$, $\mathcal{N}$ are coupled factors in Q; $\mathcal{M}$, $\mathcal{N}$ are finite (II_1 or I_m) factors; $\mathcal{N}$ is $(+, ., *)$ isomorphic to $\mathfrak{B}$ under $\tilde{L}_u \longleftrightarrow u$ and $\mathcal{M}$ is $(+, ., *)$

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anti-isomorphic to $\mathfrak{B}$ under $\tilde{R}_u \longleftrightarrow u$.

Let $\mathfrak{U}(\mathfrak{M})$ denote the set of all closed linear operators X on Q with domain dense in Q for which $VXV^{-1} = X$ for all unitary operators V in $\mathfrak{N}$. Then $\mathfrak{U}(\mathfrak{M})$ is the set of all $\tilde{R}_x$ with x in $\mathfrak{R}$ and $\mathfrak{R}$ is (+, ., *) anti-isomorphic to $\mathfrak{U}(\mathfrak{M})$ under the correspondence $x \longleftrightarrow \tilde{R}_x$. Similarly $\mathfrak{R}$ is (+, ., *) isomorphic to $\mathfrak{U}(\mathfrak{N})$ under $x \longleftrightarrow \tilde{L}_x$.

Finally, let $L(\mathfrak{N})$ be the orthocomplemented lattice of all projections F in $\mathfrak{N}$ (so $F = \tilde{L}_f$ with f hermitian idempotent) and let $P(F_1, F_2)$ be defined as $(f_1, f_2)/|f_1|^2 = \dfrac{|f_1 f_2|^2}{|f_1|^2}$. Then L is (+, -, P) isomorphic to $L(\mathfrak{N})$. Thus every L satisfying Axioms I-XIII can be described as above in terms of some finite (II_1 or I_m) factor $\mathfrak{N}$ in some finite or infinite dimensional Hilbert space Q.

If the original L is a projective geometry of dimension $n \geq 4$, then $\mathfrak{R} = \mathfrak{B} = Q$; $\mathfrak{R}$ is the ring of $(n + 1) \times (n + 1)$ matrices $x = (x^{ij})$ with x^{ij} real numbers (or complex numbers, or real quaternionic numbers); and $|x| = \sqrt{\frac{1}{n+1} \Sigma |x^{ij}|^2}$. All such matrix rings $\mathfrak{R}$ do give systems L satisfying Axioms I-XIII.

Chapter IX. This begins the construction of canonical examples of systems satisfying Axioms I-XIII so that every such system L is isomorphic to one of the canonical examples.

The manuscript breaks off at this point but von Neumann inserted a note that Axiom IX can (and must) be weakened in order that the case: L is a continuous geometry (that is, not a projective geometry) be not excluded.

In his colloquium lecture von Neumann stated that all systems L satisfying Axioms I-XIII can be obtained as follows:

"Let Q be an arbitrary real or complex Hilbert space of arbitrary dimension, let $\mathfrak{N}$ be a finite factor (then there exists a trace t(A) with t(A) real (or complex) and satisfying $t(A + B) = t(A) + t(B)$, $t(AB) = t(BA)$, $t(A^*) = \overline{t(A)}$, $t(A^*A) > 0$ if $A \neq 0$, $t(1) = 1$ for all A, B in $\mathfrak{N}$). Now L can be chosen to be the set of projections E, F, ... in $\mathfrak{N}$ partially ordered by: $E \leq F$ if $E = FE$, with $(-E) = 1 - E$ and $P(E, F) = t(EF)/t(E)$."

Clearly Axiom IX should be replaced by Axiom IX'; then the preceding assertion follows easily from Chapters I to VIII.

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QUANTUM LOGICS. (STRICT- AND PROBABILITY-LOGICS.)

Reviewed by A. H. Taub

In an unfinished manuscript, written about 1937, von Neumann proposes to deal with the question, How does the system of logics apply to those mathematical models, which various current physical theories use to describe the physical world, i.e. which they substitute in its place? The system of logics is to be understood ... to include the propositional calculus only.

"Let S be the physical system, or rather the mathematical model of a physical system, to which we wish to apply logics. The system L of logics is then the set of all statements $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ which can be made concerning S. Such a statement is always one concerning the outcome of a certain measurement, which is to be performed on S. (For a more detailed discussion, cf. L.Q.M.[1], §§ 1.-4.) The fundamental relations and operations for elements of L are these:

(I) The relation of 'implication': $\mathfrak{A} \leq \mathfrak{B}$.
$\mathfrak{A} \leq \mathfrak{B}$ means this: If a measurement of $\mathfrak{A}$ on S has shown $\mathfrak{A}$ to be true, then an immediately subsequent measurement of $\mathfrak{B}$ on S will certainly show $\mathfrak{B}$ to be true.

(II) The operation of 'negation': $-\mathfrak{A}$.
$-\mathfrak{A}$ obtains as follows: The same measurement which is used to decide about the validity of $\mathfrak{A}$ is also used to decide about the validity of $-\mathfrak{A}$, but when the result concerning the validity of $\mathfrak{A}$ is 'yes,' then the one concerning $-\mathfrak{A}$ is 'no,' and conversely.

$\mathfrak{A} \leq \mathfrak{b}$ has clearly the following properties:

(A) $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{B} \leq \mathfrak{A}$ together are equivalent to $\mathfrak{A} = \mathfrak{B}$.

(B) $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{B} \leq \mathfrak{C}$ together imply $\mathfrak{A} \leq \mathfrak{C}$.

We also define:

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(III) $\mathfrak{A} \geq \mathfrak{B}$ means $\mathfrak{B} \leq \mathfrak{A}$. $\mathfrak{A} < \mathfrak{b}$ ($\mathfrak{A} > \mathfrak{B}$) means $\mathfrak{A} \leq \mathfrak{b}$ ($\mathfrak{A} \geq \mathfrak{B}$) but $\mathfrak{A} \neq \mathfrak{B}$.

Now (A), (B) and (III) permit us to infer immediately:

(C) If $\circledast$ stands for any one of the four relations $\leq, \geq, <, >$, then $\mathfrak{A} \circledast \mathfrak{B}$ and $\mathfrak{B} \circledast \mathfrak{C}$ together imply $\mathfrak{A} \circledast \mathfrak{C}$.

(C) shows, that $\mathfrak{A} < \mathfrak{B}$ (that is $\mathfrak{A} \leq \mathfrak{B}$) defines a 'partial ordering' of L. We also infer from (A) - (C):

(D) Given $\mathfrak{A}, \mathfrak{B}$ a $\mathfrak{C}$ is called a 'greatest lower bound' of $\mathfrak{A}, \mathfrak{B}$, if for every $\mathfrak{M}$ (in L) $\mathfrak{M} \leq \mathfrak{A}$ and $\mathfrak{M} \leq \mathfrak{B}$ together are equivalent to $\mathfrak{M} \leq \mathfrak{C}$.

If such a $\mathfrak{C}$ exists at all, then it is unique.

(E) Given $\mathfrak{A}, \mathfrak{B}$, a $\mathfrak{D}$ is called a 'least upper bound' of $\mathfrak{A}, \mathfrak{B}$ if for every $\mathfrak{M}$ (in L) $\mathfrak{M} \geq \mathfrak{A}$ and $\mathfrak{M} \geq \mathfrak{B}$ together are equivalent to $\mathfrak{M} \geq \mathfrak{D}$.

If such a $\mathfrak{D}$ exists at all, then it is unique.

The existence of the logical operations of 'conjunction' ('$\mathfrak{A}$ and $\mathfrak{B}$') and 'disjunction' ('$\mathfrak{A}$ or $\mathfrak{B}$') induces us to postulate:

(IV) Any two $\mathfrak{A}, \mathfrak{B}$ possess a (unique, cf. (D)) greatest lower bound $\mathfrak{C}$, to be denoted by $\mathfrak{A}\mathfrak{B}$ -- this is '$\mathfrak{A}$ and $\mathfrak{B}$.'

(V) Any two $\mathfrak{A}, \mathfrak{B}$ possess a (unique, cf. (E)) least upper bound $\mathfrak{D}$, to be denoted by $\mathfrak{A} + \mathfrak{B}$ -- this is '$\mathfrak{A}$ or $\mathfrak{B}$.'

"These considerations lay the foundations for an axiomatic treatment of L. But we are not prepared yet to take this up systematically. We must first discuss the influence of another basic constituent of logics -- as we wish to see it --, and also discuss several examples.

"I. 3. So far the only structure L possesses is defined by the 'primitive notions' $\mathfrak{A} \leq \mathfrak{B}$, $-\mathfrak{A}$, and the 'derived notions' $\mathfrak{A} + \mathfrak{B}$, $\mathfrak{A}\mathfrak{B}$. To this extent we will call L the system of 'strict logics.' In applications of L to actual physical reality, however, a further structure of L appears, which can only be expressed in terms of 'probability.' In other words: For any well defined state of our knowledge concerning the mathematical description of physical reality, that is for any reasonable mathematical model S, a probability function exists. So we have in L:

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(VI) The (real number-valued) function called 'probability function' : $P(\mathfrak{A}, \mathfrak{B})$.

$P(\mathfrak{A}, \mathfrak{B}) = \theta$ (θ a real number) means this: If a measurement of $\mathfrak{A}$ on S has shown $\mathfrak{A}$ to be true, then the probability of an immediately subsequent measurement of $\mathfrak{B}$ on S showing $\mathfrak{B}$ to be true, is equal to θ.

If we consider the structure which L acquires by making use of the function $P(\mathfrak{A}, \mathfrak{B})$ -- that is of all relations $P(\mathfrak{A}, \mathfrak{B}) = \theta$ (of course necessarily $0 \leqq \theta \leqq 1$) -- then L appears as a new system, which we will call the system of 'probability logics.'

"It is easy to see, that the system of strict logics is part of the system of probability logics, since $\mathfrak{A} \leqq \mathfrak{B}$, $-\mathfrak{A}$ can be defined in terms of $P(\mathfrak{A}, \mathfrak{B}) = \theta$ $(0 \leqq \theta \leqq 1)$. Indeed (VI) makes the following statements obvious:

(F) $P(\mathfrak{A}, \mathfrak{B}) = 1$ is equivalent to $\mathfrak{A} \leqq \mathfrak{B}$.

(G) $P(\mathfrak{A}, \mathfrak{B}) = 0$ is equivalent to $\mathfrak{A} \leqq -\mathfrak{B}$.

Hence $\mathfrak{A} \leqq \mathfrak{B}$ is directly defined by (F), that is by $P(\mathfrak{A}, \mathfrak{B}) = 1$; and $-\mathfrak{A}$ is indirectly defined by (G), as the (unique) $\mathfrak{C}$, for which $\mathfrak{U} \leqq \mathfrak{C}$ is equivalent to $\mathfrak{U} \leqq -\mathfrak{A}$ that is, for which $P(\mathfrak{U}, \mathfrak{C}) = 1$ is equivalent to $P(\mathfrak{U}, \mathfrak{A}) = 0$.

"Conversely (F), (G) define $P(\mathfrak{A}, \mathfrak{B}) = \theta$ for $\theta = 0, 1$ in terms of strict logics.

"For a θ with $0 < \theta < 1$, however, no such 'reduction' of $P(\mathfrak{A}, \mathfrak{B}) = \theta$ to strict logics seems possible. It is of course feasible, to perform the procedure described in (VI) above on a large number N of specimens $S_1^*, \ldots, S_N^*$ of S_1 and then to interpret $P(\mathfrak{A}, \mathfrak{B}) = \theta$ as a frequency statement. I. e.: If we measure on each $S_1^*, \ldots, S_N^*$ first $\mathfrak{A}$, and then in immediate succession $\mathfrak{B}$, and if then number those among $S_1^*, \ldots, S_N^*$ where $\mathfrak{A}$ is found to be true is M, and the number of those where $\mathfrak{A}, \mathfrak{B}$ are both found to be true is M', then

(H) $P(\mathfrak{A}, \mathfrak{B}) = \theta$ means, that $\frac{M'}{M} \to \theta$ for $N \to \infty$.

This view, the so-called 'frequency theory of probability' has been very brilliantly upheld and expounded by R. v. Mises. This view, however,

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is not acceptable to us, at least not in the present, 'logical' context. We do not think, that (H) really expresses a convergence-statement in the strict mathematical sense of the word -- at least not without extending the physical terminology and ideology to infinite systems (namely, to the entirety of an infinite sequence $S_1^*, S_2^*, \ldots$) -- and we are not prepared to carry out such an extension already now. The approximative forms of (H), on the other hand, are mere probability-statements, e.g. 'Bernouilli's law of great numbers':

($H_{appr.}$) For any two $\epsilon, \delta > 0$ there exists an $N_0 = N_0(\epsilon, \delta)$, such that

for $N \geq N_0$ the probability of $|\frac{M'}{M} - \theta| < \epsilon$ is $> 1 - \delta$.

(We assume, that $N \to \infty$ implies $M \to \infty$, that is we exclude 'absurd' $\mathcal{U}$'s.) And such probability-statements are again of the same nature, as the relation $P(\mathcal{U}, \mathcal{E}) = \theta$, which they should interpret.

"We prefer therefore to disclaim any intention to interpret the relations $P(\mathcal{U}, \mathcal{E}) = \theta$ $(0 < \theta < 1)$ in terms of strict logics. In other words, we admit:

"Probability logics cannot be reduced to strict logics, but constitutes an essentially wider system than the latter, and statements of the form $P(\mathcal{U}, \mathcal{E}) = \theta$ $(0 < \theta < 1)$ are perfectly new and sui generis aspects of physical reality.

"So probability logics appear as an essential extension of strict logics. This view, the so-called 'logical theory of probability' is the foundation of J. N. Keynes' work on this subject."

von Neumann had intended to discuss four examples:

1. A system "which behaves in the sense of classical physics (in particular mechanics) and which possesses only a finite number of possible different states."

2. A system similar to 1 in which the number of states is discretely infinite.

3. A system similar to 1 in which the number of states is continuously infinite.

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4. A system where the finiteness of the number of states remains essentially untouched by the "classical" way of looking at things ... is replaced by a "quantum mechanical" one.

He also intended to give a final synthesis by combining these two kinds of extensions.

Unfortunately the manuscript contains only a discussion of example 1.

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[1] L. Q. M.: Garrett Birkhoff and John von Neumann, The Logics of Quantum Mechanics, Ann. Math. 37 (1936), 823-843.

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LATTICE ABELIAN GROUPS

Reviewed by Garrett Birkhoff

This manuscript covers about half the work presented orally by von Neumann in Spring 1940, at a Seminar on Lattice Theory given with the reviewer at the Institute for Advanced Study. The concept of a lattice-ordered Abelian group (alias "commutative ℓ-group") has been discussed briefly in a half-forgotten paper by Dedekind [1, p. 219]. But most of the available knowledge was contained in papers on vector lattices by H. Freudenthal [2], L. Kantorovitch [3], F. Riesz [4], and in the first edition of [1].

Sections I-III of the manuscript under review contain a rapid resume of some already known basic facts. The formulation of the general law $\sum_{i=1}^{n} \bigcup S_i = \bigcup_{a_i \in S_i} (\Sigma a_i)$ for arbitrary subsets when the joins exist, and the proof that the correspondence $x \to nx$ was a lattice-homomorphism (isomorphism), were especially noteworthy.

Section IV deals with complete commutative ℓ-groups; in 1940, it was not suspected that every complete ℓ-group was commutative (Theorem of Iwasawa-Ogasawara)--i.e., that the hypothesis of commutativity could be omitted. The main results are those of [1, Ch. XIV, S11]. Technically new at the time (though suggested by [4]), they concern the complete distributive lattice Σ_n of ℓ-ideals ("normal subgroups") and the complete Boolean algebra Σ_{cn} of closed ℓ-ideals ("complete normal subgroups") in a complete ℓ-group G. Being in Σ_n is a closure property of finite character, and this fact is exploited in Sections V-VI to describe the relation of Σ_{cn} to Σ_n. (The argument could be greatly shortened by an appeal to the Theorem of Glivenko [1, pp. 146-8], whose proof it closely parallels.)

Particular attention is paid to "dense" ℓ-ideals J of a closed ℓ-ideal K, defined by the relation $J^{**} = K$.

Section VII (pp. 52-81) deals with an important concept of "absolute completeness," which has not yet been systematically treated elsewhere. An ℓ-group G is defined as "absolutely complete" when it is

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its own closure ($G = G^{**}$) in any complete ℓ-group H which contains G as an ℓ-ideal. He shows that every complete ℓ-group G has a unique natural "absolute completion" $\hat{G}$, and constructs this explicitly.

The reviewer recalls (somewhat dimly, after 20 years), that von Neumann proved much more orally: that any "absolutely complete" (Abelian) ℓ-group could be represented by continuous, real-valued functions on a Boolean space I, modulo nowhere dense sets. He decomposed I into components S and S', so that functions were integer-valued on S and real-valued on S'. The MS. under review does not mention this result, although it is shown that any complete ℓ-group is reducible or simply ordered [1, p. 224, Lemma 2] in a 24-page supplement.

It may be hoped that, in time, von Neumann's complete results will be found or reconstructed. Later results of M. H. Stone [5] and F. Maeda-T. Ogasawara [6], though related, are less general. It should be stressed that to assume the existence of a unit, a norm, or even of vector operations makes the proof of representation theorems much easier.

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[1] G. Birkhoff, "Lattice theory," rev. ed., New York, 1948.

[2] H. Freudenthal, "Teilweise geordnete Moduln," Proc. Akad. Wet. Amsterdam 39 (1936), 641-51.

[3] L. Kantorovitch, "Lineare halbgeordnete Räume," Mat. Sbornik 2 (44) (1937), 121-68.

[4] F. Riesz, "Sur la théorie générale des opérations linéaires," Annals of Math. 41 (1940), 174-206.

[5] M. H. Stone, "A general theory of spectra. II.," Proc. Nat. Acad. Sci. 27 (1941), 83-7.

[6] F. Maeda and T. Ogasawara, reviewed in Math. Revs. 10 (1949), pp. 544-5.

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MEASURE IN FUNCTIONAL SPACES

Reviewed by I. Halperin

1. This manuscript is closely related to:

(i) Norbert Wiener's construction of a measure on the set Ω of all functions f(t) continuous on the interval (0, 1) and with f(0) = 0 (see Acta Mathematica, vol. 55 (1930), pp. 117-258),

(ii) V. Fock's description of states (in second quantization quantum mechanics) by sequences of symmetric functions:

$$\Psi = (\psi_0, \psi_1(x_1), \ldots, \psi_n(x_1, \ldots, x_n), \ldots)$$

with ψ_0 constant and ψ_n a symmetric function of n variables (see Physikalische Zeitschrift der Sowjetunion, vol. 6 (1934), pp. 425-469).

But in this manuscript the discussion is in abstract form, i.e., in terms of abstract inner product spaces.

2. L denotes an inner product space with complex numbers as scalars, L_{cp} denotes its completion, F = F(L) denotes the set of all linear functionals Ψ on L, $F_{co} = F_{co}(L)$ denotes the set of all continuous linear functionals on L (L_{cp} and F_{co} are isomorphic as inner product spaces).

A Lebesgue measure on F is constructed in the following way:

A subset $\mathcal{R}$ of F will be called a rectangle if

(*) for some $\phi_1, \ldots, \phi_p$ orthonormal in L and some Lebesgue measurable set A in 2p dimensional real Euclidean space $\bar{E}_{2p}$, $\mathcal{R}$ consists of precisely all Ψ for which $(\Psi(\phi_1), \ldots, \Psi(\phi_p))$ is in A.

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Here $(z_1, \ldots, z_p)$ with complex numbers z_i is to be identified with $(u_1, v_1, \ldots, u_p, v_p)$ where u_i, v_i are the real and imaginary parts respectively, of z_i. Also $\int \ldots dz$ shall mean $\int \ldots du_1 dv_1 \ldots du_p dv_p$.

For such rectangles $\mathcal{R}$, the value of

$$(**) \qquad \frac{1}{\pi^p} \int_A e^{-(|z_1|^2+\ldots+|z_p|^2} dz$$

is shown to depend on $\mathcal{R}$ but not on its particular representation as a rectangle. This value, denoted as $\mu(\mathcal{R})$, is shown to be a non-negative totally additive measure on the ring of rectangles. Hence, μ on the rectangles determines (in the manner of Carathéodory) a Lebesgue measure on F, to be denoted again as μ.

It is shown that $\mu(F) = 1$ and for every function $f(z_1, \ldots, z_p)$ which is Lebesgue measurable on $\overline{E}_{2p}$, and for all $\phi_1, \ldots, \phi_p$ orthonormal in L,

(***) $f(\Psi(\phi_1), \ldots, \Psi(\phi_p))$ is Ψ-measurable on F and its Ψ-integral equals

$$\int f(z_1, \ldots, z_p)\, e^{-(|z_1|^2+\ldots+|z_p|^2)} dz$$

(i.e., if either integral has a finite value then both have finite values and are equal).

3. Let $L_2(F)$ denote the inner product space consisting of all square summable functions of Ψ (of course, $L_2(F)$ is necessarily complete). Then the correspondence: g in L $\longleftrightarrow$ $\Psi(g)$ is an inner product space isomorphism of L on a subset of $L_2(F)$ (we write $\Psi(g)$ also as $g(\Psi)$); in particular, $\|g\|^2 = \int |\Psi(g)|^2 \, d\Psi$.

Then L_{cp} is also isomorphic to a closed subspace of $L_2(F)$; g in L_{cp}

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corresponds to $g(\Psi)$ (written also $\Psi(g)$) means: whenever g_m in L are such that $g_m \longrightarrow g$ in L_{cp} then $g_m(\Psi) \longrightarrow g(\Psi)$ in mean of order 2. Thus, for g in L_{cp}, $g(\Psi)$ is defined only up to a Ψ-null set.

Call $\mathcal{A}$ a 'generalized' rectangle in F if $\mathcal{A}$ can be described, up to a Ψ-null set by (*), with the ϕ_i in L_{cp}. Such a generalized rectangle is always Ψ-measurable and its μ-measure is again given by (**). Moreover, (***) holds, even when the ϕ_i are in L_{cp}.

Form the spaces $L_2(F(L))$ and $L_2(F(L_{cp}))$. They are isomorphic; in fact, an isomorphism between them is generated by the correspondence: for every $f(z_1, \ldots, z_p)$ measurable in $\overline{E}_{2p}$ with finite

$$\int |f(z_1, \ldots, z_p)|^2 e^{-(|z_1|^2+\ldots+|z_p|^2)} dz$$

and for arbitrary $\phi_1, \ldots, \phi_p$ orthonormal in L_{cp}, the Ψ-function $f(\Psi(\phi_1), \ldots, \Psi(\phi_p))$ in $L_2(F(L))$ shall correspond to the Ψ-function $f(\Psi(\phi_1), \ldots, \Psi(\phi_p))$ in $L_2(F(L_{cp}))$.

If L is of dimension n (finite), $F = F_{co}$ and the measure μ on F can be identified with ordinary Lebesgue measure on $\overline{E}_{2n}$ with weight factor $e^{-(|u_1|^2+\ldots+|v_n|^2)}$. If L is not of finite measure then F_{co} is a μ-null set.

If L has a countable basis (in the usual non-topological sense), F can be identified with the set of all sequences of complex numbers, i.e., with the product space of a countable family of factor spaces, each factor space coinciding with $\overline{E}_1$; μ on F then becomes a product measure where the factor measure μ_1 on each $\overline{E}_1$ is ordinary Lebesgue measure with weight factor $e^{-|t|^2}$.

4. For each fixed n = 1, 2, ..., consider the Ψ-functions which are

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products of the form: $f(\Psi) = g_1(\Psi) \ldots g_n(\Psi)$ with all g_i in L. These $f(\Psi)$ are all in $L_2(F)$ and their linear closure is a closed subspace of $L_2(F)$, to be denoted $L_2^{\,n}(F)$. The functions $\overline{f(\Psi)}$, which are complex conjugates of the $f(\Psi)$ in $L_2^{\,n}(F)$, also form a closed subspace (anti-isomorphic to $L_2^{\,n}(F)$), to be denoted as $L_2^{\,-n}(F)$.

Let $L_2^{\,o}(F)$ denote the set of functions which are constant on F.

The different $L_2^{\,m}(F)$ are shown to be mutually orthogonal. But $L_2(F) \neq \Sigma \oplus (L_2^{\,m}(F); m = 0, \pm 1, \pm 2, \ldots)$; in fact if g is a unit vector in L then $|g(\Psi)|^2 - |$ is in $L_2(F)$ but it is orthogonal to all $L_2^{\,m}(F)$.

5. The product function $g_1(\Psi) \ldots g_n(\Psi)$ is unchanged under a permutation of $g_1, \ldots, g_n$. This suggests that $L_2^{\,n}(F)$ is isomorphic to a set of symmetric elements in a direct product space. In fact, $L_2^{\,n}(F)$ is isomorphic to each of $Sy(\pi_{i=1}^{n} \otimes L)$, $Sy(\pi_{i=1}^{n} \otimes L_{cp})$ (n factors in each case), with the following definitions:

When $L_1, \ldots, L_n$ are inner product spaces, $\pi_{i=1}^{n} \circledast L_i$ denotes the linear space of all n-variable functions $\theta(\Psi_1, \ldots, \Psi_n)$ which are defined for all Ψ_i in $F_{co}(L_i)$ and are linear and continuous in each Ψ_i. $\pi_{i=1}^{n\,\prime} \otimes L_i$, which denotes the set of functions θ of the form $\theta = \Sigma_{j=1}^{M} (f_{j1} \otimes \ldots \otimes f_{jn})$,

$$\text{i.e., } \theta(\Psi_1, \ldots, \Psi_n) = \Sigma_{j=1}^{M} (\pi_{i=1}^{n} \Psi_i(f_{ji}))$$

is shown to be an inner product space if inner products are defined by:

$$(\Sigma_{j=1}^{M} \pi_{i=1}^{n} f_{ji}, \Sigma_{k=1}^{N} \pi_{i=1}^{n} g_{ki}) = \Sigma_{j=1}^{M} \Sigma_{k=1}^{N} \pi_{i=1}^{n} (f_{ji}, g_{ki}) .$$

The completion of $\pi_{i=1}^{n\,\prime} \otimes L_i$, denoted also as $\pi_{i=1}^{n} \otimes L_i$, can be identified with

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a subset of $\pi_{i=1}^{n} \otimes L_i$.

If all L_i are identical with L then all Ψ_i above are in $F_{co}(L)$. And for each θ in $\pi_{i=1}^{n} \otimes L$ and for every permutation P of (1, 2, ..., n) the function

$$(P\theta)\,(\Psi_1, \ldots, \Psi_n) = \theta(\Psi_{P(1)}, \ldots, \Psi_{P(n)})$$

is again in $\pi_{i=1}^{n} \otimes L$. Call θ symmetric if $\theta = P\theta$ for all P. Sy $\theta \equiv \frac{1}{n!} \Sigma_P (P\theta)$ is called the symmetric part of θ; it is always symmetric and coincides with θ if θ is symmetric.

Now for every θ in $\pi_{i=1}^{n'} \otimes L$, the Ψ-function $\theta(\Psi, \ldots, \Psi) \equiv (\text{Sy } \theta)$ $(\Psi, \ldots, \Psi)$ is in $L_2^{\,n}(F)$ (note: $\theta(\Psi, \ldots, \Psi)$ is defined originally only for Ψ in $F_{co}(L)$, but if θ is in $\pi_{i=1}^{n'} \otimes L$ it has an obvious extension to all Ψ in F).

Let (i) $\text{Sy}(\pi_{i=1}^{n'} \otimes L)$ and (ii) $\text{Sy}(\pi_{i=1}^{n} \otimes L)$ denote the symmetrical θ in the respective spaces. Then (i) is dense in (ii) and (ii) is a closed subspace of $\pi_{i=1}^{n} \otimes L$.

It is shown that an isomorphism between $\text{Sy}(\pi_{i=1}^{n} \otimes L)$ and $L_2^{\,n}(F)$ is generated by the correspondence: θ in $\text{Sy}(\pi_{i=1}^{n'} \otimes L)$ shall correspond to the function $(n!)^{-\frac{1}{2}}\, \theta(\Psi, \ldots, \Psi)$ in $L_2^{\,n}(F)$.

6. The previous paragraphs apply in particular when L arises as follows.

γ is a given set of points x, R is a ring of subsets of γ, $\nu_o(A)$ is defined for each A in R and is a non-negative real number and ν_o is totally additive on R, (i.e., A_n disjoint, A_n, $\Sigma_{n=1}^{\infty} A_n$ all in R implies $\nu_o(\Sigma_{n=1} A_n) = \Sigma_{n=1} \nu_o(A_n)$). $L = L_R'$ then consists of all functions f(x) of the form $f(x) = \Sigma_{i=1}^{M} c_i 1_{A_i}(x)$ (here c_i should be an arbitrary complex number, A_i should

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be in R and $1_{A_i}(x) = 1$ or 0 according as x is in A_i or is not in A_i; two functions f(x), g(x) are to be identified, if ν_o (set where $f(x) \neq g(x)) = 0$). Inner product in L_R' shall be defined by

$$(\sum_{i=1}^{M} c_i 1_{A_i}, \sum_{j=1}^{N} d_j 1_{B_j}) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_i \bar{d}_j \nu_o(A_i B_j) .$$

In the case of such an L, every element Ψ of F: $\Psi(\sum_{i=1}^{M} c_i 1_{A_i}) = \sum_{i=1}^{M} c_i \Psi(1_{A_i})$ can be characterized by the function $\nu(A) = \Psi(1_A)$ which is an (arbitrary) complex-valued additive measure on R satisfying $\nu_o(A) = 0$ implies $\nu(A) = 0$.

Paragraph 2 above then defines a measure on the space of all such additive measures.

Let B be any subset of γ which is Carathéodory-measurable with respect to ν_o, and has finite $\nu_o(B)$. Then 1_B is in L_{cp} and paragraph 3 above shows that for almost all ν (originally defined only for A in R) the value $\nu(B)$ can be defined so that whenever A_m are all in R and satisfy: $\nu_o(A_m - A_m B) + \nu_o(B - A_m B) \longrightarrow 0$ as $m \longrightarrow \infty$, then

$$\int |\nu(A_m) - \nu(B)|^2 \, d\nu \longrightarrow 0 \text{ as } m \longrightarrow \infty \quad .$$

Suppose, as a special case, that γ is the interval $0 \leq x \leq 1$, R is the set of finite sums of dyadic rational half-open intervals (i.e., $\alpha \leq x < \beta$ with each of α, β of the form $\frac{m}{2^n}$, $m = 0, 1, \ldots, 2^n$) and $\nu_o(A)$ is the sum of the interval lengths in A. For each additive measure ν on R let $\nu(x)$ denote ν(interval: $0 \leq t < x$) if x is rational dyadic and $0 \leq x \leq 1$. Then the additive measures ν on R are in (1, 1) correspondence with the arbitrary complex-

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valued functions $\nu(x)$ defined for dyadic rational x, $0 \leq x \leq 1$ and satisfying $\nu(0) = 0$. Thus μ determines a measure on the space of all such functions ν (μ can be described as a product measure on a product space as in paragraph 3 above). It was shown by Norbert Wiener that these functions $\nu(x)$, except for a μ-null set, are the restrictions to dyadic x of (arbitrary) functions f(x) continuous on $0 \leq x \leq 1$ and satisfying $f(0) = 0$. Now for this special γ, R, ν_o, μ is identical with Norbert Wiener's measure on the space Ω of such continuous f(x) (then the measure space Ω can be mapped in a (1,1) measure preserving way on the interval (0, 1)).

7. Returning to the general γ, R, ν_o, and L_R', suppose for each $i = 1, \ldots, N$ that γ_i, R_i, $(\nu_o)_i$ are as above. Let $\pi_{i=1}^N \circledast \gamma_i$ be the set of sequence $(x_1, \ldots, x_N)$ with x_i in γ_i, let $\pi_{i=1}^N \circledast R_i$ be the set of all finite sums of the form $\alpha = \Sigma_{j=1}^m \pi_{i=1}^N \alpha_{ji}$ with α_{ji} in R_i and all $\pi_{i=1}^N \alpha_{ji}$ disjoint, and let $(\pi_{i=1}^N \circledast (\nu_o)_i)_o (\alpha) = \Sigma_{j=1}^m \pi_{i=1}^N (\nu_o)_i(\alpha_{ji})$. Then $\pi_{i=1}^N \circledast R_i$ is a ring of subsets of $\pi_{i=1}^N \circledast \gamma_i$ and $(\pi_{i=1}^N \circledast (\nu_o)_i)_o$ is non-negative totally additive measure on this ring.

An isomorphism between $\pi_{i=1}^N \circledast L'_{R_i}$ and $(L'_{\pi_{i=1}^N \circledast R_i})_{cp}$ is generated by the correspondence: $\pi_{i=1}^N \circledast f_i (f_i \in L_i)$ in $\pi_{i=1}^N \circledast L'_{R_i}$ shall correspond to f in $L'_{\pi_{i=1}^N \circledast R_i}$ if $f(x_1, \ldots, x_N) = \pi_{i=1}^N f_i(x_i)$. Also, $\pi_{i=1}^N \circledast L'_{R_i}$ is isomorphic to $L_2(\pi_{i=1}^N \circledast \gamma_i)$.

Paragraphs 4 and 5 above can now be specialized to the case $L = L'_R$, using paragraph 7 with all γ_i, R_i, $(\nu_o)_i$, identical to γ, R, ν_o respectively.

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In particular, $Sy(\pi_{i=1}^{N} \otimes L'_R)$, hence $L_2^n(F)$ too (except for a factor $(n!)^{-\frac{1}{2}}$ are isomorphic to the complete inner product space of symmetric functions $f(x_1, \ldots, x_n)$, all x_i in γ, square summable on $\gamma \otimes \ldots \otimes \gamma$.

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REPRESENTATION OF CERTAIN LINEAR GROUPS BY UNITARY OPERATORS IN HILBERT SPACE

Reviewed by G. W. Mackey

This paper, when completed, was to have been a rigorous study of the continuous unitary representations of the members of a certain class of Lie groups--the semi-direct products V⊛G where V is a finite dimensional vector group and G is a connected Lie group of transformations of V. It was to have been published immediately following Wigner's 1939 Annals paper on the inhomogeneous Lorentz group and was designed to supply certain points of rigor left open in that paper. However it was never finished and in the meantime both the results it contains and the results it was presumably meant to contain have appeared in the works of other authors.

The incomplete manuscript is not quite sixty-two pages long. The introduction and various detailed preliminaries occupy two short chapters whose total length is nine pages. The third chapter (thirty pages) gives a complete analysis of the possible continuous unitary representations of a finite dimensional vector group V. The remaining twenty-three pages comprise Chapters IV and V and a fragment of Chapter VI. The results are formulated as lemmas 6 through 14 and deal mainly with the nature of the particular representations of V one gets by restricting a factor representation of V⊛G.

Let L be a factor representation of V⊛G and let H(L) denote the Hilbert space in which it acts (we do not follow the author's notation). Let $\tilde{L}$ denote the restriction of L to V. By the general theory of Chapter III $\tilde{L}$ defines a measure μ in V* the dual of V and a homorphism $f \to A_f$ of the ring of all equivalence classes of bounded μ measurable functions on V* into the ring of all bounded operators on $H(L) = H(\tilde{L})$. Moreover each $y \in G$ defines a non-singular linear transformation in V* and hence a linear transformation $f \to y(f)$ in the linear space of complex valued functions on V*. It is shown (lemma 7) that each A_f is in the weakly closed algebra generated by the L_x and satisfies $L_y A_f L_y^{-1} = A_{y(f)}$ for all y in G. In addition (lemma 8) if $y(f) = f$ for each y

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almost everywhere with respect to μ, then f is a constant almost everywhere with respect to μ. Lemmas 10 through 13 deal with the important special case in which the measure μ is concentrated in an orbit of V* under G. Lemma 9 states (falsely) that μ is always so concentrated. In lemma 10 it is shown that the null sets of μ are invariant under G and in lemma 11 that any two measures concentrated in the same orbit and having G invariant null sets have the same null sets. According to lemma 12 the orbit in which μ is concentrated is uniquely determined by $\tilde{L}$ and according to lemma 13 every orbit arises from some $\tilde{L}$. Lemma 14 applies the Radon Nikodynm theorem to express the integral of y(f) with respect to μ in terms of the integral of f with respect to μ. The work in Chapters IV and V is carried out under the somewhat generalized assumption that representation L is defined only in a connected neighborhood of the identity. Lemma 6 contains a proof that L may be extended uniquely to all of V. Otherwise the generalization in question causes no difficulties.

The spectral theorem for unitary representations of locally compact Abelian groups [1], [3] and [10] combined with Nakano's formulation of multiplicity theory [8] as a theory of Boolean algebras of projections yields at once a generalization of the analysis of Chapter III from vector groups to arbitrary separable locally compact Abelian groups. When the facts about $\tilde{L}$ are formulated in these terms lemmas 7, 8, 10 and 12 follow almost immediately as indicated in the first paragraph of section 7 of [5]. Generalizations of lemmas 6 and 11 appear as lemma 3.1 of [2] and Theorem 1.1 of [6] respectively. Theorem 1 of [5] gives a sufficient condition that lemma 9 should hold. A detailed proof of Theorem 1 is given as Theorem 6.3 of [7]. Lemma 13 is also a consequence of the considerations of [5] and is a part of the statement of Theorem 3 of that paper. Presumably von Neumann meant to prove Theorem 3 of [5] in the special case he was considering. However, the manuscript ends at the beginning of the part of the argument taken care of in [5] by the main theorem (Theorem 2) of that paper. A considerable generalization of Theorem 3 of [5] will be found in [7].

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We conclude with some remarks indicating the extent to which von Neumann anticipated and influenced the later developments sketched above. In the introduction one finds the following passages: "But this (this discussion of all representations 'up to a numerical factor') could have been done in full operator theoretical generality, and it would have led to results with some interest of their own," and "Many results could be extended so as to cover all 'Haar groups'--that is all groups in which a right invariant general Lebesgue measure exists."

Theorem 2 of [5] is a generalization of the main theorem of [4] and its proof depends in part on arguments given in [4]. As acknowledged in [4] the reviewer is indebted to von Neumann for suggesting the applicability of his direct integral theory [9] and for permitting him to examine this paper in manuscript form. The basic lemmas in Section 4 of [4] are,independently arrived at,reformulations of some of von Neumann's results in [9], a paper written contemporaneously with the manuscript being reviewed here.

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[9] J. von Neumann, On rings of operators. Reduction theory, Ann. Math. 50 (1949), pp. 401-485.

[10] M. Neumark, Positive definite operator functions on a commutative group, Bull. Acad. Sci. USSR (Izvestiya) 7 (1943), pp. 237-244.

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ON THE PRINCIPLES OF LARGE SCALE COMPUTING MACHINES

This folder has an unpublishable part of the original manuscript from which this paper was taken. It also has a folder which contains correspondence concerning, and a copy of a speech at a meeting of the Mathematical Computing Advisory Panel 15 May 1946.

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NON-LINEAR CAPACITANCE OR INDUCTANCE SWITCHING, AMPLIFYING AND MEMORY DEVICES

This folder contains copies of Patent 2, 815, 488 granted to John von Neumann on December 3, 1957 and assigned by him to I. B. M. This patent has to do with non-linear capacitance or inductance switching, amplifying and memory devices. Preliminary versions of the description of the claims made are also included in the folder.

Permission to publish any and all material in this folder has been obtained from the I. B. M. legal department. The letter giving this permission is in this folder.

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NOTES ON THE PHOTON-DISEQUILIBRIUM-AMPLIFICATION SCHEME

Dated September 16, 1953

Reviewed by John Bardeen

The possibility of making a light amplifier by use of stimulated emission in a semiconductor is considered. By various methods, for example by injection of minority carriers from a p-n junction, it is possible to upset the equilibrium concentrations of electrons in the conduction band and holes in the valence band. Recombination of excess carriers may occur primarily by radiation, with an electron dropping from the conduction to the valence band and the energy emitted as a photon with an energy slightly greater than the energy gap. The rate of radiation may be enhanced by incident radiation of the same frequency in such a way as to make an amplifier. The basic principle was used later by Townes and by Bloembergen in the MASER (Microwave Amplification by Stimulated Emission of Radiation), although not with recombination radiation in a semiconductor.

Calculations are made for an idealized semiconductor with effective masses in both valence and conduction bands equal to the ordinary electron mass. It is concluded that a very large concentration of excess carriers is required ($\sim 10^{21}/cm^3$). No method is given for obtaining and maintaining large excess carrier concentrations in this concentration range. It is concluded that the large amount of heat released by the radiation might be carried away without requiring an excessive rise in temperature.

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FIRST AND SECOND REPORTS ON THE NUMERICAL CALCULATION OF FLOW PROBLEMS

von Neumann never published a general discussion of the method he used for the stability analysis of numerical methods for solving partial differential equations of the hyperbolic and parabolic type. However, he gave a number of lectures on this topic in which he discussed a method of obtaining stability criteria for difference-equations, which approximate partial differential equations, by a linear-small-perturbation procedure and subsequent Fourier-analysis. This method was developed by R. Courant, K. Friedrichs and H. Lewy in a paper entitled, "Über die Partiellen Differenzengleichungen der Mathematischen Physik," published in Math. Ann. 100, pp. 32-74 (1928). They used it to derive the stability and convergence criteria that are characteristic of this method for the conventional difference-equation treatment of the general quasi-linear hyperbolic and parabolic differential equations for one dependent variable and two independent ones.

von Neumann applied the Courant, Friedrichs and Lewy method to various more complicated systems: those with more than one dependent variable, with more than two independent variables, with order higher than two, and with various other differences.

While the Courant, Friedrichs and Lewy method, in the range treated by these authors, was rigorous, his procedure was, as he always emphasized, at all times heuristic. It was used in various "practical" situations for practical purposes but, as he one wrote (letter of November 10, 1950 to Werner Leutert), "... as far as any evidence that is known to me (J. v. N.) goes, ... it has not led to false results so far."

On August 27, 1947 von Neumann talked about the solution of partial differential equations by finite difference methods to the Mechanics Division of the Naval Ordnance Laboratory at White Oak, Silver Spring, Md. Mr. M. A. Hyman wrote a resume of this talk, which was issued as a Mechanics

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Division Technical Note No. SB/Tm - 18.2 - 11, entitled "Stability of Finite Difference Representation," dated October 16, 1947.

Later, von Neumann allowed George C. O'Brien, Mortan A. Hyman and Sidney Kaplan to incorporate a discussion of his heuristic technique of stability analysis (adopted from the Courant, Friedrich, Lewy method and described at White Oak in 1947) in their paper, "A Study of the Numerical Solution of Partial Differential Equations," which appeared in the Journal of Mathematics and Physics 29, pp. 223-239 (1950). Although there are many references in the literature to the "von Neumann method of stability analysis," there is no detailed published description of this method other than the paper O'Brien, Hyman and Kaplan mentioned above.

These papers are included in the collected works partly because they contain excellent examples of von Neumann's use of his methods of stability analysis. They were written in connection with work done for the Standard Oil Development Company and are reprinted here with permission of that organization.

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DISCUSSION OF A MAXIMUM PROBLEM

A typescript of this paper was circulated privately during 1948 and had an important influence on the early theoretical development of linear programming. Although it is clearly a working paper and was not intended for publication, it has played a great role in initiating the theory of duality. Minor stylistic changes and the correction of obvious typographical errors have been made by H. W. Kuhn and A. W. Tucker. They have also provided two footnotes.

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A NUMERICAL METHOD FOR DETERMINATION OF THE VALUE AND THE BEST STRATEGIES OF A ZERO-SUM TWO-PERSON GAME WITH LARGE NUMBERS OF STRATEGIES

Reviewed by H. W. Kuhn and A. W. Tucker

This paper was written and circulated privately during a period when practically nothing was known about how to compute solutions of zero-sum two-person games. It proposed an algorithm which leads to approximate solutions by an iterative procedure. To describe the essentials of the method, let $A = (a_{ij})$ be the m by n payoff matrix of a zero-sum two-person game which has been scaled so that $-1 \leqq a_{ij} \leqq 1$ for all i and j. Positive constants θ and ζ being given, the first stage is to compute either a mixed strategy x such that (1) $\min xA > -\theta$, or a mixed strategy y such that (2) $\max Ay < \zeta$. If (2) holds for a given y, then this stage ends. Otherwise, form the vector $u = \max (Ay, 0)$ from the positive components of Ay. Then $x = u/\Sigma_i u_i$ is a mixed strategy for the first player. If (1) holds for this x, then this stage ends. Otherwise, pick j_o such that $\Sigma_i x_i a_{ij_o} \leqq -\theta$ and set

$$\epsilon = -\Sigma_i x_i a_{ij_o} / \Sigma_i a_{ij_o}^2 .$$

Define y' by

$$(1+\epsilon) y_j' = \begin{cases} y_i + \epsilon & \text{for } j = j_o, \\ y_j & \text{for } j \neq j_o . \end{cases}$$

(This procedure may be described informally as forming a mixed strategy for your opponent that weights his pure strategies in proportions equal to your losses above a fixed level. The relative weight of your pure strategy that is best against this fictitious mixed strategy for your opponent is then increased. The choice of ϵ, which determines this increase in weight, minimizes a measure of the losses incurred by y' which is closely related to $\Sigma_i u_i'^2$.) von Neumann obtained

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$$\frac{\log (mn\zeta_1^2/\zeta^2)}{-\log (1-\theta^2/m)} + 1 \quad (\text{where } \zeta_1 = \max Ay)$$

as a bound on the number of times this procedure may have to be repeated before either (1) or (2) holds. The interval $(-\theta, \zeta)$ is then narrowed and the next stage begins.

The final estimate of the time required for this method, for a matrix scaled to lie between 0 and 1, is that the total number of multiplications required for an accuracy of γ in the value is majorized by $17.56m^2n\gamma^{-2} \log (mn)$. Thus, allowing one millisecond for a multiplication, to obtain the value of a 100 by 200 game to an accuracy of 10 percent would require not more than 400 days of computation. As impractical as this may seem, it must be emphasized that this is a generally valid bound and should be contrasted with the number of steps that might be used on any single game. In such light, it is not unreasonable.

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SYMMETRIC SOLUTIONS OF SOME GENERAL N-PERSON GAMES (with O. Morgenstern)

Reviewed by O. Morgenstern

The results obtained in this unpublished paper relate to solutions of the general n-person games in which every coalition of n-1 or more (less than n-1) players wins (loses). The solutions investigated are those which are symmetric with respect to all but one of the players. Detailed information is found concerning the amount paid to this player by the imputations of the solutions. R. Bott ("Symmetric Solutions to Majority Games," Contributions to the Theory of Games, vol. II, 1953, p. 319-323) obtains the fully symmetric solutions to a more general class of majority games. D. B. Gillies ("Discriminatory and Bargaining Solutions to a Class of Symmetric n-Person Games," Contributions to the Theory of Games, vol. II, 1953, p. 325-342) determines a large number of unsymmetric solutions to these games. Although the solutions in the present paper are actually unsymmetric, they are not of the type investigated by Gillies.

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STATIC SOLUTIONS OF EINSTEIN FIELD EQUATIONS FOR PERFECT FLUID WITH $T_{\rho}^{\ \rho} = 0$

Reviewed by A. H. Taub

These notes are concerned with the nature of the solutions of the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

where the space-time is assumed to be a static spherically symmetric one with the line element

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + e^{\nu} dt^2$$

and the stress energy tensor is assumed to satisfy

$$T^{\mu}_{\ \mu} = 0.$$

The discussion of the solution of the field equations is reduced to the discussion of the ordinary differential equation

$$\frac{d\tau}{dz} = \frac{\tau(\tau+5) - 2(\tau+3)z}{(1+z)(3\tau-4z)}$$

where

$$r = e^{\int \frac{dz}{(1+z)(3\tau-4z)}}$$

$$e^{\lambda} = 1 + z$$

$$e^{\nu} = e^{\int \frac{\tau dz}{(1+z)(3\tau-4z)}}$$

$$r^2 p = \frac{1}{3} r^2 \rho = \frac{\tau - z}{1+z}$$

and p and ρ are the pressure and density distributions which satisfy

$$\rho = 3p.$$

The solutions of the differential equation are classified in terms of their behavior around the singular points α: $(\tau, z) = (1, 3/4)$, β: $(\tau, z) = (0, 0)$, γ: $(\tau, z) = (-1, -1)$ and the point at infinity.

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It is concluded that if the empty, "flat" solutions are disregarded there is only one solution which is regular for all finite r. For $r \to \infty$ this solution is asymptotic to a singular solution which is not-flat at infinity. It is further shown that for the main singular solution the mean density of matter is the same in each sphere and is 3/7 of the maximum density of matter. This result is compared with the result that is obtained in the classical theory which is shown to be the theory of an "Emden-polytrope for $n = \infty$."

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ON RELATIVISTIC GAS-DEGENERACY AND THE COLLAPSED CONFIGURATIONS OF STARS

Reviewed by A. H. Taub

In these notes, von Neumann points out that the equation of state of degenerate matter of the type that is presumed to occur in white dwarf stars may be approximated by the equations

$$P = K_3\rho$$

$$P = K_2\rho^{4/3}$$

$$P = K_1\rho^{5/3}$$

for various ranges of ρ, the density, where P is the pressure. The first equation holding for the largest values of ρ (at the innermost core of the star) the second for intermediate values of ρ and the third for the smallest values of ρ.

In this paper, the arguments of relativistic statistical mechanics are used to derive the equation of state for a mixture of gases, and the conditions under which the laws listed above obtain are given and the values of the constants K_1, K_2 and K_3 are determined.

The classical equations of equilibrium, namely

$$\frac{dM}{dr} = 4\pi r^2\rho$$

$$\frac{dP}{dr} = -Gr^{-2}\rho M$$

are then discussed. In these equations M is the mass inside a radius r and G is the Newtonian constant of gravitation. P is related to ρ by the equations given above; each equation holding for a limited range of ρ. (Note that although special relativity is taken into account in deriving the equation of state, general relativity is ignored in studying the equilibrium between gravitational and hydrostatic forces. This omission may be a serious one.)

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It is shown that if the mean molecular weight is taken to be 2, then the radius of the relativistically degenerate core of a star cannot exceed $\frac{1}{2}$ km while its mass cannot exceed 2 sun masses.

After the structure of the relativistically degenerate core is determined, the solution of the above differential equations, in which P and ρ are taken to be related by the second of the equations of state, is determined. The initial conditions for these equations are obtained from the requirement that the density be continuous at the interface between the core and outer material and that the mass enclosed by this interface be the mass of the core. The equation $P = K_2 \rho^{4/3}$ is assumed to hold until the outer boundary ($\rho = 0$) is reached.

It is shown that if the mean molecular weight is 2, then in the limiting configuration the star has a radius of 9 km, the core having one of $\frac{1}{2}$ km, and its total mass is 6 sun-masses, of which 2 sun-masses are within the core.

If all stellar material is broken up into electrons and protons, then the mean molecular weight is $\frac{1}{2}$ and the above figures must be multiplied by a factor of 16.

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THE POINT-SOURCE MODEL

Reviewed by A. H. Taub

In this unpublished manuscript von Neumann discusses the solutions of the system of equations:

A) $$p = RT\rho$$

B) $$q = \alpha T^4$$

C) $$\frac{d}{dr}(p+q) = -GM\rho\, r^{-2}$$

D) $$\frac{dM}{dr} = 4\pi r^2 \rho$$

Here, ρ the density, T the temperature, M the mass inside a shell of radius r, p the gas pressure and q the radiation pressure are five unknown functions of r.

The method involves reducing the system of equations A to D to the system

α) $$\frac{dx}{dz} = y^{-1} z^{1/4}$$

β) $$\frac{d^2y}{dz^2} = -x^4$$

and discussing the functions x(z) and y(z).

This discussion appears in "An Introduction to the Study of Stellar Structure," by S. Chandrasekhar, pp. 332-349.

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THE POINT-SOURCE-SOLUTION, ASSUMING A DEGENERACY OF THE SEMI-RELATIVISTIC TYPE, $p = K\rho^{4/3}$, OVER THE ENTIRE STAR

Reviewed by A. H. Taub

In this manuscript von Neumann discusses the system of equations

A) $$p = K\rho^{4/3}$$

B) $$q = \alpha T^4$$

C) $$\frac{d}{dr}(p+q) = -GM\rho r^{-2}$$

D) $$\frac{dq}{dr} = -\frac{K}{c}\frac{L_o}{4\pi r^2}\rho$$

E) $$\frac{dM}{dr} = 4\pi r^2 \rho$$

These equations are reduced to the equation

α) $$r^{-2}\frac{d}{dr}\left(r^2\frac{d}{dr}\theta\right) = -\frac{\pi G}{K}\theta^3$$

the equation for an Emden polytrope of index 3. Equation α is studied in the u, v plane where

$$u = r\theta$$

$$v = r^2\frac{d}{dr}\theta$$

and α) becomes

$$\frac{dv}{du} = -\frac{v^3}{u+v}$$

An extremely elegant derivation of the asymptotic behavior of the solution of α is given.

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DISCUSSION OF DE SITTER'S SPACE AND OF DIRAC'S EQUATION IN IT

Reviewed by A. H. Taub

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von Neumann had been interested in De Sitter space for a number of years. In a letter dated January 27, 1934, to Dirac he discussed wave equations proposed by Dirac as those holding in De Sitter space and shows that four operators behave as quantized spatial and time coordinate operators. Further, he showed that the time operator has a continuous spectrum whereas the spatial coordinates have a discrete one.

In 1940 he wrote a manuscript in which the geometry of De Sitter space S is defined as that of a 4-dimensional manifold with an indefinite Riemannian metric on it. The general point of S is defined by 5 coordinates x_i with one relation

$$\sum_{i=1}^{5} (x_i)^2 = 1$$

with x_i real for $i \neq 4$ and x_4 pure imaginary.

Conditions for vector fields and spinor fields defined in the space of coordinates $x_i (i = 1, 2, \ldots, 5)$ are given. In particular von Neumann showed that the electron wave functions should be considered homogeneous spinor functions of degree zero satisfying the wave equation

$$\alpha_i \frac{\partial}{\partial x^i} \psi = K \alpha_i x_i \psi$$

with

$$K = 2 - iRmc,$$

R the radius of the De Sitter space and α_i a set of five 4x4 "spinor" matrices satisfying

$$\alpha_i \alpha_j + \alpha_j \alpha_i = \delta_{ij}$$

$$\alpha_i^* = \alpha_i$$

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 = 1$$

and the * denotes the hermitian conjugate.

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von Neumann constructs a divergence free current vector from spinors satisfying this wave equation and then shows that it is possible to define a (not incomplete) Hilbert space of spinor functions which satisfy the wave equation and are homogeneous of degree zero.

The wave equation is one given by Dirac [Ann. Math. 36, pp. 657-669 (1935)] and the method used to construct the Hilbert space is very similar to that used by E. P. Wigner for the Dirac equation in the special theory of relativity.

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THE TAYLOR INSTABILITY PROBLEM

Reviewed by H. H. Goldstine

Consider two ideal gases of different density with one superimposed on the other, separated by an interface at which gliding can occur, when a body force is allowed to act. The study of the flows of these gases is the subject of the well-known Taylor instability problem.[1] The problem has been studied by a number of mathematicians and physicists, both analytically and numerically by means of various approximations. None of these studies has, however, proved definitive, essentially since the restrictions or approximations have proved too gross.

Professor von Neumann decided to attack the problem numerically without any approximations or idealizations to mar the results. His only restrictions were to assume the gases to be ideal and to avoid the emergence of irrelevant shocks by the introduction of his so-called pseudo-viscosity.[2] The differential equations expressing the conservation of mass, momentum and energy of a two-dimensional motion are written in terms of the Lagranian coordinates x, y; X, Y are the Eulerian coordinates with U, V the corresponding velocities, t the time, p the pressure, E the internal energy per unit mass, v the specific volume per unit mass with v_o the initial specific volume per unit mass.

In the numerical formulation each function f(x, y, t) is evaluated on a finite lattice: $x = i\Delta x$, $y = j\Delta y$, $t = h\Delta t$, with i, j, h integers and Δx, Δy, Δt fixed quantities. We write $f(i\Delta x, j\Delta y, h\Delta t)$ as $f^h_{i\,j}$. Certain quantities are evaluated at integers plus $\frac{1}{2}$ instead of integer values. The need for such intermediate lattice points arises when we center quantities properly in the equations describing the flow.

The unique insight von Neumann had into this problem lies in his treatment of the interface. He ingeniously put down two independent x, y-lattices, one in each fluid so that the interface between the two fluids corresponds to two lines, one in each lattice. This raises a complication

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since the two fluids are free to glide past one another. To make the notational problem tractable he fixed his conventions so that the upper region was defined as $j \leq j_o$ for some fixed j_o and the lower as $j \geq j_o + 1$. The interface in the one region is defined by $j = j_o$ and in the other by $j = j_o + 1$. (No quantities are allowed to exist having an index $j_o + \frac{1}{2}$.)

The main problem now is to identify correctly quantities f^h_{i,j_o} and f^h_{i',j_o+1}, i.e., to establish the correspondence between i and i' which associates corresponding quantities. This is done by making use of the fact that the Eulerian coordinates of the interface must give a unique curve. Thus there must exist an $i' = i'(i)$ such that $X^h_{i,j_o} = X^h_{i'(i),j_o+1}$, $Y^h_{i,j_o} = Y^h_{i'(i),j_o+1}$. It should be noted that for i integral $i' = i'(i)$ need not be; also for the inverse function $i = i''(i')$, for i' integral i need not be.

The remaining problem left is the treatment of the hydrodynamical and thermodynamical quantities at the interface. Inside each gas there is essentially no problem since the ordinary equations of hydrodynamics apply. On the boundary between the gases, however, these equations are not applicable. Instead one must postulate the continuity of pressure and of normal velocity across the interface. When these are expressed properly it is not difficult to see that all relevant quantities are well-defined everywhere. (In the original account by von Neumann these continuity conditions were not correctly formulated, and the reader must make the necessary changes.)

Professor von Neumann and the reviewer made some preliminary test calculations using these methods without obtaining any conclusive results.

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[1] G. I. Taylor, The Instability of Liquid Surfaces When Accelerated in a Direction Perpendicular to Their Planes. I. Proc. Roy. Soc. London. Le. A, 201, (1950), 192-196.

[2] R. D. Richtmyer and J. von Neumann, A Method for the Numerical Calculation of Hydrodynamic Shocks. J. App. Phys. 21, (1950), 232-237.

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RECENT THEORIES OF TURBULENCE

This report to the Office of Naval Research, written by von Neumann in 1949, was not considered by the author as suitable for publication for reasons which will be given below. However, it is considered by a number of workers in the field to be one of the most illuminating discussions of turbulence extant. It is therefore being printed in the collected works.

In letters transmitting copies of this report von Neumann has stated:

"It was not written for publication, and it therefore contains a number of inadequacies which I would certainly correct before I let it circulate widely or go into print. The main corrections are:

1. The references to, and the assessments of, the literature require a very careful going over.

2. If I wrote the paper now, I would discuss the latest work of Burgers (Proceedings of the Royal Dutch Academy of Sciences, Vol. 53 (1950) No. 2, page 122, No. 3, page 247, No. 4, page 393.) much more fully. It seems to me that it is the most promising development along Burgers' line, that has occurred to date, and I think in particular, that the first one of the above references has considerable potentialities.

3. The remarks at the end of the report regarding magneto-hydrodynamic turbulence should be revised and corrected, in view of the recent work of Batchelor and Chandrasekhar."

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DESCRIPTION OF THE CONFORMAL MAPPING METHOD FOR THE INTEGRATION OF PARTIAL DIFFERENTIAL EQUATION SYSTEMS WITH 1+2 INDEPENDENT VARIABLES

Reviewed by A. H. Taub

This unpublished paper of von Neumann's, written in the period December 16, 1950 - January 8, 1951, describe a method for computing hydrodynamic flows with shocks. The novel feature of the method is the introduction (and determination) of new independent variables called a, b, in place of the spatial coordinates or independent variables simply related to these, which expand the areas of rapid change and of irregular behavior, i.e. the sites of shocks and of shock intersections, relatively to other areas, i.e. to the ordinary regions.

The discussion begins with a formulation of the partial differential equations of hydrodynamics in two dimensions in terms of the independent variables: t and the Eulerian coordinates x and y. Next a pseudo-viscous pressure p_v is introduced where

$$p_v = -\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\mu = \lambda \left|\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right| ,$$

u and v all the x and y components of the Eulerian velocity and $\lambda > 0$ is a constant pseudo-viscosity coefficient. This pseudo-viscosity is analogous to the one introduced by Richtmyer and von Neumann in one-dimensional problems. (Cf. A Method for the Numerical Calculation of Hydrodynamic Shocks, with R. D. Richtmyer. J. App. Phys. 21 (1950), pp. 232-237.) Next the equations are transformed to those using

$$s = \log_e t$$

$$\xi = \frac{x}{t}$$

$$\eta = \frac{y}{t}$$

as independent variables and

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$$z^1 = \sigma = \log \rho$$

$$z^2 = U = \xi - u$$

$$z^3 = V = \eta - v$$

$$z^4 = E = (\gamma-1)^{-1} \{p/\rho + \zeta |\Delta|\Delta\}$$

as dependent variables, where p is the pressure, ρ the density, $\zeta = t^{-2} \rho^{-1} \lambda$ and

$$\Delta = 2 - (\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta}).$$

It is proposed to treat ζ as constant.

Next, von Neumann considers a problem with three independent variables, s, ξ, and η, where s has the essential characteristics of time and I dependent variables z^i (i = 1, 2, ... I). It is assumed that I partial differential equations exist which express $\frac{\partial z^i}{\partial s}$ in terms of the z^i and their first and second partial derivatives with respect to ξ and η. It is pointed out that the computational difficulties of a problem of this type are to an important extent caused by the possibility of discrepancies between what one might call the natural metric of the problem and the actual (Euclidean) metric of the ξ, η-plane. The latter is, of course, given by

$$(d\Xi)^2 = (d\xi)^2 + (d\eta)^2,$$

while the former may be tentatively described by

$$(dZ)^2 = \sum_{i=1}^{I} i \, (dz^i)^2 .$$

(Generalizations of the expression for $(dZ)^2$ are also treated in the paper.) In those regions where the problem behaves irregularly, the metrics $(dz)^2$ and $(d\Xi)^2$ will diverge from each other.

This suggests the introduction of a new set of independent variables in place of the ξ, η (keeping however s), such that their (Euclidean) metric

$$(dA)^2 = \omega^2 \, da^2 + db^2$$

where ω is a function of s alone, and is closer to the natural metric $(dz)^2$ of the problem than is the metric $(d\Xi)^2$.

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The strongest fulfillable requirement that can be made is (*)

(*) $$(dZ)^2 = \mathcal{G}(dA)^2$$

where $\mathcal{G}$ is an arbitrary function of a, b and s. This requirement states that the surface with the metric $(dZ)^2$ is conformal to the flat surface with the metric $(dA)^2$.

von Neumann then solves the problem of determining the mapping

$$\xi = \xi(s; a, b) \quad \eta = \eta(s; ab)$$

such that (*) holds for arbitrary s if it holds some initial $s = s_0$. A vector field in the a, b plane with components α, β, is introduced. This vector field is defined by the equations

$$\frac{\partial \xi}{\partial s} = \alpha \frac{\partial \xi}{\partial a} + \beta \frac{\partial \xi}{\partial b}$$

$$\frac{\partial \eta}{\partial s} = \alpha \frac{\partial \eta}{\partial a} + \beta \frac{\partial \eta}{\partial b} \quad .$$

These equations state that

$$\xi(s+ds; a, b) = \xi(s; a+\alpha ds, b+\beta ds) + 0((ds)^2).$$

Thus their solution determines $\xi(s; a, b)$ in terms of $\xi(s_0; a, b)$. It follows by differentiating (*) with respect to s that the vector field (α, β) must satisfy the differential equations

(**)
$$\frac{\partial \alpha}{\partial a} - \frac{\partial \beta}{\partial b} = \frac{(\mathcal{C}/\omega^2) + \mathcal{L}}{\mathcal{G}} + \nu$$

$$\omega^2 \frac{\partial \alpha}{\partial b} + \frac{\partial \beta}{\partial a} = \frac{-\mathcal{L}}{\mathcal{G}}$$

where

$$\nu = \frac{1}{\omega} \frac{d\omega}{ds}$$

(and is determined later), and

$$\mathcal{C} = \sum_1^I i \; \frac{\partial z^i}{\partial a} \frac{\partial^* z^i}{\partial a}$$

$$\mathcal{L} = \sum_1^I i \left(\frac{\partial z^i}{\partial a} \frac{\partial^* z^i}{\partial b} + \frac{\partial z^i}{\partial b} \frac{\partial^* z^i}{\partial a} \right)$$

$$\mathcal{L} = \sum_1^I i \; \frac{\partial z^i}{\partial b} \frac{\partial^* z^i}{\partial b}$$

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where ${}^{*}z^{i}$ is the partial derivative of z^{i} with respect to s when ξ and η are kept constant. It may be expressed in terms of the partial derivatives of z^{i} with respect to a and b virtue of the partial differential equations satisfied by the dependent variables z^{i}. It may be shown that

$$\frac{\partial z^{i}}{\partial s} = \alpha \frac{\partial z^{i}}{\partial a} + \beta \frac{\partial z^{i}}{\partial b} + {}^{*}z^{i}.$$

Hence a knowledge of α and β enables us to determine z^{i} at $s + \Delta s$ from z^{i} at s, i.e. to integrate the original differential equations.

The boundary conditions that the functions α and β must satisfy are shown to be

$$\alpha = 0 \quad \text{for } a = 0 \text{ or } 1$$
$$\beta = 0 \quad \text{for } b = 0 \text{ or } 1.$$

The solution of equations (**) which satisfy these boundary conditions are obtained by means of Fourier series.

Since numerical methods are to be used in applying the method, the a, b representation of ξ, η and the z^{i} will be approximate. Therefore attention must be paid to the method of calculating the key quantities and ω^{2} so that these have optimum properties and to correcting the errors that arise in using these optimum values instead of the rigorously correct ones. These two topics are discussed in detail.

The paper concludes with a list of formulas which show how the method is to be applied to the specific hydrodynamical problem discussed at the beginning of the paper. The problem of determining the initial mapping

$$\xi = \xi(s_0; a, b) \qquad \eta = \eta(s_0; a, b)$$

is not treated, nor is there any detailed analysis of the accuracy of the method.

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ATTACHMENT B

Listing of Contents of Folders in

FILE B: Lecture Notes, Drafts of These, and Material Absorbed in These

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B. 1 Methods of Mathematical Physics. Lectures by E. U. Condon and J. von Neumann. Notes by J. D. Stewart. Princeton Univ. 1931-32. 1. Fourier Series, by J. von Neumann.

B. 2 Quantum Electrodynamics. Lectures by J. von Neumann. Notes by A. H. Taub. Princeton Univ. 1933.

B. 3 Lattices and Dimension. (Preliminary notes on Continuous Geometry, 1935.)

A folder with preliminary notes. These were in an envelope labelled by von Neumann "Lattices and Dimension." Parts (1)-(4), part of (5): Nov. 12, 1935. There are actually parts (1) through (12) and pages labeled (10)-(1) and (6)-(1), supposedly these are addenda to parts (10) and (6). These would seem to be superseded by the lecture notes (B. 5) and the notes in the Proceedings of National Academy of Sciences.

B. 4 Normal Hypercomplex Number Systems (rings) and Continuous Geometries. Misscellaneous notes, 1936.

A folder of preliminary notes presumably written in the summer of 1936. There are 9 parts plus a part ϵ, plus an extra part. Pages 1-12 of Part IV are not there. There is also a small piece of paper with an itinerary and dates on it. Kaplansky has dismissed these as miscellaneous preliminary notes, most probably superseded by the lecture notes on Continuous Geometry, Parts II and III.

B. 5 Continuous Geometry, Parts I, II and III. Lectures by J. von Neumann. Notes by L. Roy Wilcox. Institute for Advanced Study, 1936.

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B. 6 Introduction of a Ring for a Complemented Modular Lattice of Order $n \geq 4$. 1936.

Kaplansky has said: "This is superseded by mimeographed lectures."

B. 7 First Draft of A. M. S. Colloquium Publication on Continuous Geometry. About 1937.

Of this paper Kaplansky has said: "It is 102 pages of very detailed lattice-theoretic preliminaries to continuous geometry. Ends abruptly."

B. 8 Continuous Geometry. Lectures by J. von Neumann. Pennsylvania State College, 1937.

B. 9 Quantum Mechanics of Infinite Systems. 1937.

Mimeographed version of a lecture given at Pauli's seminar held at the Institute for Advanced Study in 1937; also a handwritten manuscript related to this lecture.

B. 10 Quantum Logics. Summary of Institute for Advanced Study lectures 1938.

B. 11 Invariant Measures. Lectures by J. von Neumann. Notes by Paul R. Halmos. Institute for Advanced Study 1940-41.

B. 12 Unsolved Problems in Mathematics. Speech by J. von Neumann. International Congress of Mathematicians, Amsterdam, Netherlands, 1954.

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ATTACHMENT C

Listing of Contents of Folders in

FILE C: Unpublishable Manuscripts

C. 1 An Inequality Concerning Rational Functions of Non-Hermitean Matrices.

This file contains the first part on a draft of a paper entitled: "An inequality concerning rational functions of non-Hermitean matrices."

The basic idea is contained in the definition:

Let A be an arbitrary linear operator, and $\mathscr{S}$ a set of complex numbers. $\mathscr{S}$ is a spectral set of A if and only if it possesses the following property:

Let f(z) be an arbitrary rational function. If for every z in $\mathscr{S}$, f(z) is defined and $|f(z)| \leqq 1$, then f(A) is defined and $|||f(A)||| \leqq 1$.

The manuscript states Theorem 1:

The set of all complex numbers z with $|z| \leq 1$ (the unit circular disc) is a spectral set of the operator A, if and only if $|||A||| \leqq 1$.

The manuscript terminates before the proof of the first lemma involved is given. The first lemma is, however, stated.

C. 2 On Generalized Matrix-Algebras With a Unique Trace.

This file contains seven handwritten pages entitled, "On generalized matrix-algebras with a unique trace."

Of this manuscript F. J. Murray has said: "This is the beginning of an effort to motivate the whole rings of operator theory. Unfortunately it is far too incomplete to be of any use."

C. 3 Aleph-Boolean Algebra.

This file contains a 12-page handwritten manuscript on "Aleph-Boolean Algebras." An Aleph-Boolean algebra B is one, in which every set $S \subseteq B$ with a power less than aleph possesses a l. u. b. $\Sigma(S)$ and a gr.

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l.b. $\pi(S)$. An isomorphism between two aleph Boolean algebras $B \bullet\!\!\rightarrow C$ is a (possibly) many-to-one mapping of B on a (not necessarily proper) subset of C, which leaves the notions of l.u.b. and gr.l.b. for sets of power less than aleph invariant. A function $C \Longleftrightarrow B$ is an isomorphism $B \bullet\!\!\rightarrow C$. The set of all these functions is denoted by F(B, C). These definitions are shown to include a variety of mathematical objects. Seven examples of mathematical objects that can be described in terms of the above definitions are listed and discussed.

The manuscript ends with the sentences, "Yet there is nothing in our general F(B, C)-mechanism which corresponds to the 'two-variable-substitution' in such a natural and obvious way, as the correspondence ... which correlates the 'one-variable-substituion' h(y) = f(g(y)) to the composition rule ... . Let us therefore analyse the two-variable-substitution h(y) = f(g'(y), g''(y)) somewhat more closely."

C.4 Notes on Operator Theory

This folder contains:

(1) An outline of a lecture.

(2) Some very rough notes on lattices.

(3) 34 pages of notes discussing a particular set of operators in the Hilbert space of all measurable (complex-valued) functions f(x) defined for $-\infty < x < +\infty$ with a finite

$$\int_{\infty}^{\infty} |f(x)|^2 dx \quad .$$

The operators discussed are shown to generate a "factor" of the ring of all operators in this Hilbert space which is of type II_1.

C.5 Paper Dealing With Decomposition of a Simple Ring with d.c.c. into a Tensor Product of Subalgebra and Centralizer. About 1933.

Concerning this file I. Kaplansky has said: "Would guess that these are notes (written in 1933 or so) collecting thoughts on tensor products, motivated by rings of operators. Quite outdated now."

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C.6 Real Analytic Functions and Elliptic Operators.

Concerning this file, G. Hunt has said:

"Consider an open set D in n-space, and on D an elliptic differential operator Δ of second order with constant coefficients. The formal operator Δ can be extended to a definite selfadjoint operator $\breve{\Delta}$ on a suitable Hilbert space H of functions defined on D. The manuscript is devoted to proving that a function in H is an analytic function on D (after alteration on a null set) if it belongs to a bounded spectral interval for $\breve{\Delta}$. A more general result is contained in Theorem 8 of the paper 'Analytic Vectors' by E. Nelson (Ann. Math. 70 (1959), pp. 572-615)."

C.7 Notes on Closures and Adjoints of Linear Differential Operators.

Concerning this file, G. Hunt has said:

"The contents of this manuscript appear in I. Halperin's paper 'Closures and adjoints of linear differential operators' (Ann. Math. 38 (1937), pp. 880-919) as Lemma 2.1 and, somewhat generalized, as Lemma 2.2"

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C. 8 Group Theory, Topology. About 1934.

Of this file, Deane Montgomery has said:

"This folder contains (1) a manuscript outlining a possible approach to the problem of showing that a locally Euclidean group does not contain arbitrarily small subgroups, a problem which has since been solved by other methods. It is first shown that the problem can be reduced to the consideration of p-adic groups. The idea is then to use a similar method to one used by Newman for the case of finite cyclic groups of order p. The paper carries this approach quite far but the argument is left incomplete and, incidentally, no one has ever been able to fill in the details for this approach, although it still is tempting.

"(2) This paper studies the action of a compact group on a finite dimensional space, and considers the question of whether or not the orbit space has dimension at most equal to that of the original space. However, the argument is broken off without being complete and the question remains open to the present."

C. 9 Notes Related to Rings of Operators. About 1937.

Concerning this file I. Kaplansky has said: "This manuscript looks like a paper promised in Rings of Operators IV (with F. J. Murray. Ann. Math., 44 (1943), pp. 716-808). It starts with an abstract algebra with a trace and winds up with a factor of type II_1."

C. 10 Arithmetic of Continuous Rings.

Concerning this file I. Kaplansky has said: "This manuscript concerns the 'arithmetic of continuous rings' as sketched in PNAS, 1937, pp. 341-349." (Continuous Rings and Their Arithmetics.)

C. 11 Ergodic Theory. 1941-42.

Concerning (1) of this file S. Kakutani has said: "Discussion of an example in the ergodic theory which was never published. It was

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

written during the winter term 1941-42."

Concerning (2) of this file S. Kakutani has said: "Probably introduction to Gibbs Lecture 11/24/44, or book on same which was never written up."

And P. Halmos has said: "Makes contact with abstract on random ergodic theorems (with Ulam). In any case, however, there is nothing in the notes."

C. 12 Shock Waves. About 1943-44.

This file contains an introduction to a report which was to be written for Los Alamos. Calkin, Metropolis and others were involved in this work. It may have been completed by other workers. Calkin suspects that A. E. Roberts of Arlington, Virginia may have written a report on this.

C. 13 Motion of a Cavity in a Fluid-Collapse and Rebound. About 1944.

This manuscript begins with the discussion of the spherically symmetric motion of an incompressible fluid during the time

$$-\infty < t \leq t_0$$

subject to the boundary conditions

$$p_{(y=\infty)} = 0$$

$$p_{(y=y_0)} = 0$$

where p is the pressure and y is the distance from a point called the center. The free surface $y = y_0(t)$ bounds a cavity whose collapse and rebound is the subject of discussion.

After a complete discussion of this problem is given the "composite case" is taken up. This case is one in which

$$\rho = \rho_1 = \text{constant for all } p < P$$

$$\rho = \rho_2 = \text{constant for all } p > P$$

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

$P > 0$, $\rho_2 > \rho_1 > 0$.

It is assumed that P is reached first at $t = 0$ according to the theory given in the first part. For $t > 0$ three zones will develop:

Zone 1: $y_0 < y < y_1$, $p < P$, $\rho = \rho_1$

Zone 2: $y_1 < y < y_2$, $p > P$, $\rho = \rho_2$

Zone 3: $y_2 < y < \infty$, $p < P$, $\rho = \rho_1$.

The boundary conditions for the surface separating zones 1 and 2 and zones 2 and 3 are taken to be those holding across shocks. That is, the relation between the jump in velocity across this boundary and the jump in pressure and density is taken to be that which follows from the equations of conservation of mass and momentum applied to the surface of discontinuity.

The solution derived for a single fluid is applied to the composite case and studied in detail in the neighborhood of t_0, the time interval for which $y_0(t) \approx 0$ but $y_2(t) \not\approx 0$. The time t_0 is said to be the time of total collapse of the central cavity.

The final part of the manuscript discusses the composite case for times greater than t_0 and is entitled, "The composite case. The rebound up to the reversal of the flow of matter from zone 3 into zone 2."

C. 14 Theory and Techniques for Design of Electronic Digital Computers. Moore School Lecture 8/13/46.

These notes constitute the transcription of the recording of a lecture given by von Neumann at the Moore School of Electrical Engineering, The University of Pennsylvania, on August 13, 1946. von Neumann never took the time to put these notes, which are almost unintelligible, into publishable shape. In a letter dated November 11, 1946 to Carl C. Chambers, he expressed the inability to find time to do so.

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From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

In the lecture von Neumann discussed problems on which computers will not make a significant difference in their solution, as well as problems of the opposite sort. In the first category he lists:

(1) Mathematical theorem searching

(2) Combinatorial problems such as those arising in playing chess.

As problems on which computers will make a great contribution he lists and discusses:

(1) Ordinary and partial differential equations including those of

a) hydrodynamics

b) turbulence

c) meteorology

(2) Systems of linear equations.

These notes are so rough that they should not be made public. Moreover, many of the things said here are said better in later speeches and reports. (Cf. A. 9.)

C. 15 Discussion of the Cylinder Symmetrical Geometry of Points and Trajectories. 195-?

This file contains a seven-page discussion of the representation of points and trajectories inside a cylinder. It ends with a section 7 which in turn begins with the sentence: "These equations that follow control the behavior of a photon in the inverse Compton effect." It may be that this work is connected with a Monte Carlo calculation made on some Los Alamos or other classified problem.

C. 16 The Fourier Transformation Method for 2-Dimensional (Baratropic) Meteorological Prediction. 195-?

Concerning this file J. G. Charney has said: "Johnny's report entitled 'The Fourier transformation method ... prediction' has never been published to my knowledge. However, we did use the method in

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

a joint paper entitled 'Numerical integration of the barotropic vorticity equation' which appeared in Tellus, Vol. 2, No. 4, 1950. In this paper the method is set forth in detail; hence I see no point in publishing the report."

C. 17 Remarks Concerning Various Forms of the "Relaxation Method." 1951.

The contents of this file are best described by von Neumann's summary given in the file:

"This is an application of the 'extrapolated Liebman method' of 'relaxation.' It is done, for the sake of simplicity, in one dimension, but the real interest would, of course, lie in an extension to $\nu(= 2, 3, \ldots)$ dimensions.

"The 'elementary step' considered consists of applying the 'e. L. m.' consecutively in two opposite directions. (First 'increasing i' and then 'decreasing i,' cf. (1), (2).) This step is represented by the operator σ. (Cf. (3).)

"The proper-value spectrum of σ is determined, and it is found to lie in two narrow regions: N-2 values in an interval essentially from $1 - \frac{\pi}{N}$ to $1 - \frac{2\pi}{N}$, and an additional value essentially at $-\delta$, where $\delta = e^{-4\pi} = 3.7 \times 10^{-6}$. (Cf. after (50).)

"Now operators of the form

(A) $$\mathcal{P} = \sum_{0}^{n} m \; a_m \sigma^m \qquad (\sum_{0}^{n} m \; a_m = 1)$$

may be used instead of σ. The freedom inherent in (A) (i.e. in the choice of n and of the a_m) may be used to compress the spectrum of σ nearer to 0."

The file should not be published for the following two reasons:

(1) H. H. Goldstine states that there are errors in the work. (This is one reason that the paper that he and von Neumann were writing never got completed.)

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

(2) J. W. Sheldon published a paper entitled, "On the Numerical Solution of Elliptic Difference Equations," in Vol. 9 of MTAC (1955), pp. 101-111, which discussed the material in the file. This paper reaches substantially the same results given by von Neumann and seems to be error free. This would indicate that the errors in the notes by von Neumann are not serious ones. Nevertheless, the main ideas are in the Sheldon paper and although von Neumann gives a more detailed analysis of a proper value problem that occurs, this does not warrant publication at this date.

C.18 Imbedding of $F(E_m)$ in $\mathfrak{H}$.

In the paper entitled, "Fourier Integrals and Metric Geometry," by von Neumann and Schoenberg [Vol. IV, No. 22] the following statement is made:

"In conclusion we want to say that the operator method of Part II, dealing with the embedding of $F(E_1)$ in $\mathfrak{H}$, may also be extended to cover the general case of the embedding of $F(E_m)$ in $\mathfrak{H}$. Although this extension has been fully worked out, for reasons of conciseness we treat in Part II only the case of screw lines ($m = 1$)."

This file contains the manuscript which confirms the last sentence: "This extension has been fully worked out." The file also contains some correspondence with I. Schoenberg.

C.19 Carter Oil Company Reports to which J. v. Neumann contributed.

This file contains three items:

(1) A preprint of a paper by Edward F. Johnson and Henry J. Welge entitled, "An Analysis of the Linear Displacement of Oil by Gas-Driven Solvent."

(2) An internal Company report by John von Neumann entitled, "Deviation of Mathematical Relations for Underground Combustion."

(3) An internal Company report entitled, "A Theoretical Study of Miscible Fluid Displacement in the Five-Spot Pattern," by Edward F. Johnson, Henry J. Welge and John von Neumann.

There is also contained in this file a letter to A. H. Taub from Max C. Sons, Vice President of the Jersey Production Research Co., transmitting and commenting on these reports.

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

C. 20 An Adaptation of the MADDIDA, A Digital Differential Analyser of Northrup Aircraft, Inc.

Concerning this file W. H. Ware of the Rand Corporation, Santa Monica, has said: "Someone has guessed that the paper was presented to the Cowles Commission Conference, Univ of Chicago, 1949 (included in George Washington Univ. Logistics Research Project.)"

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

ATTACHMENT D

Listing of Contents of Folders in

FILE D: Notes Not in Manuscript Form

D. 1 Economic Theory. Early 1930's.

This file contains miscellaneous notes on mathematical economics. Of these notes O. Morgenstern has said: "I doubt that it would be possible to publish any of the different notes."

D. 2 Operator Theory. Analysis. Lattices.

This file contains four sets of handwritten notes. These notes contain detailed calculations concerning special problems in operator theory. The problems are not clearly stated nor are there conclusions drawn from the calculations. One of these sets of notes is written in German and one is written on stationery marked "The Seville, Daytona Beach, Florida."

The file also contains notes headed "T. Carleman: Application de la théorie des équations intégrales singulières aux équations differentielles de la dynamique. Akiv für Mat., Astr. och Fysik, 22B (1932), No. 7." These notes contain von Neumann's comments (in German) on Carleman's paper.

D. 3 Lebesgue-Type Measure Theory.

Rough notes in a folder entitled, "Simplified treatment of Lebesque-type measure theory, based on the open sets."

D. 4 Discussion with R. Post and J. Tuck, Jan. 9, 1953.

Notes on a discussion of R. Post, J. Tuck and J. von Neumann on Jan. 9, 1953. Estimates are made concerning a system consisting of N particles of charge -e and N particles of charge e in a Mathien field of amplitude a and frequency ω.

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

B.5 Performance Statistics for Meteorology. 1950.

Two pages of computer performance statistics on meteorology problem for the period 1500 Nov. 24, 1950 to 1500 Nov. 25, 1950.

B.6 General Relativity. Quantum Mechanics.

(1) In German (JvN) and English (CCT) calculations by J. von Neumann and another person (C. C. Torrance?) on the Ricci tensor for a space time with a non-diagonal metric tensor. The actual form of the metric tensor is not given. Instead its value at (0, 0, 0, t) is given as well as its first and second derivations at this point. The purpose of the calculation is not indicated nor are any physical conclusions drawn from the calculation.

(2) Six pages (in German) entitled "Zusammenstellung über den De Sitterschen Raum," with one figure. The De Sitter space is considered as the hypersphere $x^2 + y^2 + z^2 + u^2 - v^2 = 1$ in a flat five space. Explicit formlas are given for the (non-null and null) geodesics and the null geodesic joining two non-null geodesics is computed. The known static solution to the Einstein field equations with the cosmological constant present is described as a new metrization of the hypersphere in the five-dimensional space and compared to the "empty" De Sitter universe. Remarks are made concerning the nature of the singularity of the new metrization of the hypersphere.

(3) Twelve handwritten pages (in German) on Relativistic Cosmology. These notes begin with a discussion of the function R(t) which enters the line element of cosmological spaces. It continues with a comparison of these spaces and the constant values of $\alpha = R^3\rho_0$, $\beta = R^4 p$, $\Gamma = \dot{R}/R$ and T, the age of the universe where ρ_0 is the density, p the pressure, $\dot{R} = dR/dt$ and T is taken to be the propagation time for light signal to go from a point of the universe to the antipodal point. Most of the results herein are to be found in Tolman's book and articles referred to there. The paper concludes with a calculation of the amount

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

that light will be deflected by N mass points contained in a cylinder of radius R.

(4) Seven handwritten pages of notes (in German) on a series of transformations of coordinates in a space-time with a Schwarschild metric. In the final coordinate system there is a limiting surface on which the metric becomes singular.

D. 7 Discussion of $a = \left|\frac{\mathrm{sh}z}{x}\right|$, $a > 0$ real, z complex.

On pages [1'] and [2'] von Neumann gives some rough notes containing a differential difference equation approximation of the equations of motion of a small progressive disturbance in a fluid. The equation is

$$\frac{f(\xi+1) + f(\xi-1) - 2f(\xi)}{\frac{d^2 f}{d\xi^2}} = a^2 \quad .$$

Solutions of this equation of the form

$$f(\xi) = ce^{2z\xi}$$

may be obtained if z and a are related by

(*) $$\left|\frac{\sinh z}{z}\right| = a > 0$$

with $a > 0$ and z complex. In pages (1) to (6) von Neumann discusses the nature of the solutions of equation (*). No application of those solutions to the original problem is made. There are two pages of penciled figures in this folder that seem to have no bearing on the other material.

D. 8 Stability of Finite Difference Schemes for Wave Equation in 2 Variables.

These very rough notes (12 pages) contain various difference approximations to the wave equation in two variables:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0 \quad .$$

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Approximations are found which involve implicit difference equations which are unconditionally stable. These results are now in the literature.

D. 9 Hydrodynamics With Energy Equation and Momentum Equation Having "Source" Terms.

These seven handwritten pages contain a discussion of the one-dimensional equations of hydrodynamics with a body force proportional to the velocity and an energy source which is a linear function of the internal energy plus a multiple of sin(x-ct). An approximate solution of these equations is obtained.

D. 10 Algebraic Treatment of 3-Shock Configurations for Weak Incidence and Reflected Shocks.

A small pocket notebook containing various equations pertaining to the 3-shock configuration of Mach reflection. The three-shock solution corresponding to a weak incident wave is studied. The results herein contained are known and more besides.

D. 11 Weak Shocks and Weak Prandtl-Meyer Waves.

A fragmentary manuscript related to a war-time report written with V. Bargmann and D. Montgomery for Division 2 of NDRC, and superseded by that report.

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

ATTACHMENT E

BIBLIOGRAPHY OF JOHN VON NEUMANN

Books

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Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
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Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

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Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
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†With most items listed there is given a volume and a number. These denote the volume and the order number in that volume where the item may be found. Thus, [Vol. I, No. 2] at the end of item 1 implies that item 1 is the second article in Volume I.

*Translated by S. Bargmann. On the Theory of Games of Strategy. In: Contributions to the Theory of Games, Vol. IV (Ann. Math. Studies, No. 40, Princeton Univ. Press 1959), pp. 13-42.

**These items will not be found in the collected works. The Appendix to the bibliography states the nature of the material omitted.

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Supplement to the Bibliography of John von Neumann

The bibliography given above should have the following items added:

Eigenwerteproblem symmetrischer Functionaloperatoren. Jber. Deutch. Math. Verein. 37, pp. 11-14 (1928).

Sur la Theorie des jeux. C. R. Acad. Sci. Paris 186, pp. 1689-91 (1928).

These papers were not reprinted in the Collected Works because their contents are contained in other papers. Thus the first paper, which is an abstract of a speech presented at the Vierter Deutscher Mathematikertag in Bad Kissingen September 18-24, 1927, mentions results discussed in items 28 and 29 of the bibliography (Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Vol. II, p. 3 and Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren, Vol. II, p. 86). The second paper is a short note which mentions results obtained in item 15, Zur Theorie der Gesellschaftsspiele, Vol. VI, p. 1.

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Appendix to
Bibliography of John von Neumann

The following paragraphs describe the nature of the articles in the bibliography which are omitted from the collected works. The articles are identified by the number appearing in the bibliography.

72. Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes.

This paper has been translated and published as item number 98 of the bibliography, which is included in Vol. VI, No. 3.

77. With R. H. Kent. The Estimation of the Probable Error from Successive Differences.

The results contained in this report are given in item 78, which is included in Vol. IV, No. 33.

81. Shock Waves Started by an Infinitesimally Short Detonation of Given (Positive and Finite) Energy.

Item 109 of the bibliography, which is in Vol. VI, No. 21, is another, more polished version of this report.

92. With R. J. Seeger. On Oblique Reflection and Collision of Shock Waves.

The contents of this short memorandum are contained in item 94 of the bibliography. The latter report is in Vol. VI, No. 22.

96. Introductory Remarks (Sec. I), Theory of the Spinning Detonation (Sec. XII), Theory of the Intermediate Product (Sec. XIII). In: Report of Informal Technical Conference on the Mechanism of Detonation.

The remarks made by von Neumann at the conference summarized in this document are in the main contained in item 88 of the bibliography. The latter report is in Vol. VI, No. 20.

97. Rieman Method; Shock Waves and Discontinuities (one-dimensional); Two Dimensional Hydrodynamics. Lectures in: Shock Hydrodynamics and Blast Waves, by H A. Bethe, K. Fuchs, J. von Neumann, R. Peierls and W. G. Penney. Notes by J. O. Hirschfelder.

This report contains notes of lectures given by von Neumann at Los Alamos

Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

Scientific Laboratory. The material in these notes is described in a number of treatises and text books.

100. With A. H. Taub. Flying Wind Tunnel Experiments.

This report describes the first steps of an experimental program which was not carried to completion.

114. On the Theory of Stationary Detonation Waves.

Item 88 of the bibliography, which is in Vol. VI, No. 20, contains all the results described in this report and the mathematical derivation of the results.

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Addendum to

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i. Surface Water Waves Excited by an Underwater Explosion. (Memorandum to J. R. Oppenheimer, Aug. 28, 1944.) LAMS-128. 5 pp.

ii. Digest of J. von Neumann's Lecture at Meeting on Optimum Heights, on Sept. 22, 1944. (Memorandum to N. F. Ramsey from B. Waldman, Dec. 7, 1944.) LAMD-46. 8 pp. SRD

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vi. By Ralph W. Gerard. Some of the Problems Concerning Digital Notions in the Central Nervous System. In: Cybernetics (New York, Josiah Macy, Jr. Foundation). Remarks by John von Neumann in the discussion, pp. 19-31.

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Records of the Office of the Director / Faculty Files / Box 35 / von Neumann, John / Publication of Collected Works - ONR Contract - Report
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

In 1945 J. von Neumann wrote a series of letters to H. H. Goldstine which the latter put together and entitled First Draft of a Report on the Edvac. This was issued by the Moore School of Electrical Engineering on June 30, 1945 as a report on Contract No. W-670-ORD-4926 between the United States Army Ordnance Department and the University of Pennsylvania. A photocopy of this report is in Fine Hall Library.

The report contains a detailed design of a synchronous machine and discusses the roles of the various organs. Much of the latter discussion is to be found in the Burks, Goldstine, von Neumann report and the Goldstine and von Neumann ones.

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ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

ADIUVANTIBUS

L. KALMÁR, L. RÉDEI ET K. TANDORI

REDIGIT

B. SZ.-NAGY

TOMUS XXIII

FASC. 1—2

I. Halperin

Von Neumann's Arithmetics of Continuous Rings

This article refers to C.10

SZEGED, 1962

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

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A SZEGEDI TUDOMÁNYEGYETEM KÖZLEMÉNYEI

ACTA SCIENTIARUM MATHEMATICARUM

KALMÁR LÁSZLÓ, RÉDEI LÁSZLÓ ÉS TANDORI KÁROLY

KÖZREMŰKÖDÉSÉVEL

SZERKESZTI

SZŐKEFALVI-NAGY BÉLA

23. KÖTET

1—2. FÜZET

SZEGED, 1962. JÚNIUS HÓ

SZEGEDI TUDOMÁNYEGYETEM BOLYAI-INTÉZETE

Von Neumann's Arithmetics of Continuous Rings

By ISRAEL HALPERIN in Kingston (Canada)

This paper is a revision of an unpublished manuscript *Arithmetics of Regular Rings Derived From Continuous Geometries* written by J. VON NEUMANN in 1937[1]) and summarized in §§ 6, 7 of his note [3]. I am grateful for permission to present this work of VON NEUMANN. The material has been freely re-arranged, the introduction, footnotes and Lemma 6. 1 have been added and I have strengthened Lemmas 2. 1, 2. 2, 3. 1, 6. 2 and 6. 3 by using different proofs. Any faults in the present exposition are, of course, mine.

1. Introduction

1. 1. We use terminology close to that of [4]. $\mathfrak{R}$ will usually denote a fixed regular ring with unit, $\overline{R}_{\mathfrak{R}}$ = lattice of all principal right ideals of $\mathfrak{R}$ and $\overline{L}_{\mathfrak{R}}$ = lattice of all principal left ideals of $\mathfrak{R}$. For any idempotent e in $\mathfrak{R}$, $\mathfrak{R}(e)$ will denote the ring of all eae, $a \in \mathfrak{R}$. We shall say "a has a reciprocal in $\mathfrak{R}(e)$" if $a \in \mathfrak{R}(e)$ and $ab = ba = e$ for some $b \in \mathfrak{R}(e)$.

If $\mathfrak{R}$ is a complete rank ring, $R(a)$ will denote the unique normalized rank, defined for all a in $\mathfrak{R}$ and related to the dimension functions D in $\overline{R}_{\mathfrak{R}}$ and D' in $\overline{L}_{\mathfrak{R}}$ by:

$$R(a) = D((a)_r) = D'((a)_l).$$

A complete rank ring $\mathfrak{R}$ is (cf. [4]) either a discrete or a continuous ring according as the range of R is $0, 1/n, \ldots, n/n$ for some integer $n \geqq 1$ or the set of all real numbers $0 \leqq t \leqq 1$. If $e \neq 0$, $\mathfrak{R}(e)$ is discrete or continuous along with $\mathfrak{R}$ and the rank function of $\mathfrak{R}(e)$ coincides with $R(a)/R(e)$, a in $\mathfrak{R}(e)$.

Z will denote the center of $\mathfrak{R}$ and we let $P = P(\mathfrak{R})$ denote the set of all polynomials

$$p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0 t^0$$

with all z_i in Z and $m \geqq 1$. When $p(a)$ is calculated, t^0 is ordinarily to be replaced by the unit of $\mathfrak{R}$. But whenever we write "$p(a)$ in $\mathfrak{R}(e)$" we shall mean that t^0 is to be replaced by e.

We shall show below (cf. Lemma 2. 1) that if $\mathfrak{R}$ satisfies a weak condition (in particular, if $\mathfrak{R}$ is discrete or continuous) then the center of $\mathfrak{R}(e)$ consists of

[1]) The original manuscript is kept in the VON NEUMANN file of the Institute for Advanced Study in Princeton and may be seen there.

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all $ez, z\in Z$. Hence if $a\in\Re(e)$ and $q\in P(\Re(e))$ then $q(a)$ coincides with $(p(a)$ in $\Re(e))$ for some $p\in P(\Re)$.

Suppose $\Re$ is a discrete or continuous ring. If $p\in P$ and ε is a real number >0, we shall say a is *ε-p-algebraic* or simply *ε-algebraic* if $R(p(a))<\varepsilon$. If a is *ε-algebraic* for every $\varepsilon>0$ we shall say that a is *almost-algebraic*. If $R(p^m(a))\to 0$ as $m\to\infty$ we shall say a is *limiting p-algebraic* or simply *limiting algebraic*. If $R(p(a))=0$, equivalently $p(a)=0$, we shall say a is *p-algebraic* or simply *algebraic*.

On the other hand, if $R(p(a))=1$ for every $p\in P$ we shall say a is *purely transcendental.*[2])

1. 2. In this paper we shall prove the theorem:

(1. 2. 1) *In a continuous ring the algebraic elements are everywhere dense in the sense of rank-metric.*

(1. 2. 1) means: for every $a\in\Re$ and every real $\varepsilon>0$ there exists an algebraic $b\in\Re$ such that $R(a-b)<\varepsilon$. It is easy to see that (1. 2. 1) follows from (1. 2. 2)—(1. 2. 5) below:

(1. 2. 2) If a is ε-algebraic then $R(a-b)<\varepsilon$ for some algebraic b (cf. Lemma 3. 2).

(1. 2. 3) If a is limiting p-algebraic then for every real $\varepsilon>0$, a is ε-q-algebraic when $q=p^m$ for sufficiently large integer m (cf. § 4).

(1. 2. 4) If a is purely transcendental then for every real $\varepsilon>0$ and every $p\in P$ with p of degree greater than $1/\varepsilon$, $R(a-b)<\varepsilon$ for some ε-p-algebraic b (cf. Lemma 5. 3).

(1. 2. 5) If (1. 2. 3), (1. 2. 4) hold then for every a and every real $\varepsilon>0$, $R(a-b)<\varepsilon$ for some ε-algebraic b (cf. Lemma 7. 1 and Theorem 8. 1).

Before going into the detailed proof (cf. sections 2—8) we give a brief indication of its principal ideas.

1. 3. To prove (1. 2. 2) we suppose $R(p(a))<\varepsilon$ and let $(e)_r=(p(a))_r$. Set $b=(1-e)a+x$ with x arbitrary in $\Re(e)$. We show that $(1-e)a$ is in $\Re(1-e)$ and $p(b)=(p(x)$ in $\Re(e))$. Since $R(a-b)=R(ea-ex)\leqq R(e)<\varepsilon$, (1. 2. 2) will be verified if for given $p\in P$ and given e we can find $x\in\Re(e)$ such that $(p(x)$ in $\Re(e))=0$. In the Corollary to Lemma 3. 1 we show that this is possible.

1. 4. The statement (1. 2. 3) is almost trivial (cf. § 4).

1. 5. The proof of (1. 2. 4) is technically the most difficult part of the entire proof. We show first that if a is purely transcendental then for every integer $N\geqq 1$ there exists a decomposition into independent elements: $\Re=\Sigma((a^ie_j)_r;$ $i=1, \ldots, j; j\geqq N)$. Now choose $m>1/\varepsilon$, choose any integer $N>\frac{m}{\varepsilon}$ and define an idempotent g so that $(g)_r=\Sigma((a^ie_j)_r; j\geqq N;\ i=m, 2m, \ldots,$ but $i\leqq j)$, $(1-g)_r=$ $=\Sigma((a^ie_j)_r;\ j\geqq N,\ i\leqq j$ but $i\neq m, 2m, \ldots)$. We set[3]) $b\equiv a-a^{-m+1}p(a)g$ and show: $p(b)a^ie_j=0$ for all $j\geqq N$ and all i satisfying $i\leqq ms$ for some $ms\leqq j$. It follows that $R(p(b))\leqq\frac{m}{N}<\varepsilon$ and $R(a-b)\leqq R(g)\leqq\frac{1}{m}<\varepsilon$, hence this b verifies (1. 2. 4).

[2]) VON NEUMANN discovered [3] that in every continuous ring there exist purely transcendental elements but the manuscript (see 1)) gives no details of his proof. See [2] for a proof.

[3]) The symbol $\equiv$ means "equality by definition".

1. 6. To verify (1. 2. 5) we define $P'=P'(a)$ to the set of irreducible polynomials p in P for which $R(p(a))<1$. We show that P' is finite or denumerable, $P'\equiv(p_1, p_2, \ldots)$. Then we determine a "resolution of the identity" for a, that is, a sequence of orthogonal idempotents $e_0, e_1, \ldots$, each of which commutes with a, such that $\Re=\Sigma(e_i)_r$ and:

(1. 6. 1) $ae_0=e_0a$ is purely transcendental in the ring $\Re(e_0)$,

(1. 6. 2) for $i\geqq 1$, $ae_i=e_ia$ is limiting p_i-algebric in the ring $\Re(e_i)$.

Suppose now (1. 2. 3) and (1. 2. 4) hold for every continuous ring. Then if n_i is sufficiently large, $R(p_i^{n_i}(ae_i)$ in $\Re(e_i))<\frac{\varepsilon}{2^{i+2}}$. And if p_0 is any polynomial of sufficiently high degree there exists b_0 in $\Re(e_0)$ such that $R(ae_0-b_0)<\frac{\varepsilon}{2}$ and $R(p_0(b_0)$ in $\Re(e_0))<\frac{\varepsilon}{4}$. Now we choose j so large that $R(1-(e_0+\ldots+e_j))<$ $<\frac{\varepsilon}{2}$ and set

$$b=b_0+ae_1+\ldots+ae_j,\quad p=p_0p_1^{n_1}\ldots p_j^{n_j}.$$

It follows that $R(a-b)\leqq R(ae_0-b_0)+R(1-(e_0+\ldots+e_j))<\varepsilon$, and $R(p(b))<\varepsilon$. This verifies (1. 2. 5).

1. 7. Some of our Lemmas are proved under hypotheses weaker than the requirement that $\Re$ be a continuous ring; in particular, irreducibility of $\Re$ is frequently not required. This will facilitate an extension (to the reducible case) of Theorem (1. 2. 1).

2. Preliminary lemmas

Lemma 2. 1. *Suppose e is an idempotent in an associative ring $\Re$ and that* (i): *$e=0$ or* (ii): *$\Re$ possesses a set of matrix units $s_{ij}, i, j=1, \ldots, k$ for some $k=1,2, \ldots$ with $es_{11}=s_{11}e=s_{11}$. Then the center of $\Re(e)$ is the set of all ze, $z\in Z$.*

Proof. The Lemma is trivially true if $e=0$. Consider the case $e\neq 0$ and suppose a is any element in the center of $\Re(e)$. Then clearly $\bar a\equiv as_{11}$ is in the center of $\Re(s_{11})$. Let $z=\sum_{i=1}^{k}s_{i1}\bar as_{1i}$. Then z is in Z: for if x is in $\Re$, $x=\sum_{i=1}^{k}xs_{ii}=\sum_{i=1}^{k}s_{ii}x$, so (since $s_{1j}xs_{i1}$ is in $\Re(s_{11})$ and $\bar a$ is in the center of $\Re(s_{11})$)

$$xz=\sum_{i=1}^{k}xs_{i1}\bar as_{1i}=\sum_{j=1}^{k}s_{j1}\Big(\sum_{i=1}^{k}s_{1j}xs_{i1}\bar as_{1i}\Big)=\sum_{j=1}^{k}s_{j1}\Big(\sum_{i=1}^{k}\bar as_{1j}xs_{i1}s_{1i}\Big)=$$

$$=\Big(\sum_{j=1}^{k}s_{j1}\bar as_{1j}\Big)x=zx.$$

We shall show that $y\equiv ze-a$ satisfies $y=0$. Clearly, y is in the center of

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$\Re(e)$ and $ys_{11}=\bar{a}s_{11}-as_{11}=0$. Hence for all u, v in $\Re$, $yus_{11}v=(yeues_{11})v=$ $=(eue)ys_{11}v=0$. Then for each $i=1, \ldots, k$, $ys_{ii}=ys_{i1}s_{11}s_{1i}=0$ so $y=\sum_{i=1}^{k} ys_{ii}=0$.

Thus, if a is in the center of $\Re(e)$ then $a=ze$ for some z in Z. Conversely if z is in Z and x is in $\Re(e)$ then $(ze)x=zx=xz=(xe)z=x(ze)$ so ze is in the centre of $\Re(e)$. This completes the proof of Lemma 2. 1.

Corollary. *If e is an idempotent in a continuous or discrete ring $\Re$, then the center of $\Re(e)$ consists of all ze, $z\in Z$. And if $q(t)=t^m+z_{m-1}t^{m-1}+\ldots+z_0t^0$ with all z_i in the center of $\Re(e)$ then for some p in P, for every a in $\Re(e)$, $(p(a)$ in $\Re(e))$ coincides with $ep(a)=q(a)$.*

Proof. If $e\neq 0$, $\Re$ does possess a set of matrix units s_{ij}, $i, j=1, \ldots, k$ for some $k=1, 2, \ldots$ with $es_{11}=s_{11}e=s_{11}$. Hence Lemma 2. 1 applies and the Corollary follows.

Lemma 2. 2. *Suppose $\Re$ is an associative ring possessing a set of matrix units, s_{ij}, $i, j=1, \ldots, k$ for some $k\geqq 2$. Then for any a in $\Re$, if $faf=af$ for every idempotent f in $\Re$, a is in the center of $\Re$.*

Proof. The condition $faf=af$ can be written: $(1-f)af=0$. With f in place of $1-f$ this gives: $fa(1-f)=0$, hence $fa=faf=af$.

Now for any x in $\Re$, $f+fx(1-f)$ is idempotent along with f, so $(f+fx(1-f))a=$ $=a(f+fx(1-f))$. But $fa=af$, and so we obtain

$$(fx(1-f))a=a(fx(1-f))$$

for every x in $\Re$ and every idempotent f in $\Re$.

Thus, $xa=ax$ whenever $x=fx(1-f)$ for some idempotent f, in particular whenever $x=s_{ii}x=xs_{jj}$ for some $i\neq j$(use $f=s_{ii}$). Hence, for every x, if $i\neq j$ then $s_{ii}xs_{jj}$ commutes with a, and for any i, using some $j\neq i$(here we use the hypothesis $k\geqq 2$),

$$a(s_{ii}xs_{ii})=a(s_{ii}xs_{ij})s_{ji}=(s_{ii}xs_{ij})as_{ji}=(s_{ii}xs_{ij})s_{ji}a=(s_{ii}xs_{ii})a,$$

so

$$xa=\sum_{i,j=1}^{k}(s_{ii}xs_{jj})a=\sum_{i,j=1}^{k}a(s_{ii}xs_{jj})=ax$$

showing that a is in the center of $\Re$, as stated.

Corollary. *Suppose a is an element in $\Re(e)$ with e idempotent in an associative ring $\Re$ and suppose $faf=af$ for every idempotent f in $\Re(e)$. Then $a=ze$ for some z in the center of $\Re$ if* (i): *$e=0$ or* (ii): *$\Re(e)$ possesses a set of matrix units s_{ij}, $i, j=1, \ldots, k$ for some $k\geqq 2$ and $\Re$ possesses some set of matrix units S_{ij}, $i, j=1, \ldots, K$, for some $K\geqq 1$ with $eS_{11}=S_{11}e=S_{11}$.*

Proof. The Corollary follows at once from Lemma 2. 2 and Lemma 2. 1.

Remark. The conditions of the Corollary to Lemma 2. 2 are always satisfied if $\Re$ is a continuous ring. But if $\Re$ is a discrete ring (then $\Re$ is the ring of $n\times n$ matrices over some division ring $\mathfrak{D}$) and $(e)_r$ is an atom in $\overline{R}_{\Re}$ (so $\Re(e)$ is ring isomorphic to $\mathfrak{D}$) the conditions fail to hold, and in fact if $\mathfrak{D}$ is not commutative, the Corollary actually fails to hold.

3. Proof of (1. 2. 1)

Lemma 3. 1. *Suppose $\Re$ is any associative ring with unit (not assumed regular) and let $\Re_N$ denote the ring of all $N\times N$ matrices over $\Re$. If $N\geqq 1$ and*

$$p(t)=t^N+z_{N-1}t^{N-1}+\ldots+z_0$$

is any polynomial of degree N with all z_i in the centre of $\Re$ then there exists a matrix M in $\Re_N$ such that $p(M)=0$.

Proof. For $i, j=1, \ldots, N$, let S_{ij} be the matrix with ij-th entry equal to 1 and all other entries 0. Then our requirements are satisfied by the matrix $M=\sum_{i=1}^{N-1}S_{i+1,1}-\sum_{i=0}^{N-1}z_iS_{i+1,N}$. We have

$$M=\begin{vmatrix} 0 & 0 & \ldots & 0 & -z_0 \\ 1 & 0 & \ldots & 0 & -z_1 \\ 0 & 1 & \ldots & 0 & -z_2 \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ 0 & 0 & \ldots & 0 & -z_{N-2} \\ 0 & 0 & \ldots & 1 & -z_{N-1} \end{vmatrix}.$$

In fact, $MS_{i1}=S_{i+1,1}$ for $i=1, \ldots, N-1$, hence $M^hS_{11}=S_{h+1,1}$ for $h=0, \ldots, N-1$. Now,

$$M^NS_{11}=MM^{N-1}S_{11}=MS_{N,1}=-\sum_{i=0}^{N-1}z_iS_{i+1,1}=$$

$$=-\sum_{i=0}^{N-1}z_iM^iS_{11}=-(z_{N-1}M^{N-1}+\ldots+z_0)S_{11}.$$

Thus $p(M)\cdot S_{11}=0$ and for $i>1$, $p(M)S_{i1}=p(M)M^{i-1}S_{11}=M^{i-1}p(M)S_{11}=0$. Hence, $p(M)S_{ii}=p(M)S_{i1}S_{1i}=0$ for $i=1, \ldots, N$ and summation over i gives $p(M)=0$. This completes the proof.

Corollary. *If e is an idempotent in a continuous ring $\Re$ and p is a polynomial in P then for some x in $\Re(e)$, $(p(x)$ in $\Re(e))=0$.*

Proof. The statement is obvious if $e=0$ (with $x=0$). So we may assume $e\neq 0$, and then by replacing $\Re$ by $\Re(e)$ we may suppose $e=1$. Since $\Re$ is a *continuous* ring it can be represented (cf. [4, page 99, Theorem 3. 3 and page 93, Definition 3. 2]) as a ring of $n\times n$ matrices over an associative ring $\mathfrak{S}$ with unit for *every* n (the center of $\Re$ can be identified with center of $\mathfrak{S}$). The Corollary now follows from Lemma 3. 1.

Remark. In the proof of this Corollary we made use of the fact that $\Re$ is continuous, not discrete.

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Lemma 3. 2. *For arbitrary $\varepsilon>0$, if a is ε-algebraic in a continuous ring $\mathfrak{R}$ then $R(a-b)<\varepsilon$ for some algebraic b in $\mathfrak{R}$.*

Proof. Suppose $R(p(a))<\varepsilon$, choose an idempotent e with $(e)_r=(p(a))_r$ and let $b=(1-e)a+x$ with x arbitrary in $\mathfrak{R}(e)$. Then $e=p(a)\cdot u$ for some u in $\mathfrak{R}$ so $ae=ap(a)u=p(a)\cdot au=ep(a)\cdot au=eap(a)u=eae$; thus $(1-e)ae=0$, $(1-e)a=(1-e)a(1-e)\in\mathfrak{R}(1-e)$. Then $p(b)=p\big((1-e)a(1-e)+exe\big)=(1-e)\cdot\cdot p(a)+\big(p(x)$ in $\mathfrak{R}(e)\big)=\big(p(x)$ in $\mathfrak{R}(e)\big)$, since $(1-e)p(a)=(1-e)ep(a)=0$.

Now $a-b=ea-x=e(a-x)$ so $R(a-b)\leqq R(e)=R(p(a))<\varepsilon$. And $p(b)=0$ if x is chosen in $\mathfrak{R}(e)$ so that $\big(p(x)$ in $\mathfrak{R}(e)\big)=0$. Now x can be so chosen by the Corollary to Lemma 3. 1 and then the Lemma is proved.

4. Proof of (1. 2. 3)

If $R\big((p(a))^m\big)\to 0$ as $m\to\infty$ then the polynomial $q(t)=p^m(t)$ has the property $R\big(q(a)\big)<\varepsilon$ if m is sufficiently large. This proves (1. 2. 3).

5. Proof of (1. 2. 4)

We shall use the symbol $E_n(e)$ to denote the expression $\Sigma(A, (a^ie)_r; i=1, \ldots, n)$.

Lemma 5. 1. *Suppose a is purely transcendental in a continuous ring $\mathfrak{R}$, let N be any integer $\geqq 1$, and suppose A is a principal right ideal such that $A\neq\mathfrak{R}$, $aA\leqq A$. Then there exists an idempotent $e\neq 0$ which is a solution for*

(5. 1) $$(A, (ae)_r, \ldots, (a^Ne)_r)\perp .\ ^{4})$$

Proof. We note for future use that if d, c are in $\mathfrak{R}$ then

$$D\big(d(c)_r\big)=R(dc)\leqq R(c)=D\big((c)_r\big).$$

If $R(d)=1$ and $dA\leqq A$ (in particular if $d=p(a)$ for some p in P) then $dc\in A$ implies $d\big((c)_r+A\big)\leqq A$, hence $(c)_r+A\leqq d^{-1}A$ and so $D\big((c)_r+A\big)\leqq D(A)$, and so $(c)_r\leqq A$. Thus for such d, if $dc\in A$ then $c\in A$.

Now suppose $N=1$. Choose B to be a complement of A in $\mathfrak{R}$. Then $B=a(a^{-1}B)$ (a^{-1} exists since $R(a)=1$). Choose an idempotent e such that $(e)_r=a^{-1}B$. Then $e\neq 0$ since $A\neq\mathfrak{R}$, hence $B\neq 0$. Also (5. 1) holds since it asserts only that $A(ae)_r=0$ and this is true since $(ae)_r=B$. So the Lemma holds for $N=1$.

Next, suppose the Lemma is established and $f_0\neq 0$ is a solution for the case $N=n$ for some integer $n\geqq 1$. We shall prove below:

(5. 2) $(a^{n+1}f)_r\leqq E_n(f)$ is false for some idempotent $f\neq 0$, $f\in(f_0)_r$.

Assuming (5. 2) we have

(5. 3) $$(a^{n+1}f)_rE_n(f)\neq(a^{n+1}f)_r$$

4) The symbol $\perp$ signifies independence of the lattice elements (cf. [4, page 8]).

so we can choose an idempotent $e'\neq 0$ with $(e')_r$ a relative complement of the left side of (5. 3) in the right side of (5. 3).

Now choose an idempotent e with $(e)_r=(a^{-(n+1)}e')_r$. We shall prove: *this e is a solution of* (5. 1) *for* $N=n+1$. In fact, $(e')_r\leqq(a^{n+1}f)_r$ so $(e)_r\leqq(f)_r$ and $(a^{n+1}e)_r=(e')_r$. Thus $e\neq 0$ since $e'\neq 0$, and $(A, (ae)_r, \ldots, (a^ne)_r)\perp$ since $(e)_r\leqq$ $\leqq(f)_r\leqq(f_0)_r$ and f_0 is a solution for (5. 1) with $N=n$. Furthermore, $(a^{n+1}e)_rE_n(e)\leqq$ $\leqq(e')_rE_n(f)=0$. Thus (5. 1) holds with $N=n+1$, as required. Thus by induction on N the Lemma would be proved for all N if (5. 2) were verified.

Assume (5. 2) false, if possible. Then for every f in $(f_0)_r$, $(a^{n+1}f)_r\leqq E_n(f)$. This implies

(5. 4) $$a^{n+1}f=y+\sum_{i=1}^{n}a^ifv_i$$

for some y in A and some v_i in $\mathfrak{R}$. Using right multiplication by f we could suppose $y=yf$ and $v_i=fv_if$ for all i. Choose, in particular, $f=f_0$ and let the resulting y, v_i in (5. 4) be denoted by x, u_i respectively. Then (5. 4) becomes

(5. 5) $$a^{n+1}f_0=x+\sum_{i=1}^{n}a^iu_i.$$

Right multiplication of (5. 5) by f and subtraction from (5. 4) yields

(5. 6) $$0=(y-xf)+\sum_{i=1}^{n}a^i(v_i-u_if).$$

Since the addends in (5. 6) are in the principal right ideals A, $(a^if_0)_r$, $i=1, \ldots, n$, respectively, and $(A, (af_0)_r, \ldots, (a^nf_0)_r)\perp$, therefore all of $y-xf$, $a^i(v_i-u_if)$ must be 0. Then $v_i-u_if=(a^{-1})^ia^i(v_i-u_if)=0$, so $v_i=u_if$. But $fv_i=v_i$ so $fu_if=u_if$ for every idempotent f in $(f_0)_r$, in particular for every idempotent f in $\mathfrak{R}(f_0)$. Hence, by the Corollary to Lemma 2. 2, $u_i=z_if_0$ for some z_i in Z. Now (5. 5) becomes

$$a^{n+1}f_0=x+\sum_{i=1}^{n}a^iz_if_0,$$

$$\Big(a^{n+1}-\sum_{i=1}^{n}z_ia^i\Big)f_0=x.$$

Put $p(t)=t^{n+1}+\sum_{i=1}^{n}(-z_i)t^i$. Then p is in P and $p(a)f_0=x\in A$. Since $R\big(p(a)\big)=1$, $f_0\in A$, $f_0\in(f_0)_rA=0$, $f_0=0$. This contradicts $f_0\neq 0$, so (5. 2) cannot be false and this completes the proof of the Lemma.

Corollary. *Under the hypotheses of Lemma* 5. 1 *there exists a maximal solution e. This means: if f is a solution and $(e)_r\leqq(f)_r$ then $(e)_r=(f)_r$.*

Proof. By transfinite induction there exists an ordinal number Ω (not necessarily a limit ordinal) and a set of solutions of (5. 1) e_α, $\alpha<\Omega$, such that $(e_\alpha)_r<(e_\beta)_r$ whenever $\alpha<\beta<\Omega$ and such that no solution f satisfies $(e_\alpha)_r<(f)_r$ for all $\alpha<\Omega$.

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Let $(e)_r=\Sigma_\alpha(e_\alpha)_r$. Then by Axiom III of [4, page 2] (the continuity of lattice operations) assumed for $\bar{R}_\Re$, we have: for $n=1, 2, \ldots, N$

$$(a^n e)_r E_{n-1}(e)=\sum_\alpha\big((a^n e_\alpha)_r E_{n-1}(e)\big)=\sum_{\alpha,\beta}(a^n e_\alpha)_r E_{n-1}(e_\beta)=$$
$$=\sum_\alpha(a^n e_\alpha)_r E_{n-1}(e_\alpha)=\sum_\alpha 0=0.$$

This implies that e is a solution of (5. 1). Since $(e)_r\geqq(e_\alpha)_r$ for all $\alpha<\Omega$ it follows that no solution f can satisfy $(f)_r>(e)_r$.

Lemma 5. 2. *Let $\Re$, a, N, A be as in Lemma* 5. 1. *Then there exists a sequence of idempotents $e_N, e_{N+1}, \ldots$ with the properties:*

$$(A, (a^i e_j)_r;\ i=1, \ldots, j;\ j\geqq N)\perp. \tag{5.7}$$

$$\sum(A, (a^i e_j)_r;\ i=1, \ldots, j;\ j\geqq N)=\Re. \tag{5.8}$$

Proof. Let $\bar{e}_N$ be a maximal solution of (5. 1) (existing by the Corollary to Lemma 5. 1) and for each $j\geqq N$ define idempotents $e_j, \bar{e}_{j+1}$ by induction on j so that:

$$(e_j)_r=a^{-(j+1)}\sum_{n=N}^{j}E_n(\bar{e}_n)(\bar{e}_j)_r,$$

$$(\bar{e}_{j+1})_r=\text{relative complement of } (e_j)_r \text{ in } (\bar{e}_j)_r.$$

Then for all $k\geqq j\geqq N$ and all i,

$$(a^i e_j)_r+(a^i\bar{e}_{j+1})_r=(a^i\bar{e}_j)_r,$$
$$(a^{j+1}\bar{e}_{j+1})_r\sum_{n=N}^{j}E_n(\bar{e}_n)=0,$$
$$(a^{j+1}e_j)_r\leqq\sum_{n=N}^{j}E_n(\bar{e}_n),$$
$$E_j(\bar{e}_j)=\sum_{n=j}^{k}E_j(e_n)+\sum_{i=1}^{j}(a^i\bar{e}_{k+1})_r.$$

Since $R(\bar{e}_{k+1})\leqq\frac{1}{k+1}$, $D\left(\sum_{i=1}^{j}(a^i\bar{e}_{k+1})_r\right)\leqq\frac{j}{k+1}$ and, consequently, converges to 0 as $k\to\infty$ for fixed j; hence $E_j(\bar{e}_j)=\sum(E_j(e_n);\ n\geqq j)$.

It is now easily verified that (5. 7) holds. We need only show $(A, (a^i e_j)_r;\ i=1, \ldots, j; k\geqq j\geqq N)\perp$ for all $k\geqq N$, and it is therefore sufficient to prove

$$(A, (a^i e_j)_r;\ i=1, \ldots, j;\ k-1\geqq j\geqq N;\ (a^i\bar{e}_k)_r;\ i=1, \ldots, k)\perp \tag{5.9}$$

for all $k\geqq N$. But (5. 9) holds for $k=N$, (by the definition of $\bar{e}_N$), and if (5.9) holds for some $k\geqq N$ then it holds also for $k+1$ since

(i): $(a^i\bar{e}_k)_r$ is the union of the independent elements $(a^i e_k)_r$, $(a^i\bar{e}_{k+1})_r$ for $i=1, \ldots, k$ and

(ii): $$(a^{k+1}\bar{e}_{k+1})_r\big(A+\sum((a^i e_j)_r;\ i=1, \ldots, j;\ k\geqq j\geqq N)+$$
$$+\sum((a^i\bar{e}_{k+1})_r;\ i=1, \ldots, k)\big)\geqq(a^{k+1}\bar{e}_{k+1})_r\sum_{j=N}^{k}E_j(\bar{e}_j)=0.$$

Thus, by induction on k, (5. 9) holds for all $k\geqq N$ and so (5. 7) holds.

If the left side of (5. 8) is denoted by E then for each addend S on the left side of (5. 8) we have $aS\leqq E$. From this we deduce $aE\leqq E$ since the mapping: $x\to ax(x\in\bar{R}_\Re)$ is an order-isomorphism of $\bar{R}_\Re$ onto itself with inverse mapping: $x\to a^{-1}x$(use: $x=(e)_r$ for some e, and then $ax=(ae)_r$).

Now if $E=\Re$ were false we could apply Lemma 5. 1 to obtain an $f\neq 0$ with $(E, (af)_r, \ldots, (a_N f)_r)\perp$. Since $(af)_r(a\bar{e}_N)_r\leqq(af)_r E=0$, so $(f)_r(\bar{e}_N)_r=0$; choosing an idempotent e' with $(e')_r=(f)_r+(\bar{e}_N)_r$ we would have a solution e' of (5. 1) with $(e')_r>(\bar{e}_N)_r$, contradicting the choice of $\bar{e}_N$ as a maximal solution. Thus $E=\Re$ and (5. 8) holds. This completes the proof of the Lemma.

Lemma 5. 3. *Suppose a is purely transcendental in a continuous ring $\Re$. Then for every real $\varepsilon>0$ and for every $p(t)=t^m+z_{m-1}t^{m-1}+\ldots+z_0 t^0$ with all z_i in Z, and $m>1/\varepsilon$ there exists b in $\Re$ such that $R(p(b))<\varepsilon$ and $R(a-b)<\varepsilon$.*

Proof. Choose $N>\frac{m}{\varepsilon}$ and apply Lemma 5. 2 (with $A=0$) to obtain $\Re=$ $=\Sigma((a^i e_j)_r;\ i=1, \ldots, j; j=N, N+1, \ldots)$. For each j let t_j be the largest integer with $mt_j\leqq j$ and set

$$A=\sum((a^{ms}e_j)_r;\ j\geqq N;\ s=1, \ldots, t_j),$$
$$B=\sum((a^i e_j)_r;\ j\geqq N;\ i\neq ms \text{ for all } s=1, \ldots, t_j).$$

Then A and B are complementary in $\Re$ and

$$(g)_r=A,\quad (1-g)_r=B \text{ for some idempotent } g$$

and $R(g)=D(A)\leqq\frac{1}{m}$.

Choose $b=a-a^{-m+1}p(a)g$. Then

$$R(a-b)\leqq R(g)\leqq\frac{1}{m}<\varepsilon.$$

Now we shall prove below:

$$p(b)a^i e_j=0 \text{ for } j\geqq N \text{ and } i=1, \ldots, mt_j. \tag{5.10}$$

From (5. 10) it will follow at once that

$$R(p(b))=D'(p(b))_l=1-D(p(b))_l^r\leqq$$
$$\leqq D(\Sigma((a^i e_j)_r;\ j\geqq N;\ mt_j<i<m(t_j+1)))\leqq\frac{m}{N}<\varepsilon$$

as required. Thus we need only prove (5. 10).

Clearly $bx=ax$ whenever $gx=0$, that is, whenever $x\in B$. By repetition $b^h(a^i e_j)=a^h(a^i e_j)$ whenever $a^i e_j, a^{i+1}e_j, \ldots, a^{i+h-1}e_j$ are all in B.

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Hence, if $i=ms+1$ for some s with $0\leqq s<t_j$ we can deduce:

$$b^h(a^ie_j)=a^h(a^ie_j) \text{ for } h=0,1,\ldots,m-1,$$

$$b^m(a^ie_j)=bb^{m-1}(a^ie_j)=b(a^{m-1+i}e_j)=$$

$$=(a-a^{-m+1}p(a)g)a^{m(s+1)}e_j=$$

$$=a^{m(s+1)+1}e_j-p(a)a^{ms+1}e_j \quad (\text{use } ga^{m(s+1)}=a^{m(s+1)})=$$

$$=a^m(a^ie_j)-p(a)a^ie_j;$$

and so $p(b)a^ie_j=-p(a)a^ie_j+p(a)a^ie_j=0$. Therefore $p(b)a^ie_j=0$ for all $i=ms+1$, $0\leqq s<t_j$. But this implies $p(b)a^h(a^ie_j)=p(b)b^h(a^ie_j)=b^hp(b)a^ie_j=0$ for such i and all $h=0,1,\ldots,m-1$. This proves (5.10) and establishes the Lemma.

6. Preliminary Lemmas

In this section L will be assumed to be a relatively complemented modular lattice which is $\aleph_0$-complete and $\aleph_0$-continuous. This means: whenever I is countable,

(6.1) $$\Sigma(a_\alpha;\ \alpha\in I) \quad \text{and} \quad \Pi(a_\alpha;\ \alpha\in I) \quad \text{exist},$$

(6.2) $$b\Sigma(a_\alpha;\ \alpha\in I)=\Sigma_F(b\Sigma(a_\alpha;\ \alpha\in F)),$$

$$b+\Pi(a_\alpha;\ \alpha\in I)=\Pi_F(b+\Pi(a_\alpha;\ \alpha\in F))$$

(where F varies over all finite subsets of I). $\mathfrak{R}$ will be assumed to be a regular ring not necessarily with unit such that

(6.3) $$\bar{R}_{\mathfrak{R}} \text{ is } \aleph_0\text{-complete and } \aleph_0\text{-continuous.}$$

If $\mathfrak{R}$ has a unit $\bar{R}_{\mathfrak{R}}$ and $\bar{L}_{\mathfrak{R}}$ are anti-isomorphic under the mutually inverse mappings

(6.4) $$(a)_r\to(a)_r^l\equiv(x;\ xa=0), \quad (a)_l\to(a)_l^r\equiv(x;\ ax=0)$$

(cf. [4, page 71, Corollary 2 to Lemma 2.2]). Since $\aleph_0$-completeness and $\aleph_0$-continuity are self-dual properties it follows that they are possessed by $\bar{L}_{\mathfrak{R}}$ if by $\bar{R}_{\mathfrak{R}}$.

Lemma 6.1. *Let L be an $\aleph_0$-complete $\aleph_0$-continuous relatively complemented modular lattice. Suppose a, b are in L and suppose $Tx=y$ defines a $(1,1)$ mapping of $L(0,a)$ onto $L(0,b)$ such that (i): $a_1\leqq a_2\leqq a$ holds if and only if $Ta_1\leqq Ta_2\leqq Ta(=b)$ and (ii): $a_1\sim Ta_1$ whenever $a_1(Ta_1)=0$. Then $a\sim b$.*

Proof. (i) If $Ta_1\leqq a_1$ let d be a relative complement of Ta_1 in a_1. Then $d, Td, \ldots, T^nd, \ldots$ are all defined, independent and mutually perspective. To prove this, note firstly that $Ta_1\leqq a_1$ implies that $T^2a_1=T(Ta_1)$ is defined and $T^2a_1\leqq Ta_1\leqq a_1$. By induction, T^na_1 is defined for all n and so T^nd is defined for all n since $d\leqq a_1$. Then for all $n\geqq 0$, $m\geqq 1$,

$$(T^nd)(T^{n+1}d+\ldots+T^{n+m}d)=T^n(d(Td+\ldots+T^md))\leqq T^n(d(Ta_1))=T^n(0)=0,$$

so $d, Td,\ldots$ are independent Finally, $T(T^nd)=T^{n+1}d$ and $(T^nd)\ (T^{n+1}d)=0$, so the

hypotheses of Lemma 6.1 yield $T^nd\sim T^{n+1}d$. From this it follows that $d, Td, \ldots$ are mutually perspective.

Now the argument given in [4, page 21, Theorem 3.8] is valid with our present hypothesis on L, so $d=0$. Thus $Ta_1\leqq a_1$ *implies* $Ta_1=a_1$.

(ii) Set $a_1'=a$, $b_1'=b$. Define $a_{n+1}', a_n, b_{n+1}', b_n$ for all $n\geqq 1$ by induction on n so that:

$$a_{n+1}'=a_n'b_n', \qquad a_n=\text{relative complement of } a_{n+1}' \text{ in } a_n',$$

$$b_{n+1}'=Ta_{n+1}', \qquad b_n=Ta_n.$$

Then (cf. [1, page 543, Lemma 2.11]),

$$a_n'=a_{n+1}'+a_n, \quad a_0'\geqq a_1'\geqq\ldots,$$

$$a=\Pi a_n'+\Sigma a_n, \quad (\Pi a_n', a_1, a_2, \ldots)\perp,$$

$$b_n'=b_{n+1}'+b_n, \quad b_0'\geqq b_1'\geqq\ldots,$$

$$b=\Pi b_n'+\Sigma b_n, \quad (\Pi b_n', b_1, b_2, \ldots)\perp,$$

$$T(\Pi a_n')=\Pi b_n'\geqq\Pi a_n'.$$

Applying the preceding paragraph (with T^{-1} in place of T) it follows that $T(\Pi a_n')=\Pi a_n'=\Pi b_n'$. Since $a_nb_n=0$ for $n\geqq 1$, so $a_n\sim b_n$. Now under the present hypothesis on L, perspectivity is additive for countable independent families (cf. [1, page 561, Theorem 6.2]). Then $a=\Pi a_n'+\Sigma a_n$ is perspective to $b=\Pi b_n'+\Sigma b_n$, proving the Lemma.

Lemma 6.2.[5] *Let $\mathfrak{R}$ be a regular ring with unit such that $\bar{R}_{\mathfrak{R}}$ (hence also $\bar{L}_{\mathfrak{R}}$) is $\aleph_0$-complete and $\aleph_0$-continuous. Then for all a, b in $\mathfrak{R}$:*

(6.5) $$(a)_r\sim(b)_r \quad \textit{if and only if} \quad (a)_l\sim(b)_l.$$

(6.6) $$(a)_r\precsim(b)_r \quad \textit{if and only if} \quad (a)_l\precsim(b)_l.$$

(6.7) $$(a)_l^r\Pi((a^n)_r;\ n\geqq 1)=(a)_r^l\Pi((a^n)_l;\ n\geqq 1)=0.$$

(6.8) *There exists a unique idempotent e such that*

$$(e)_r=\Pi((a^n)_r;\ n\geqq 1) \quad \text{and} \quad (e)_l=\Pi((a^n)_l;\ n\geqq 1).$$ [6]

Proof of (6.5). Suppose $(a)_r$ and $(b)_r$ have a common complement in $\mathfrak{R}$. Then by [4, page 69, Theorem 2.1] there exist idempotents f, g such that $(a)_r=(f)_r$, $(b)_r=(g)_r$ and the common complement is $(1-f)_r=(1-g)_r$. This implies $f=fg$, $g=gf$ so $(f)_l=(g)_l$. Now perspectivity is transitive in $\bar{L}_{\mathfrak{R}}$ under our present hypotheses on $\mathfrak{R}$ (see [1, page 550, Theorem 5.1]), so if we can show $(a)_r=(f)_r$ implies $(a)_l\sim(f)_l$, the same result for $(b)_r=(g)_r$ will yield $(a)_l\sim(f)_l$, $(f)_l=(g)_l$, $(g)_l\sim(b)_l$,

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[5] (6.5) was proved in [4, page 223] for the special case of complete rank rings.

[6] Although not required for this paper, the following remark may be of interest to the reader. Assume the hypotheses of Lemma 6.2 and suppose $x\in\mathfrak{R}$. From (6.6), (6.7) and (6.8) it follows that $(x)_r=(e)_r$, $(x)_l=(e)_l$ for some idempotent e (necessarily unique) if and only if $(x^2)_r=(x)_r$, equivalently: for $a\in\mathfrak{R}$, $x^2a=0$ implies $xa=0$.

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hence $(a)_l \sim (b)_l$. This will prove: $(a)_r \sim (b)_r$ implies $(a)_l \sim (b)_l$. We shall show now that $(a)_r = (f)_r$ implies $(a)_l \sim (f)_l$.

Since $(a)_r = (f)_r$, so $a = fa$, $f = ad$ for some d. Replacing d by df we can suppose $d = df$ so $(d)_l = (f)_l$. Now define the mappings T, T_1 by:

$$T(x)_l = (xd)_l \quad \text{for} \quad (x)_l \leqq (a)_l,$$

$$T_1(x)_l = (xa)_l \quad \text{for} \quad (x)_l \leqq (f)_l.$$

If $(x)_l \leqq (a)_l$ then $x = ua$, $xda = uada = ufa = ua = x$. It follows that T and T_1 are mutually inverse (1, 1) order preserving mappings between $L(0, (a)_l)$ and $L(0, (f)_l)$. Moreover, if $(x)_l \leqq (a)_l$ and $(x)_l\ (xd)_l = 0$ then $(x)_l \sim T(x)_l = (xd)_l$ with axis $(x + xd)_l$; for

(i) $(x)_l + (x + xd)_l = (x)_l + (xd)_l = (xd)_l + (x + xd)_l$;

(ii) $(x)_l (x + xd)_l = 0$ since $y = ux = v(x + xd)$ implies $(u - v)x = v(xd) \in \in (x)_l (xd)_l = 0$, $vxd = 0$, $vx = vxda = 0$, $y = vx + vxd = 0 + 0 = 0$.

(iii) $(xd)_l (x + xd)_l = 0$ since $y = uxd = v(x + xd)$ implies $vx = (u - v)xd \in \in (x)_l (xd)_l = 0$, $vx = 0$, $y = vx + vxd = 0$.

Now Lemma 6. 1 applies and shows that $(a)_l \sim (f)_l$.

Thus $(a)_r \sim (b)_r$ implies $(a)_l \sim (b)_l$. This result is equivalent to its dual: $(a)_l \sim (b)_l$ implies $(a)_r \sim (b)_r$. This proves (6. 5).

Proof of (6. 6). If $(a)_r \sim (c)_r \leqq (b)_r$ there exist orthogonal idempotents e, f such that $(e)_r = (c)_r$, $(f)_r =$ relative complement of $(c)_r$ in $(b)_r$. Then $(a)_r \sim (e)_r$, $ef = 0$, $(e + f)_r = (e)_r + (f)_r = (b)_r$.

By (6. 5) $(e + f)_l \sim (b)_l$. Since $e = e(e + f)$, so $(e)_l \leqq (e + f)_l$, $(e)_l \precsim (b)_l$. Using (6. 5) again and the transitivity of perspectivity [1, page 550, Theorem 5. 1] it follows that $(a)_l \sim (e)_l \precsim (b)_l$, and $(a)_l \precsim (b)_l$. So $(a)_r \precsim (b)_r$ implies $(a)_l \precsim (b)_l$. Combining this result with its dual we obtain (6. 6).

Proof of (6. 7). We need only prove $x \in \big((a)_l^r \Pi((a^n)_r;\ n \geqq 1)$ implies $x = 0$ (together with its dual this yields (6. 7)). We have: $ax = 0$ but for every $n \geqq 1$, $x = a^n y_n$ for some y_n. For each $n \geqq 1$ let $(b_n)_l$ be a relative complement of $(a^{n+1})_l$ in $(a^n)_l$. Then $(a^n)_l = (a^{n+1})_l + (b_n)_l$, $a_n = ua^{n+1} + vb_n$ for some u, v;

$$x = a^n y_n = (ua^{n+1} + vb_n) y_n = uaa^n y_n + vb_n y_n = uax + vb_n y_n = 0 + vb_n y_n = vb_n y_n.$$

So $(x)_l \leqq (b_n y_n)_l$ and (6. 6) yields $(x)_r \precsim (b_n y_n)_r \leqq (b_n)_r$. Hence, again by (6. 6), $(x)_l \precsim \precsim (b_n)_l$, that is, $(x)_l \sim (x_n)_l \leqq (b_n)_l$ for some x_n. But the $(b_n)_l$ are independent, so the $(x_n)_l$ are independent. It follows that all $(x_n)_l = 0$ (cf. [4, page 21, Theorem 3. 8]) and so $(x)_l = 0$, $x = 0$ as required.[7])

[7]) This type of argument shows: *if* $x \in \Pi_n(a^n)_r$ *and* $\Pi_n(a^n x)_l = 0$ *then* $x = 0$. Indeed, for fixed m, $a^m = ua^{m+n} + vb_n$ with $(b_n)_l =$ relative complement of $(a^{m+n})_l$ in $(a^m)_l$. So $x = a^m y = ua^{m+n} y + + vb_n y = ua^n x + vb_n y$; $(x)_l \leqq (a^n x)_l + (b_n y)_l$; $(x)_l \leqq \Pi_n((a^n x)_l + \Sigma_m (b_m y)_l) = \Pi_n (a^n x)_l + \Sigma_n (b_n y)_l = = \Sigma_n (b_n y)_l$. But the b_n can be chosen so that $((b_n)_l;\ n \geqq 1) \perp$ and $\Sigma_n (b_n)_l =$ relative complement of $\Pi_n (a^n)_l$ in $(a^m)_l$ (use [1, Theorem 6. 2]). Then $(x)_l \leqq \Sigma_n (b_n y)_l \precsim$ relative complement of $\Pi_n (a^n)_l$ in $(a^m)_l$ and letting $m \to \infty$, we obtain $(x)_l = 0$ (use [1, Theorem 6. 1]). Hence, $x = 0$.

Proof of (6. 8). The existence of e as described is equivalent to

$$\Pi((a^n)_r;\ n \geqq 1)\big(\Pi((a^n)_l;\ n \geqq 1)\big)^r = 0, \tag{6. 9}$$

and

$$\Pi((a^n)_r;\ n \geqq 1) + \big(\Pi((a^n)_l;\ n \geqq 1)\big)^r = \mathfrak{R}, \tag{6. 10}$$

and then [4, page 69, Theorem 2. 1] shows that if e exists, it is uniquely determined.

The statement (6. 10) is equivalent to the left-right dual of (6. 9) (by (6. 3) and (6. 4)). So we need only prove (6. 9), equivalently:

$$\left(\sum_{n=1}^{\infty} (a^n)_l^r\right) \Pi((a^n)_r;\ n \geqq 1) = 0. \tag{6. 11}$$

Because of (6. 2) we need only prove (6. 11) with arbitrary finite m in place of ∞. Since $(a^1)_l^r \leqq (a^2)_l^r \leqq \ldots$ we need only prove: for each $m \geqq 1$,

$$(a^m)_l^r \Pi((a^n)_r;\ n \geqq 1) = 0. \tag{6. 12}$$

But from (6. 7), with a there replaced by a^m, the left side of (6. 22) is less than or equal to $(a^m)_l^r \Pi((a^{mn})_r;\ n \geqq 1) = 0$. This proves (6. 8).

Lemma 6. 3. *Let $\mathfrak{R}$ be a regular ring with unit such that $\overline{R}_{\mathfrak{R}}$ and $\overline{L}_{\mathfrak{R}}$ satisfy the axioms I—V of discrete or continuous geometry* (cf. [4, pages 1, 2], *irreducibility is not assumed*). *Suppose S is a subset of $\mathfrak{R}$ such that for any a, b in S there is some c in S such that $(c)_r \leqq (ab)_r (ba)_r$ and some d in S such that $(d)_l \leqq (ab)_l (ba)_l$. Then there exists an idempotent $e = e(S)$ such that*

$$(e)_r = \Pi((a)_r;\ a \in S), \qquad (e)_l = \Pi((a)_l;\ a \in S)$$

(this e is unique) and e commutes with any u which commutes with every a in S.

Remark. The hypotheses on S are satisfied if $a, b \in S$ imply $ab = ba \in S$, in particular if S consists of all $p(a), p \in P$ with a fixed, or if S consists of all $p^n(a)$, $n \geqq 1$ with $p \in P$, p fixed and a fixed.

Proof. Note that the hypotheses on S imply: for each d in S and each $n \geqq 1$, $(d^n)_r \geqq (a)_r$ for some a in S.

Now to prove e exists we need only prove (as in the proof of (6. 8)) that $\big(\Sigma((a)_l^r;\ a \in F)\big)\, \Pi((a)_r;\ a \in S) = 0$ for every finite subset F of S. But the hypotheses on S imply that for some d in S, $(d)_l \leqq (a)_l$ for all $a \in F$, hence $(d)_l^r \geqq (a)_l^r$ for all a in F. Thus we need only prove $(d)_l^r \Pi((a)_r;\ a \in S) = 0$ for every d in S. But the hypotheses on S together with (6. 7) yield, $(d)_l^r \Pi((a)_r;\ a \in S) \leqq (d)_l^r \Pi((d^n)_r;\ n \geqq 1) = 0$, so e does exist as described.

If $ua = au$ for all $a \in S$, then $(e)_r \leqq (a)_r$ yields $av = e$, $ue = uav = auv \in (a)_r$, so $ue \in \Pi((a)_r;\ a \in S)$, $ue = eue$. By the dual of this result, $eu = eue$ so $eu = ue$ as required.

Corollary. *If $ua = au$ for all a in S then there is a unique decomposition $u = u_1 + u_2$ with u_1 in $\mathfrak{R}(e)$, u_2 in $\mathfrak{R}(1 - e)$, namely $u_1 = ue$, $u_2 = u(1 - e)$ and for every p in P*

(i) $p(u) = \big(p(ue)$ *in* $\mathfrak{R}(e)\big) + \big(p(u(1 - e))$ *in* $\mathfrak{R}(1 - e)\big)$,

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(ii) *$(p(ue)$ in $\Re(e))_r$ and $(p(u(1-e))$ in $\Re(1-e))_r$ are independent and their lattice union is $(p(u))_r$.*

(iii) *If this u is also in S then $(ue)_r=(e)_r$, equivalently, if $e\neq 0$, ue has a reciprocal in $\Re(e)$.*

Proof. By Lemma 6.3, $eu=ue$ and the unique decomposition of u follows immediately. Since

$$p(u) = p(u)e+p(u)\,(1-e) = (p(ue) \text{ in } \Re(e))+(p(u(1-e)) \text{ in } \Re(1-e)),$$

(i) holds. Since the two addends on the right side of (i) are orthogonal, (ii) follows.

Finally, if u is also in S then $(u)_r\geqq(e)_r$ so $e=uv$ for some v, $e=(eu)v$, hence $(e)_r\geqq(eu)_r$. But $(eu)_r\leqq(e)_r$, so (iii) holds.

7. Decomposition into limiting algebraic and transcendental parts

7.1. Assume that $\Re$ is a regular ring with unit for which $\overline{R}_\Re$ and $\overline{L}_\Re$ satisfy the axioms I—V of discrete or continuous geometry (cf. [4, pages 1, 2]) (irreducibility is not assumed). Let a be a fixed element of $\Re$ and set $S_0=(p(a); p\in P)$ and for each p in P, $S_p=(p^n(a); n\geqq 1)$.

Clearly S_0 (and each S_p) satisfies the hypotheses of Lemma 6.3. We may set $e_0=e(S_0)$, $f_0=1-e_0$, $f_p=e(S_p)$, $e_p=1-f_p$.

Since all members of S_0 commute, it follows from Lemma 6.3 that they all commute with e_0 and with every f_p, and then, again from Lemma 6.3, that e_0, all f_p commute. So e_0, all e_p commute. Moreover $S_0\supset S_p$ so $(e_0)_r\leqq(f_p)_r$, hence $f_pe_0=e_0$, $e_pe_0=0$.

Suppose p, q in P have the property:

(7.1) $p(t)h+q(t)k=1$ for some h,k of the form $z_mt^m+\ldots+z_0$ with $m\geqq 0$ and all z_i in Z.

Then p^m, q^n have this property (7.1) also (use: $1=(ph+qk)^{m+n}=p^mh_1+q^nk_1$ for some h_1, k_1 of the form $z_jt^j+\ldots+z_0$ with $j\geqq 0$ and all z_i in Z). Hence $p^m(a)h_1(a)+q^n(a)k_1(a)=1$,

$$\begin{aligned}\Re &= (p^m(a))_r+(q^n(a))_r = \Pi((p^m(a))_r+(q^n(a))_r;\ m\geqq 1,\ n\geqq 1) =\\ &= \Pi((p^m(a))_r+\Pi((q^n(a))_r;\ n\geqq 1);\ m\geqq 1) =\\ &= \Pi((p^m(a))_r;\ m\geqq 1)+\Pi((q^n(a))_r;\ n\geqq 1) =\\ &= (f_p)_r+(f_q)_r=\Re \text{ and, consequently, } 0=\Re^l=(f_p)_r^l(f_q)_r^l=(e_p)_l(e_q)_l\end{aligned}$$

Thus (7.1) implies $e_pe_q=0$, $(e_q)_r\leqq(f_p)_r$.

7.2. From now on we assume also that $\overline{R}_\Re$ is irreducible so that $\Re$ is a complete rank ring, either discrete or continuous and so Z is a commutative division ring. Let $P'=P'(a)$ denote the set of $p\in P$ for which $(p(a))_r\neq\Re$ and p is irreducible (that is, $p\neq p_1p_2$ with $p_1, p_2\in P$). If p, q are in P' and $p\neq q$, then p, q are relatively prime with respect to coefficient domain Z so (7.1) holds, hence $e_pe_q=0$. Moreover

if $p\in P'$ then $(f_p)_r\leqq(p(a))_r\neq\Re$ so $f_p\neq 1$, $e_p\neq 0$. Since the e_p are mutually orthogonal, $1\geqq R(e_p+e_q+\ldots)=R(e_p)+R(e_q)+\ldots$ with all $R(e_p)>0$, so P' is finite or denumerable.

Let P' be enumerated: $p_1, p_2, \ldots$ (say) and from now on write e_m, f_m in place of e_p, f_p (with $p=p_m$). Since each p in P can be expressed as a product of powers of irreducible polynomials $p=p_1^{n_1}\ldots p_m^{n_m}$ for suitable $n_1, \ldots, n_m\geqq 0$, it follows that

$$\begin{aligned}(e_0)_r &= \Pi((p(a)_r;\ p\in P)=\\ &= \Pi((p_m^n(a))_r;\ m\geqq 1,\ n\geqq 1)=\Pi((f_m)_r;\ m\geqq 1),\\ (e_0)_r^l &= (f_0)_l=\Sigma((f_m)_r^l;\ m\geqq 1)=\Sigma((e_m)_l;\ m\geqq 1).\end{aligned}$$

By the dual argument, $(f_0)_r=\Sigma((e_m)_r;\ m\geqq 1)$. Since the e_i, $i\geqq 0$ are orthogonal, for all $m\geqq 0$

$$(f_m)_r=\Sigma((e_i)_r;\ i\geqq 0,\ i\neq m),$$
$$(f_m)_l=\Sigma((e_i)_l;\ i\geqq 0,\ i\neq m).$$

Also

$$\Re=\Sigma((e_m)_l;\ m\geqq 0)=\Sigma((e_m)_r;\ m\geqq 0),$$
$$\Sigma(R(e_m);\ m\geqq 0)=1,$$
$$R(1-(e_0+\ldots+e_m))\to 0 \text{ as } m\to\infty.$$

Lemma 7.1. *With notation as in the preceding paragraphs,*

(7.2) *ae_0 is purely transcendental in $\Re(e_0)$,*

(7.3) *if $i\geqq 1$, ae_i is limiting p_i-algebraic in $\Re(e_i)$,*

(7.4) *if $p=p_1^{n_1}\ldots p_m^{n_m}$ then $(1-(e_1+\ldots+e_m))_r$ and the $(p_i^{n_i}(ae_i)$ in $\Re(e_i))_r$, $i=1, \ldots, m$ are an independent set and their lattice union is $(p(a))_r$.*

Proof. For every p in P, $(p(ae_0)$ in $\Re(e_0))=e_0p(ae_0)$ and, by (iii) of the Corollary to Lemma 6.3, has a reciprocal in $\Re(e_0)$. The Corollary to Lemma 2.1 now yields (7.2).

For every $i\geqq 1$, a and e_i commute so $(p_i(ae_i)$ in $\Re(e_i))=e_ip_i(a)$. Hence

$$\Pi((p_i^n(ae_i))_r;\ n\geqq 1)=(e_i)_r\Pi((p_i^n(a))_r;\ n\geqq 1)=(e_i)_r(f_i)_r=0.$$

This proves (7.3).

Now let $g_m=1-(e_1+\ldots+e_m)=f_1\ldots f_m$. Then $((g_m)_r, (e_1)_r, \ldots, (e_m)_r)\perp$ (the g_m, e_i; $i=1, \ldots, m$ are even orthogonal) and $(p_i^{n_i}(ae_i)$ in $\Re(e_i))=e_ip_i^{n_i}(ae_i)\in(e_i)_r$. It follows that $((g_m)_r, (p_i^{n_i}(ae_i))_r;\ i=1, \ldots, m)\perp$. Also,

$$\begin{aligned}p(a) &= p(ae_1+\ldots+ae_m+ag_m) =\\ &= e_1p(ae_1)+\ldots+e_mp(ae_m)+g_mp(ag_m) =\\ &= p(ae_1)e_1+\ldots+p(ae_m)e_m+p(ag_m)g_m.\end{aligned}$$

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Since $g_m, e_1, \ldots, e_m$ are orthogonal,

$$(p(a))_r = (p(ae_1)e_1)_r + \ldots + (p(ae_m)e_m)_r + (p(ag_m)g_m)_r.$$

But $e_i p(ae_i) = e_i p(a) = e_i p_1^{n_1}(a) \ldots p_m^{n_m}(a)$. If for some $j = 1, \ldots, m$ we consider $S = S_{p_j} = (p_j^n(a);\ n \geqq 1)$ and apply (iii) of the Corollary to Lemma 6.3 with $u = p_j^{n_j}(a)$ and $e = f_j$ we see that $p_j^{n_j}(a) b_j = f_j$ for some b_j. If $j \neq i$, then $f_j e_i = e_i$ so $p_j^{n_j}(a)(b_j e_i) = f_j e_i = e_i$ which implies

$$(e_i p(ae_i))_r = (e_i p_i^{n_i}(a) \Pi(p_j^{n_j}(a);\ j \neq i,\ j = 1, \ldots, m))_r = (e_i p_i^{n_i}(a))_r$$

for $i = 1, \ldots, m$. Similarly since $f_j g_m = g_m$,

$$(g_m p(ag_m))_r = (g_m p(a))_r = (g_m)_r.$$

From this, (7.4) follows at once.

Remark. ae_0 will be called the *transcendental part* of a, $a(1-e_0)$ will be called the *almost algebraic part* of a, and for $i \geqq 1$, the ae_i will be called the p_i — *limiting algebraic parts* of a.

8. The main theorem

Theorem 8.1. *Let a be an element of a continuous ring $\mathfrak{R}$ and suppose given any real $\varepsilon > 0$. Then there exists $p \in P$ and $b \in \mathfrak{R}$ such that $p(b) = 0$ and $R(a-b) < \varepsilon$. Moreover p can be of the form $p = p_1^{n_1} \ldots p_m^{n_m}$ if $P'(a) = (p_1, p_2, \ldots)$ is not empty and p can be required to be any assigned polynomial in P of degree $> \frac{1}{\varepsilon}$ if $P'(a)$ is empty.*

Proof. We shall show first that with p as described, b' exists with $R(a-b') < \frac{\varepsilon}{2}$, $R(p(b')) < \frac{\varepsilon}{2}$. This follows from Lemma 5.3 if $P'(a)$ is empty (then a is purely transcendental) so we may suppose $P'(a)$ is not empty.

Set $g_m = 1 - (e_0 + \ldots + e_m)$. Choose m so large that $R(g_m) < \frac{\varepsilon}{4}$ and for $i = 1, \ldots, m$ choose n_i so large that $n = n_1 + \ldots + n_m$ and $p = p_1^{n_1} \ldots p_m^{n_m}$ satisfy; $n > 1$, $R(ae_0 - b_0) < \frac{\varepsilon}{4}$, $R(e_0 p(b_0)) < \frac{\varepsilon}{4(m+1)}$ for some b_0 in $\mathfrak{R}(e_0)$, and $R(p_i^{n_i}(ae_i)e_i) < \frac{\varepsilon}{4(m+1)}$ for $i = 1, \ldots, m$.

To see that such n_i exist, use the fact that for any c and any idempotent e, if $c \in e\mathfrak{R}e$ then: Rank of c in $\mathfrak{R}$ is equal to Rank of e in $\mathfrak{R}$ multiplied by (normalized) Rank of c in $e\mathfrak{R}e$. Now by (7.3), ae_i is limiting p_i-algebraic in $\mathfrak{R}(e_i)$, a fortiori, $R(e_i p_i^{n_i}(ae_i)) \to 0$ as $n_i \to \infty$; and by (7.2), ae_0 is purely transcendental in $\mathfrak{R}(e_0)$ which implies by Lemma 5.3, whenever $\eta > 0$ and p is given of degree n, $n > \frac{1}{\eta}$, there exists b_0 in $\mathfrak{R}(e_0)$ (and so in $\mathfrak{R}$), with the properties $R(e_0 p(b_0)) \leqq R(e_0)\eta$, $R(ae_0 - b_0) \leqq R(e_0)\eta$.

Set $b' = b_0 + ae_1 + \ldots + ae_m$. Then

$$R(a-b') = R(ae_0 - b_0) + R(ag_m) \leqq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

$$R(p(b')) \leqq R(e_0 p(b_0)) + R(e_1 p(ae_1)) + \ldots + R(e_m p(ae_m)) + R(g_m) \leqq$$

$$\leqq R(e_0 p(b_0)) + R(p_1^{n_1}(ae_1)e_1) + \ldots + R(p_m^{n_m}(ae_m)e_m) + R(g_m) <$$

$$< (m+1)\frac{\varepsilon}{4(m+1)} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

so b' exists as stated above.

Now with this b' let $(e)_r = (p(b'))_r$ so $e = p(b')u$ for some u. Then $b'e = b'p(b')u = p(b')b'u$ so $b'e = eb'e$, $(1-e)b'e = 0$, $(1-e)b'(1-e) = (1-e)b'$; so for all x in $\mathfrak{R}(e)$,

$$p((1-e)b' + x) = (1-e)p(b') + ep(x) =$$

$$= (1-e)ep(b') + ep(x) = ep(x).$$

By the Corollary to Lemma 3.1 we can choose x_0 in $\mathfrak{R}(e)$ so that $p(x_0) = 0$. Choose $b = (1-e)b' + x_0$. Then $p(b) = 0$ and $R(a-b) \leqq R(a-b') + R(eb' - ex_0) <$

$< \frac{\varepsilon}{2} + R(p(b')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus this b satisfies the requirements of the Theorem.

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INDEX — TARTALOM

Halperin, I. Von Neumann's Arithmetics of Continuous Rings 1
Ádám, A. Über die monotone Superposition der Wahrheitsfunktionen 18
Харазов, Д. Ф. О спектре вполне непрерывных операторов, аналитически зависящих от параметра, в линейных топологических пространствах 38
Чакань, Б. Об эквивалентности некоторых классов алгебраических систем 46
Гечег, Ф. Шрейерово расширение мультиоператорных Ω групп 58
Szász, F. Bemerkung zu meiner Arbeit „Über Gruppen, deren sämtliche nichttriviale Potenzen zyklische Untergruppen der Gruppe sind" 64
Nunke, R. J. Slender Groups 67
Wiegandt, R. Bemerkung zu einer Arbeit von Herrn Steinfeld 74
Goffman, C. On the approximate limits of a real function 76
Neugebauer, C. J. A theorem on derivates 79
Leindler, L. Über die starke Summierbarkeit der Orthogonalreihen 82
Móricz, F. Über die Rieszsche Summation der Orthogonalreihen 92
Halperin, I. The product of projection operators 96
Bruijn, de N. G. On unitary equivalence of unitary dilations of contractions in Hilbert space 100
Sz.-Nagy, B. et *Foiaş, C.* Sur les contractions de l'espace de Hilbert. V. Translations bilatérales 106
Sz.-Nagy, B. et *Foiaş, C.* Sur les contractions de l'espace de Hilbert. VI. Calcul fonctionnel 130
Itô N. und *Szép J.* Über die Quasinormalteiler von endlichen Gruppen 168
Maurer, I. Gy. Über im Endomorphismenringe einer abelschen Gruppe definierte unendliche Reihen und Produkte 171
Errata 175
Bibliographie 176

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